6/6/06 Oberwolfach G. Papanicolaou

Basic resolution theory

Analysis of the functional

$$I^{KM}(y^s) = \sum_{x_r} P(x_r, \tau(x_r, y^s))$$

when $P(x_r, t)$ comes from a point source at y.

$$\hat{P}(x_r,\omega) = \hat{f}_B(\omega-\omega_0)\, \hat{G}_0(x_r, y, \omega)$$

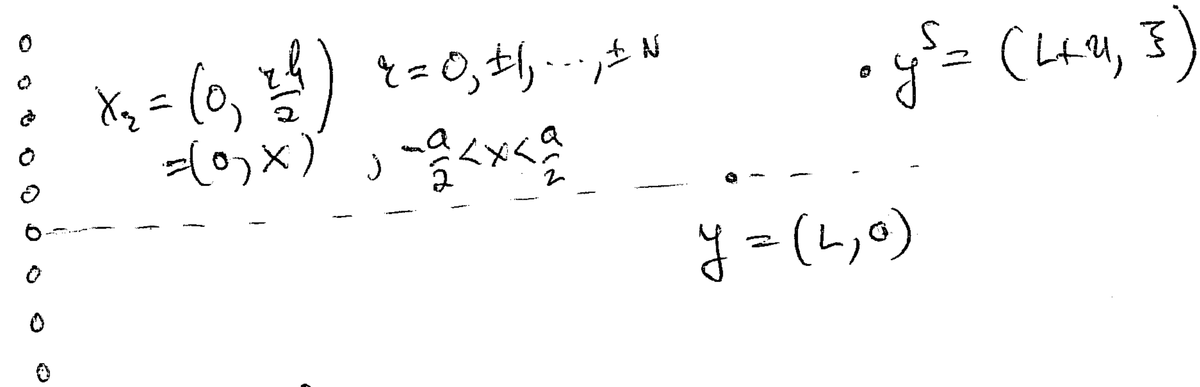
We consider first the expression.

$$\sum_{x_r} \frac{e^{i\omega(\tau(x_r,y) - \tau(x_r,y^s))}}{4\pi|x_r-y|}$$

at a fixed frequency ω. With $k=\frac{\omega}{c_0}$ we have

$$\sum_{x_r} \frac{e^{ik(|x_r-y| - |x_r-y^s|)}}{4\pi|x_r-y|}$$

We introduce coordinates as follows



$a = Nh$, $a \ll L$.

$$|x_\xi - y| = (L^2 + x^2)^{\frac{1}{2}} = L\left(1 + \left(\frac{x}{L}\right)^2\right)^{\frac{1}{2}}$$

$$\sim L\left(1 + \frac{1}{2}\left(\frac{x}{L}\right)^2\right)$$

$$= L + \frac{x^2}{2L} \qquad \text{paraxial approximat}$$

$$|x_\xi - y^s| = \left((L+u)^2 + (x-\xi)^2\right)^{\frac{1}{2}}$$

$$\sim (L+u)\left(1 + \left(\frac{x-\xi}{L+u}\right)^2\right)^{\frac{1}{2}}$$

$$\sim L + u + \frac{(x-\xi)^2}{2(L+u)}$$

$$|x_\xi - y| - |x_\xi - y^s| \sim -u + \frac{x^2}{2L} - \frac{(x-\xi)^2}{2(L+u)}$$

$$= -u + \frac{x^2}{2L} - \frac{x^2 - 2x\xi + \xi^2}{2(L+u)}$$

$$= -u - \frac{\xi^2}{2(L+u)} + \frac{x\xi}{L+u} + \frac{1}{2}\left(\frac{1}{L} - \frac{1}{L+u}\right)x^2$$

$$= -u - \frac{\xi^2}{2(L+u)} + \frac{x\xi}{L+u} + \frac{ux^2}{2L(L+u)}$$

The expression of interest can be approximated by.

$$\frac{qe^{-ik\left(u+\frac{\xi^2}{2(L+u)}\right)}}{4\pi L h} \int_{-a/2}^{a/2} e^{ik\left(\frac{x\xi}{L+u} + \frac{ux^2}{2L(L+u)}\right)} dx$$

$$= \frac{e^{-ik\left(u+\frac{\xi^2}{2(L+u)}\right)}}{2\pi L h} \int_{-a/2}^{a/2} e^{ik\left(\frac{x\xi}{L+u} + \frac{ux^2}{2L(L+u)}\right)} dx$$

at $u=0$ the absolute value of this expression is

$$\frac{1}{2\pi L h}\left|\int_{-a/2}^{a/2} e^{ik\frac{x\xi}{L}}\,dx\right| = \frac{1}{2\pi \cancel{L} h}\,\frac{\cancel{L}}{ik\xi}\left(e^{ik\frac{\xi a}{2L}} - e^{-ik\frac{\xi a}{2L}}\right)$$

$$= \frac{1}{\pi h k\xi}\sin\left(\frac{k\xi a}{2L}\right)$$

First zero of sin at $\frac{k\xi a}{2L} = \pi$, $\xi = \frac{2\pi L}{ka} = \frac{\lambda L}{a}$

We return to the general case and change variables $x = ax'$ so that we have, after dropping primes.

$$\frac{a}{2\pi L h}\left|\int_{-1/2}^{1/2} e^{ik\left(\frac{a\xi x}{L+u} + \frac{a^2 u x^2}{2L(L+u)}\right)}\,dx\right|$$

Rewrite this as, retaining only the integral.

$$\int_{-1/2}^{1/2} e^{i\pi\left(\frac{ka\xi}{\pi(L+u)}x + \frac{k}{2\pi}\frac{a^2u}{2L(L+u)}x^2\right)}\,dx$$

But $\lambda = \frac{2\pi}{k}$ so the exponent is.

$$\frac{2\xi a}{\lambda(L+u)}x + \frac{ua^2}{\lambda L(L+u)}x^2.$$

For $u \ll L$ this simplifies to

$$2\frac{\xi}{(\lambda L/a)}\, x + \frac{u}{\lambda\left(\frac{L}{a}\right)^2}\, x^2.$$

We see that the cross range resolution limit corresponds to $\xi \sim \frac{\lambda L}{a}$. So we may also take as range resolution limit $u \sim \lambda\left(\frac{L}{a}\right)^2$. But we must check that this is consistent with $u << L$, since $a << L$. In general, returning to the previous page, we may define the range resolution by

$$u = \lambda\frac{L(L+u)}{a^2} = \lambda\frac{L^2}{a^2} + \frac{\lambda L}{a}\frac{u}{a}.$$

which is an equation for u. We have

$$u\left(1 - \frac{\lambda L}{a}\frac{1}{a}\right) = \lambda\left(\frac{L}{a}\right)^2.$$

$$u = \frac{\lambda\left(\frac{L}{a}\right)^2}{1 - \frac{\lambda L}{a}\frac{1}{a}}$$

So the range resolution limit $\lambda\frac{L^2}{a^2}$ is consistent provided $\frac{\lambda L}{a^2} < 1$, or $\frac{\lambda L}{a} < a$.

Up to factor of π, $\frac{\lambda L}{a^2}$ is the Fresnel number and we want to be in the Geometrical optics regime, $\lambda L/a^2 < 1$.

If the bandwidth is not too large there resolution limits continue to hold with λ being the carrier or central ~~frequen~~ wavelength.

But when the bandwidth is very large or when the array is very large, or both, then a different analysis is needed.

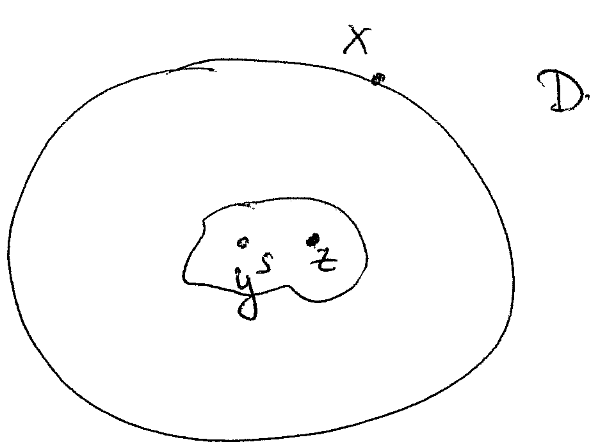
We consider the case where:

$$\hat{P}(x_r,\omega) = \int dz\, \rho(z)\, \hat{G}_0(x_r, z, \omega)$$

and consider fixed frequency time reversal over a surface D

$$\Gamma^{TR}(y^s,\omega) = \sum_{x_r \in D} \hat{P}(x_r,\omega)\, \overline{\hat{G}_0(x_r, y^s; \omega)}$$

$$\sim \int_D dS(x)\, \hat{P}(x,\omega)\, \overline{\hat{G}_0(x, y^s; \omega)}$$



We assume that D is <u>convex</u>.

We have that

$$\Gamma^{TR}(y^s,\omega) \approx \int dz\, \rho(z) \int_D \frac{e^{ik(|x-z|-|x-y^s|)}}{(4\pi)^2\,|x-z|\,|x-y^s|}\, dS(x)$$

We will assume that

$$\frac{|z-y^s|}{|x-y^s|} \ll 1$$

for all z in the support of $\rho(\cdot)$ and all $x\in D$.

We will show that we then have that

$$\Gamma^{TR}(y^s;k) \sim \int dz\, \rho(z)\, \frac{\sin k|z-y^s|}{4\pi k\,|z-y^s|}$$

We consider a fixed y^s and expand $|x-z|$ when z is near y^s:

$$|x-z| = |x-y^s| + \nabla_z |x-z|\Big|_{z=y^s} \cdot (z-y^s)$$

$$+\frac{1}{2}\nabla_z\nabla_z |x-z|\Big|_{z=y^s} (z-y^s)\cdot(z-y^s) + \cdots$$

$$\nabla_z |x-z| = \frac{z-x}{|z-x|}$$

$$\nabla_z\nabla_z |x-z| = \frac{I}{|z-x|} - \frac{(z-x)\otimes(z-x)}{|z-x|^3}$$

So

$$|x-z| = |x-y^s| + (z-y^s)\cdot\frac{y^s-x}{|y^s-x|}$$

$$+\frac{1}{2}\left(\frac{I}{\cancel{\quad\quad}} - \frac{(y^s-x)\otimes(y^s-x)}{|y^s-x|^2}\right)\frac{(z-y^s)}{|y^s-x|}\cdot(z-y^s) + \cdots$$

$$|x-z| - |x-y^s| = (z-y^s)\cdot\left(\frac{y^s-x}{|y^s-x|} + o(1)\right) \qquad \frac{|z-y^s|}{|y^s-x|} << 1.$$

also

$$|x-z| \approx |x-y^s|\left(1 + \frac{z-y^s}{|x-y^s|}\cdot\frac{y^s-x}{|y^s-x|}\right) \sim |x-y^s|, \quad \frac{|z-y^s|}{|x-y^s|} <<$$

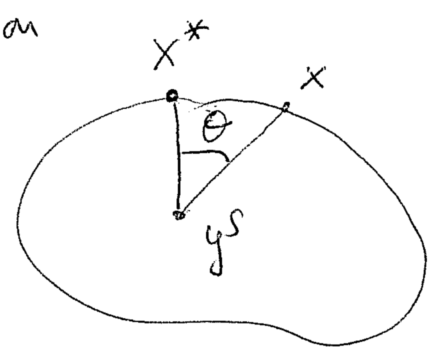
This means that.

$$I^{TR}(y^s,k) \sim \int dz\, \rho(z) \int_D \frac{e^{ik(z-y^s)\cdot\frac{y^s-x}{|y^s-x|}}}{(4\pi)^2\,|x-y^s|^2}\, ds(x)$$

Now the surface D has the representation

$$|x-y^s| = g(\theta,\phi)$$

with θ, φ polar angles relative to a fixed point x^* on the surface.



The surface element $dS(x)$ has the form.

$$dS(x) = |y^S - x|^2 \sin\theta \, d\theta \, d\varphi$$

in these coordinates. Therefore.

$$p^{TR}(y^S, k) \sim \int dz \, e(z) \int_0^{\pi} \int_0^{2\pi} \frac{e^{ik|z-y^S|\cos\theta}}{(4\pi)^2} \sin\theta \, d\theta \, d\varphi$$

$$= \int dz \, e(z) \frac{1}{8\pi} \int_0^{\pi} e^{ik|z-y^S|\cos\theta} \sin\theta \, d\theta$$

$$= \int dz \, e(z) \frac{\sin k|y^S - z|}{4\pi \, k|y^S - z|}$$

as was to be shown.

For a point source $e(z) \sim \delta(z^* - z)$ we have.

$$p^{TR}(y^S, k) \sim \frac{\sin k|y^S - z^*|}{4\pi \, k|y^S - z^*|}$$

We get the Rayleigh resolution by looking for the first zero of the sine

$$k|y^S - z^*| = \pi, \quad |y^S - z^*| = \frac{1}{2} \frac{2\pi}{k} = \frac{\lambda}{2}.$$

which means that the resolution when the source is completely enclosed by the array surface is $\frac{\lambda}{2}$, half the wavelength.

We can also consider the imaging functional

$$I^{KM}(y^s; A, B) = \int \rho(z)\,dz \int_A dx \int_{|\omega-\omega_0|\le B} d\omega \quad M(x, y^s) \cdot$$

$$\cdot \frac{e^{i\omega(\tau(x,z) - \tau(x,y^s))}}{(4\pi)^2 |x-y^s|\,|x-z|}.$$

Now A is a subset of $\mathbb{R}^2$ and tends to it. $A \to \mathbb{R}^2$ in the limit we want to consider, and $M(x, y^s)$ is a multiplier that will make the result come out with an exact reconstruction of $\rho(y^s)$.

We assume that for z in support of $\rho(\cdot)$ we have that

$$\frac{|z-y^s|}{|x-y^s|} \ll 1 \quad , \quad x \in A.$$

As we have just seen, this allows us to simplify I^{KM} !

$$I^{KM}(y^s; A, B) \sim \int_A e(z)\,dz \int dx \int_{|\omega-\omega_0|\le B} d\omega \quad \mathcal{M}(x, y^s)\,\cdot$$

$$\cdot \frac{e^{i\frac{\omega}{c_0}(z-y^s)\cdot\frac{x-y^s}{|x-y^s|}}}{(4\pi)^2\,|x-y^s|^2}$$

Now we make the change of variables.

$$(x,\omega)\in \mathbb{R}^3 \to \xi = \frac{\omega}{c_0}\frac{x-y^s}{|x-y^s|}$$

which is one to one and onto as $A\to\mathbb{R}^2$ and $B\to+\infty$. Let

$$dx\,d\omega = \frac{\partial(x,\omega)}{\partial\xi}\,d\xi = \mathcal{J}(x,y^s)\,d\xi$$

and choose $\mathcal{M}(x,y^s)$ such that

$$\frac{\mathcal{M}(x,y^s)}{(4\pi)^2\,|x-y^s|^2}\,\mathcal{J}(x,y^s) = \frac{1}{(2\pi)^3}$$

Note that $\mathcal{U}(x, y^5)$ depends only on x and y^5.

Let $Z(A,B) = \{ J \in \mathbb{R}^3 \mid x \in A, |\omega - \omega_0| \leq B \}$.

Clearly $Z(A,B) \longrightarrow \mathbb{R}^3$ as $A \to \mathbb{R}^2$, $B \to \infty$.

Therefore we have

$$I^{KM}(y^5; A, B) \sim \int \rho(z)\, dz \int_{Z(A,B)} dJ \, \frac{e^{i(z-y^5)\cdot J}}{(2\pi)^3}$$

$$\sim \rho(y^5)$$

as $A \to \mathbb{R}^2$ and $B \to \infty$.