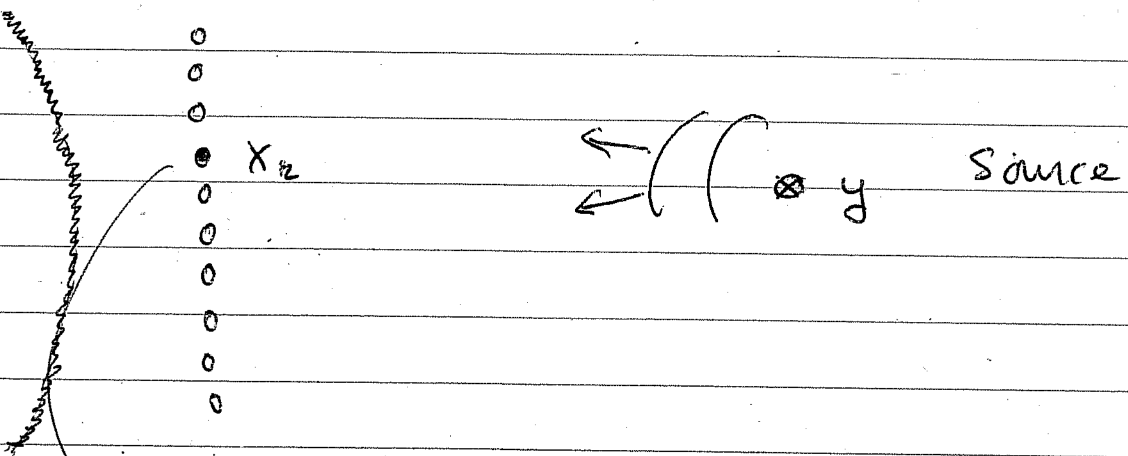
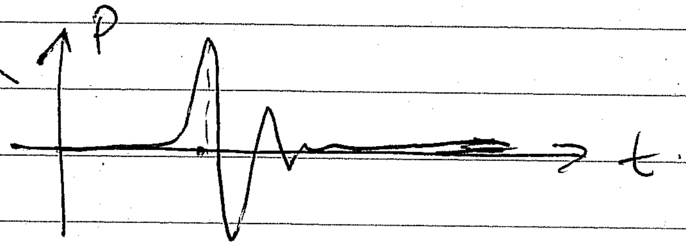
6/5/06 Oberwolfach. G. Papanicolaou

- Array imaging



signal received at x_r : $P(x_r, t)$



Time harmonic point source

$$\hat{G}_0(x,y,\omega) = \frac{e^{i\omega\tau(x,y)}}{4\pi|x-y|} \cdot \left(e^{-i\omega t}\right)$$

$\tau(x,y)$ = travel time from x to y = $\dfrac{|x-y|}{c_0}$

c_0 = speed of propagation

Time pulse : $f(t) = e^{-i\omega_0 t} f_B(t)$

ω_0 = carrier frequency

$f_B(t)$ = base-band pulse (B = bandwidth)

Fourier transform

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} g(t)\,dt.$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,\hat{g}(\omega)\,d\omega$$

Fourier transform of pulse $f(t)$

$$\hat{f}(\omega) = \hat{f}_B(\omega-\omega_0) \qquad \left(\hat{f}_B \equiv 0\ ,\ |\omega|>B\right)$$

Received signal at the array, in a homogeneous medium.

$$\hat{P}(x_r,\omega) = \hat{f}_B(\omega-\omega_0)\,\frac{e^{i\omega\tau(x,y)}}{4\pi\,|x-y|}.$$

In the time domain

$$P(x_r,t) = e^{-i\omega_0 t}\,\frac{f_B(t-\tau(x_r,y))}{4\pi\,|x_r-y|}$$

For an impulsive source $f_B(t) = \delta(t)$
and then signal arrives at

$$t = \tau(x_r,y)$$

or

$$|x_r-y|^2 = c_0^2 t^2.$$

If L is the distance of y from the array
and x_r is measured along the array from
the nearest point to y then

$$x_r^2 + L^2 = c_0^2 t^2$$

or
$$\frac{c_0^2 t^2}{L^2} - \frac{x_r^2}{L^2} = 1$$

which is a hyperbola in (x_r, t).

Reduced wave equation for $\hat{G}_0(x, y, \omega)$

$$(\Delta + k^2)\hat{G}_0 = -\delta(x-y) \qquad k = \frac{\omega}{c_0}$$

+ radiation condition (time factor $e^{-i\omega_0 t}$).

Time domain problem:

$$\frac{1}{c_0^2} P_{tt} - \Delta P = f(t)\,\delta(x-y)$$

P and P_t initially zero.

Problem:

Place point sources at x_r. Let $g(t, x_r)$ be the pulse emitted from x_r.

The signal received at a point y^s is

$$\sum_r \frac{g(t - \tau(x_r, y^s), x_r)}{4\pi |x_r - y^s|}$$

How do we choose $g(t, x_r)$ so as to beamform a pulse to y?

Use time reversal:

$$g(t, x_r) = P(x_r, -t)$$

Then the signal received at y^s is

$$e^{i\omega_0 t} \sum_{x_r} \frac{f_B\left(-t + \tau(x_r, y) - \tau(x_r, y^s)\right)}{(4\pi)^2 \, |x_r - y|^2}$$

and in the Fourier domain

$$\overline{\hat{f}_B(\omega - \omega_0)} \sum_{x_r} \overline{\hat{G}_0(x_r, y, \omega)} \, \hat{G}_0(x_r, y^s, \omega)$$

How well does time reversal work?

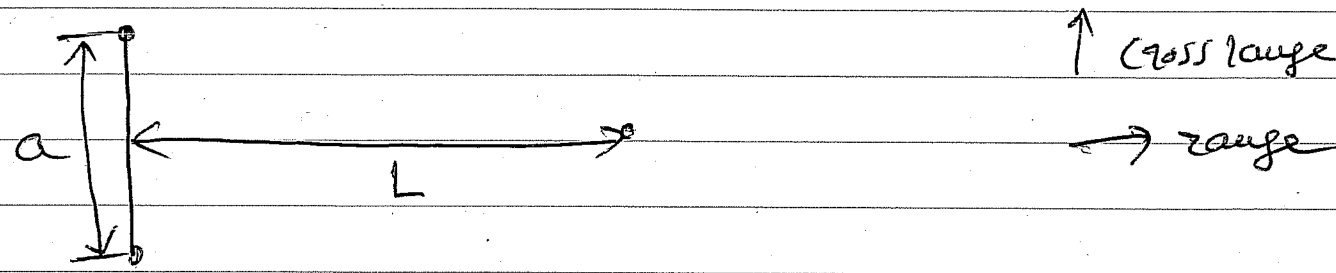
- Resolution theory for array imaging.

Imaging functional (Kirchhoff migration)

$$\int d\omega \sum_{x_r} \hat{P}(x_r, \omega) \, e^{-i\omega \tau(x_r, y^s)}$$

$$= \sum_{x_r} P(x_r, \tau(x_r, y^s)) = I^{KM}(y^s)$$

Basic quantities in the analysis of the imaging functional



- Range L.
- aperture a of the array
- wavelength of carrier $\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi c_0}{\omega_0} = \frac{c_0}{f_0}$

Regarding the aperture: The spacing between array transducers is small (less than $\lambda_0/2$) so we can consider the array as a continuum.

Cross range resolution is $\frac{\lambda_0 L}{a}$ $(L \gg a)$
(Rayleigh resolution)

Range resolution is $\lambda_0 \left(\frac{L}{a}\right)^2$ $(L \gg a)$
(monochromatic)

Range resolution is $\frac{c_0}{B}$
(broadband)

- For large arrays the resolution theory is different

What happens in random media?
Now the Green's function is random.

$$\left(\Delta + k^2 (1+\mu(x))\right) \hat{G}(x,y,\omega) = -\delta(x-y)$$

$\mu(x)$ a random function with mean zero, the fluctuation process.

- In random media time reversal and imaging are different.

<u>Time reversal</u>:

$$\Gamma^{TR}(y^s) = \int d\omega \sum_{x_r} \overline{\hat{P}(x_r,\omega)}\, \hat{G}(x_r, y^s, \omega)$$

For a point source at y

$$\hat{P}(x_q,\omega) = \hat{f}_B(\omega-\omega_0)\, \hat{G}(x_q, y, \omega)$$

<u>Imaging</u>: Kirchhoff migration or time reversal imaging.

$$I^{KM}(y^s) = \int d\omega \sum_{x_r} \hat{P}(x_r,\omega)\, e^{-i\omega \tau(x_r, y^s)}$$

$$= \sum_{x_r} P(x_r, \tau(x_r, y^s))$$

(same as for a homog. medium)

Note that random Green's function cannot be written in terms of a "random travel time".

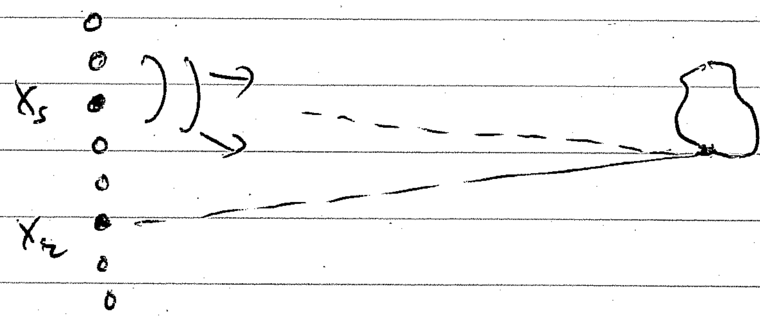
The resolution theory for Γ^{TR} and I^{KM} is very different.

Γ^{TR} gains resolution in RM.

I^{KM} loses resolution in RM.

Main objective of lectures: Articulate these differences in a quantitative way.

Active arrays:



Pulse $f(t) = e^{-i\omega_0 t} f_B(t)$ sent from x_s

Signal received at x_r : $P(x_r, x_s, t)$

For a "point" scatterer model of signal received in homogeneous medium.

$$\hat{P}(x_r, x_s, \omega) = \zeta \hat{f}_B(\omega - \omega_0)\, \hat{G}_0(x_r, y, \omega)\, \hat{G}(y, x_s, \omega)$$

(ζ is a scattering amplitude)

- Time reversal of this signal would attempt to return the pulse to x_s.

We want to either
(a) Send a signal to y.
or (b) Image y.

How do we do this?

Suppose we have the impulse response matrix at the array.

$$\Pi(x_r, x_s, t) \qquad \left(\hat{\Pi}(x_r, x_s, \omega)\right)$$

Model: $\hat{\Pi}(x_r, x_s, \omega) = \hat{\xi}(\omega)\, \hat{G}_0(x_r, y, \omega)\, \hat{G}_0(x_s, y, \omega)$

(a complex, symmetric rank one matrix)

$$\hat{\Pi}(\omega) = \hat{\xi}(\omega)\, \hat{g}_0(y,\omega)\, \hat{g}_0(y,\omega)^T, \quad \hat{g}_0(y,\omega) = \left(\hat{G}_0(x_r, y, \omega)\right)$$

right singular vector of $\hat{\Pi}(\omega)$: $\dfrac{\overline{\hat{g}_0(y,\omega)}}{\|\hat{g}_0(y,\omega)\|}$

Illuminate with it.

$\hat{\Pi}(\omega) \dfrac{\overline{\hat{g}_0(y,\omega)}}{\|\hat{g}_0(y,\omega)\|}$ is received signal at array.

$$= \hat{\xi}(\omega)\, \hat{g}_0(y,\omega)\, \|\hat{g}_0(y,\omega)\|$$

Time reverse and send dim back to y^S.

$$\int d\omega\, \hat{g}_0(y^S,\omega)^T \cdot \overline{\left(\hat{\Pi}(\omega) \frac{\overline{\hat{g}_0(y,\omega)}}{\|\hat{g}_0(y,\omega)\|}\right)} = \int \overline{\hat{\xi}(\omega)}\, \hat{g}_0(y^S,\omega)^T\, \overline{\hat{g}_0(y,\omega)}\, \|\hat{g}_0(y,\omega)\|\, d\omega$$

$$= \int \overline{\hat{\xi}(\omega)}\, \|\hat{g}_0(y,\omega)\| \cdot \hat{g}_0(y^S,\omega)^T\, \overline{\hat{g}_0(y,\omega)}\, d\omega$$

Now we note that

$$\|g_0(y,\omega)\|^2 = \sum_{x_2} \overline{\hat{G}_0(x_2,y,\omega)}\, \hat{G}_0(x_2,y,\omega)$$

$$= \sum_{x_2} \frac{1}{(4\pi|x_2-y|)^2} = \text{independent of } \omega \; \cancel{\ldots}$$

So: Illuminating with the singular vector of $\hat{\Pi}(\omega)$ and time reversing to y^S gives.

$$\Gamma^{TR/SVD}(y^S) = \|g_0(y,\omega)\| \int d\omega\, \hat{\zeta}(\omega) \sum_{x_2} \hat{G}_0(x_2,y^S,\omega)\overline{\hat{G}_0(x_2,y,\omega)}$$

This is almost the same as TR for a passive array (up to the factor $\hat{\zeta}(\omega)$ entering).

So we have found an effective way to do time reversal for active arrays as well. This will be fully explored in the third (Wednesday) lecture.