Localization and mode conversion for elastic waves in randomly layered media

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Abstract

We derive localization theory for elastic waves in plane-stratified media, a multi-mode problem complicated by the interconversion of shear and compressional waves, both in propagation and in backscatter. In the low frequency limit, i.e. when the randomness constitutes a microstructure, we give analytical expressions for the following quantities: the localization length, and another deterministic length, called the equilibration length, which gives the scale for the equilibration of compressional and shear energy in propagation; the probability density of the fraction of reflected energy which remains in the same mode (shear or compressional) as the incident field; and the probability density of the ratio of shear to compressional energy in transmission through a large slab. This last quantity is shown to be asymptotically independent of the incident field. Our main mathematical tools are: the Oseledec theorem, which establishes the existence of the localization length and other structural information, and limit theorems for stochastic differential equations with a small parameter.

Contents

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1 Introduction

In this paper we study the reflection and transmission of time-harmonic, elastic plane waves obliquely incident on a randomly layered region whose mean or effective elastic parameters are uniform. Our analysis is carried out in the asymptotic limit where the random layering is on a scale much smaller than typical propagation distances while the wavelength of the incident waves is intermediate between these two length scales. The random fluctuations in the elastic parameters are not, however, assumed to be small.

We studied in detail acoustic wave reflection and transmission in this regime in [1] and pointed out its relevance to exploration geophysics, discussed extensively in [2]. For waves in random media this asymptotic regime is interesting because localization phenomena [3] are fully developed. In its simplest form localization means that time harmonic plane waves are exponentially attenuated by multiple scattering when passing through a randomly layered medium. The rate of multiple scattering attenuation is the reciprocal of the localization length, which depends on the frequency of the waves and the statistical properties of the medium. For layered random media the localization length for acoustic wave propagation is always finite and its behavior for low frequencies or weak fluctuations, as well as some other cases, is known [4,5]. Multiple scattering attenuation is seen clearly in numerical experiments [6] and its separation from intrinsic attenuation is an important problem for which a lot of progress has been made in the acoustic case ([1] and the references therein). This paper is the first one, as far as we know, where localization phenomena for elastic waves in randomly layered media are analyzed in detail, in particular, modal wave energy conversion and Lyapounov exponents. Earlier attempts [7,8] dealt with a more general, somewhat abstract vector wave problem with mode coupling but the results were not as explicit and informative as the ones in this paper.

The vector nature of elastic waves makes the analysis of reflection-transmission much more complicated than for acoustic or even electromagnetic waves, which are vector waves. In the latter case, plane waves at an arbitrary angle of incidence can be decomposed into horizontally and vertically polarized modes that do not couple except statistically since they both propagate through the same randomly layered medium. The horizontally and vertically polarized wave modes behave like acoustic waves [9]. In the elastic case, compressional or P waves, horizontally polarized shear (SH) waves and vertically polarized shear (SV) waves decouple for normal incidence. At oblique incidence the SH mode decouples from the P and SV modes but these two stay coupled. The SH mode behaves like an acoustic wave. The P-SV coupled mode problem gives rise to four-by-four propagator matrices (section 4) while for acoustic waves the propagator matrices are two-by-two. This is an essential complication that requires special analysis and leads to new results.

The main results in this paper are as follows. We calculate explicitly, in the above mentioned asymptotic regime, (i) the probability density of backscatter mode conversion from a randomly layered half space, (ii) the probability density of transmitted mode conversion, (iii) the localization length of the waves (the reciprocal of the smallest of the two positive Lyapounov exponents) and (iv) the mode equilibration length for transmission (the reciprocal of the larger of the two positive Lyapounov exponents). The main methods in our analysis are limit theorems for stochastic equations that depend on a parameter, as in [1] and elsewhere, and the Oseledec theorem (section 6) which defines the Lyapounov exponents and the associated limit eigenspaces for the propagator matrices.

In the case of acoustic waves we can study more complicated reflection and transmission problems by randomly layered media because we do not have to deal with coupled modes, the propagators being two-by-two matrices. In [1] we studied the reflection and transmission of pulsed spherical waves in randomly layered media with varying deterministic background and with intrinsic attenuation. We also formulated and solved some inverse problems. We are now studying the solution of these more complicated problems in the case of elastic waves.

In sections 2-4 we formulate the P-SV reflection transmission problem in a manner that is convenient for the asymptotic analysis where we exploit the special structure of the propagator matrices. In section 5 we apply the asymptotics for stochastic equations to calculate the probability density of backscatter mode conversion and calculate moments by numerically evaluating the analytical formulas. In section 6 we use the Oseledec theorem to define suitably the Lyapounov exponents of the propagator matrices which we calculate in section 7. We compare our analytical results with numerical simulations and find excellent agreement between them. In section 8 we analyze mode conversion for transmission. In a number of appendices we clarify some associated special issues. In appendix A, in particular, we review the necessary modifications that convert our results to the case of small fluctuations in the medium properties when the wavelength of the incident waves is comparable to the scale of the inhomogeneities.

2 P-SV waves in a randomly-stratified medium

Let $\mathbf{u} = (u_1, u_2, u_3)^T$ be displacement, $\tau_{j\ell}$ the stress tensor, ρ the density, and λ, μ the Lamé parameters. Then for isotropic elasticity we have

$$\begin{aligned} \rho \frac{\partial^2 u_j}{\partial t^2} &= \tau_{j\ell,\ell} \\ \tau_{j\ell} &= \lambda u_{k,k}\delta_{j\ell} + \mu(u_{j,\ell} + u_{\ell,j}) \ , \end{aligned} \tag{2.1}$$

where $\delta_{j\ell}$ is the Kronecker delta and we use the summation convention. We nondimensionalize by choosing typical macroscale length $\bar{x}$, velocity $\bar{c}$, density $\bar{\rho}$, and displacement $\bar{u}$, and setting

$$\begin{aligned} x'_j &= x_j/\bar{x} \ , \ t' = (\bar{c}/\bar{x})t \\ u'_j &= u_j/\bar{u} \ , \ \tau'_{j\ell} = (\bar{x}/(\bar{\rho}\bar{u}\bar{c}^2)\tau_{j\ell} \\ \rho' &= \rho/\bar{\rho} \ , \ \lambda' = \lambda/(\bar{\rho}\bar{c}^2), \mu' = \mu/(\bar{\rho}\bar{c}^2) \ . \end{aligned} \tag{2.2}$$

Since the primed variables also satisfy (2.1) we will work with them, dropping primes. For exploration geophysics, suitable length and velocity scales are, say, $\bar{x} = 3$ km, $\bar{c} = 3$ km/sec, so that a macroscopic time scale is 1 sec. A typical $\bar{\rho}$ =2.7 gm/cm^3.

We consider a stratified medium, so that the material parameters ρ, λ, μ are functions of the coordinate $x_3 = z$ only. It can be shown that horizontal shear (SH) waves are not coupled to vertical shear (SV) or compressional (P) waves, and so can be analyzed separately. They can be represented by a two-variable system which is mathematically equivalent to the acoustic problem which has been analyzed extensively elsewhere [1]. To analyze the P–SV problem, we seek a solution of (2.1) in the form of a time-harmonic plane wave of angular frequency ϖ and horizontal slowness p

$$\begin{aligned} \mathbf{u} &= e^{-i\varpi(t-px_1)}\widehat{\mathbf{u}}(z) \\ \tau &= e^{-i\varpi(t-px_1)}\widehat{\tau}(z) \ . \end{aligned} \tag{2.3}$$

Substitution of (2.3) into (2.1) yields nine equations. Three of these, for $\widehat{u}_2, \widehat{\tau}_{12}, \widehat{\tau}_{23}$ uncouple from the rest, so that these variables may be set equal to zero. A fourth equation determines $\widehat{\tau}_{22}$ in terms of $\widehat{u}_1$ and $\partial\widehat{u}_3/\partial z$. Of the remaining five equations, $\widehat{\tau}_{11}$ may be eliminated to produce a closed system of equations for

$$\mathbf{X} = \begin{bmatrix} -i\varpi\widehat{u}_3 \\ \widehat{\tau}_{13} \\ \widehat{\tau}_{33} \\ -i\varpi\widehat{u}_1 \end{bmatrix} \tag{2.4}$$

The choice of $\mathbf{X}$ to represent the wave is suggested by the formalism of Ursin [10]. $\mathbf{X}$ satisfies the four-dimensional linear ordinary differential equation

$$\frac{d\mathbf{X}}{dz} = -i\varpi M\mathbf{X} \tag{2.5}$$

where the 4×4 matrix M has the block structure

$$M = \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix} \tag{2.6}$$

and M_1, M_2 are the 2×2 real symmetric matrices

$$M_1 = \begin{bmatrix} \frac{1}{(\lambda+2\mu)} & \frac{\lambda p}{(\lambda+2\mu)} \\ \frac{\lambda p}{(\lambda+2\mu)} & \rho - \frac{4p^2\mu(\lambda+\mu)}{(\lambda+2\mu)} \end{bmatrix} \tag{2.7}$$

$$M_2 = \begin{bmatrix} \rho & p \\ p & \frac{1}{\mu} \end{bmatrix} \ . \tag{2.8}$$

We now assume that the material parameters vary randomly as functions of z, and vary substantially over small distances, which define a microscale. To express this mathematically, let $\epsilon > 0$ be a small, dimensionless parameter, and let

$$\lambda = \lambda(z/\epsilon^2), \quad \mu = \mu(z/\epsilon^2), \quad \rho = \rho(z/\epsilon^2) \tag{2.9}$$

be stationary random functions on the microscale, ϵ^2. For exploration geophysics a typical microscale has been estimated as on the order of 3 m [2] so that $\epsilon^2 \approx 10^{-3}$ and $\epsilon \approx .03$.

While we analyze here the case of stationary random fluctuations, similar methods have been used, for the acoustic problem, if there are also deterministic background variations, i. e. non-random variations on the slow, z-scale [1].

Now if all other parameters are $O(1)$, then the rapidly-fluctuating parameters in (2.5) tend to average, so that the effective medium theory obtains. That is, we define the averaged matrix

$$\overline{M} = E[M] = \begin{bmatrix} 0 & \overline{M}_1 \\ \overline{M}_2 & 0 \end{bmatrix} , \tag{2.10}$$

where E denotes expected value. Here $\overline{M}_1, \overline{M}_2$ may be written in terms of the five effective parameters

$$
\begin{aligned}
\bar{\rho} &= E[\rho] \\
\bar{\mu} &= \left(E\left[\frac{1}{\mu}\right]\right)^{-1} \\
\gamma_1 &= E\left[\frac{1}{(\lambda+2\mu)}\right] \\
\gamma_2 &= E\left[\frac{\lambda}{(\lambda+2\mu)}\right] \\
\gamma_3 &= E\left[\frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}\right]
\end{aligned}
\tag{2.11}
$$

Then

$$
\overline{M}_1 = E[M_1] = \begin{bmatrix} \gamma_1 & \gamma_2 p \\ \gamma_2 p & \bar{\rho} - \gamma_3 p^2 \end{bmatrix} \tag{2.12}
$$

$$
\overline{M}_2 = E[M_2] = \begin{bmatrix} \bar{\rho} & p \\ p & \frac{1}{\bar{\mu}} \end{bmatrix} . \tag{2.13}
$$

We define $\overline{\mathbf{X}}$ to be the solution of the averaged system. That is, (2.5) with M replaced by $\overline{M}, \mathbf{X}$, by $\overline{\mathbf{X}}$. The random system $\mathbf{X}$ is well approximated, as $\epsilon \downarrow 0$, by the deterministic,"effective medium" solution, $\overline{\mathbf{X}}$. Stochastic effects are small, $O(\epsilon)$, in this limit, and can be computed for initial value problems by the theorem of Khasminskii [11], or for boundary value problems, by the theorem of White and Franklin [12], which treat general non-linear equations with this scaling. For an application of this method to a wave problem, see Besieris and Kohler [13].

Note that in general the effective medium is not isotropic, but that it can become isotropic when, for example, there are no fluctuations in $\mu = \overline{\mu}$. In that case, we can define $\overline{\lambda}$ so that $\gamma_1 = (\overline{\lambda} + 2\overline{\mu})^{-1}$, and then $\gamma_2 = \overline{\lambda}\gamma_1, \gamma_3 = 4\overline{\mu}(\overline{\lambda} + \overline{\mu})\gamma_1$ as in the isotropic case. This is a special instance, for layered media, of more general effective media whose microstructure have constant shear modulus [14].

For the remainder of this work, we introduce one final scaling that invalidates effective medium theory, and produces $O(1)$ random fluctuations in the wave. Following Burridge et. al. [15] and Asch et. al. [1], we let

$$
\overline{\omega} = \frac{\omega}{\epsilon} . \tag{2.14}
$$

For a time scale of 1 second and $\epsilon \approx .03$, this scaling produces a typical frequency on the order of $1/\epsilon \approx 30Hz$, which is consistent with exploration geophysics. More generally, we are assuming that the wavelength, of order $O(\epsilon)$ is much larger than the $O(\epsilon^2)$ microscale, but much smaller than the $O(1)$ macroscale, so that propagation distances will be many wavelengths. With (2.14) and (2.9), (2.5) becomes

$$
\frac{d\mathbf{X}}{dz} = -\frac{i\omega}{\epsilon} M(z/\varepsilon^2)\mathbf{X} . \tag{2.15}
$$

Now we consider that a slab of this random medium, of length $\overline{z}$, occupies the region $\{z : 0 < z < \overline{z}\}$, bounded by two half-spaces of homogeneous material, above, for $z < 0$, and below, for $z > \overline{z}$. (z is positive downward as is usual in geophysics. We will also use a second coordinate system $\zeta = -z$, positive upward.) For $z < 0$, the constant material parameters ρ^0, λ^0, μ^0 give compressional and shear wave velocities c_p^0, c_s^0, respectively, where

$$c_p^0 = \sqrt{\frac{(\lambda^0 + 2\mu^0)}{\rho^0}} \quad , \quad c_s^0 = \sqrt{\frac{\mu^0}{\rho^0}} \; , \tag{2.16}$$

and corresponding vertical slowness

$$\alpha_p^0 = \sqrt{\left(\frac{1}{c_p^0}\right)^2 - p^2} \quad , \quad \alpha_s^0 = \sqrt{\left(\frac{1}{c_s^0}\right)^2 - p^2} \; . \tag{2.17}$$

We consider that either a P or an SV wave is incident on the slab from above. For $z < 0$, the displacement of each incident wave is given by the familiar forms of $\mathbf{u}_p^{\rm inc}, \mathbf{u}_s^{\rm inc}$, respectively,

$$\begin{aligned} \mathbf{u}_p^{\rm inc} &= \epsilon \begin{bmatrix} p \\ 0 \\ \alpha_p^0 \end{bmatrix} e^{-i\frac{\omega}{\epsilon}(t - px_1 - \alpha_p^0 z)} \\ \mathbf{u}_s^{\rm inc} &= \epsilon \begin{bmatrix} -\alpha_s^0 \\ 0 \\ p \end{bmatrix} e^{-i\frac{\omega}{\epsilon}(t - px_1 - \alpha_s^0 z)} \end{aligned} \tag{2.18}$$

The multiplicative ϵ factors have been inserted in (2.18) to produce corresponding $O(1)$ wave vectors $\mathbf{X}$. At $z = 0$, the incident wave vectors are then

$$\mathbf{X}_p^{\rm inc} = i\omega \begin{bmatrix} -\alpha_p^0 \\ 2\mu^0\alpha_p^0 p \\ \lambda^0 p^2 + (\lambda^0 + 2\mu^0)(\alpha_p^0)^2 \\ -p \end{bmatrix} \tag{2.19}$$

for an incident P-wave, or

$$\mathbf{X}_s^{\rm inc} = i\omega \begin{bmatrix} -p \\ \mu^0(p^2 - (\alpha_s^0)^2) \\ 2p\alpha_s^0\mu^0 \\ \alpha_s^0 \end{bmatrix} \tag{2.20}$$

for an incident SV-wave.

3 Ups and Downs

Following Ursin [10], we diagonalize $\overline{M}$ by the following procedure: We first choose 2×2 matrices L_1, L_2 such that

$$L_1 L_2^T = L_2 L_1^T = I \tag{3.1}$$

$$\begin{aligned} \overline{M}_1 &= L_1 \Lambda L_2^{-1} \\ \overline{M}_2 &= L_2 \Lambda L_1^{-1} \end{aligned} \tag{3.2}$$

where I is the 2×2 identity, Λ is a diagonal matrix

$$\Lambda = \begin{bmatrix} \alpha_p & 0 \\ 0 & \alpha_s \end{bmatrix} \tag{3.3}$$

and $\alpha_s > \alpha_p > 0$. We define the $4{\times}4$ matrix L by the block structure

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} L_1 & L_1 \\ L_2 & -L_2 \end{bmatrix} . \tag{3.4}$$

It then follows that

$$L^{-1}\overline{M}L = \Lambda_1 \tag{3.5}$$

where Λ_1 is the $4{\times}4$ diagonal matrix

$$\Lambda_1 = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} . \tag{3.6}$$

From (3.1) and (3.4) it follows that

$$L^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} L_2^T & L_1^T \\ L_2^T & -L_1^T \end{bmatrix} . \tag{3.7}$$

The reader is referred to Ursin [10] for the details. In brief, the columns of L_1 (or L_2) are properly normalized eigenvectors of $\overline{M}_1\overline{M}_2$ (or $\overline{M}_2\overline{M}_1$). $\overline{M}_1\overline{M}_2$ and $\overline{M}_2\overline{M}_1$ have the same eigenvalues, which are α_p^2 and α_s^2, and α_p, α_s correspond to vertical slownesses for the faster and slower wave modes, respectively. Here the plus or minus signs in (3.6) correspond to up and down-going waves, respectively.

For an isotropic effective medium, α_p, α_s are given by the familiar formula (2.16), (2.17), and L_1, L_2 can be written explicitly

$$L_1^{\text{iso}} = \begin{bmatrix} (\frac{\alpha_p}{\overline{\rho}})^{1/2} & p(\overline{\rho}\alpha_s)^{-1/2} \\ -2p\overline{\mu}(\frac{\alpha_p}{\overline{\rho}})^{1/2} & (\overline{\rho} - 2\overline{\mu}p^2)(\overline{\rho}\alpha_s)^{-1/2} \end{bmatrix} . \tag{3.8}$$

$$L_2^{\text{iso}} = \begin{bmatrix} (\overline{\rho} - 2\overline{\mu}p^2)(\overline{\rho}\alpha_p)^{-1/2} & 2\overline{\mu}p(\frac{\alpha_s}{\overline{\rho}})^{1/2} \\ -p(\overline{\rho}\alpha_p)^{-1/2} & (\frac{\alpha_s}{\overline{\rho}})^{1/2} \end{bmatrix} . \tag{3.9}$$

As discussed in section 2, the effective medium is, however, usually anisotropic.

Let

$$\mathbf{y} = L^{-1}\mathbf{X} = \begin{bmatrix} \mathbf{U} \\ \mathbf{D} \end{bmatrix} . \tag{3.10}$$

where $\mathbf{U}, \mathbf{D}$ are 2-vectors with $\mathbf{U}$ representing up and $\mathbf{D}$ representing down-going waves. Note that up and down-going waves are computed here with respect to the *effective*, not the random, medium. For $z < 0$ we may use (3.8), (3.9) to decompose the alternate incident waves (2.19), (2.20). Thus at $z = 0$

$$\mathbf{U}_p^{\text{inc}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{D}_p^{\text{inc}} = -i\omega(2\rho^0\alpha_p^0)^{1/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.11}$$

$$\mathbf{U}_s^{\text{inc}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{D}_s^{\text{inc}} = -i\omega(2\rho^0\alpha_s^0)^{1/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{3.12}$$

which are purely down-going, as is appropriate.

Now $\mathbf{X} = L\mathbf{y}$ is continuous across interfaces, i.e. values of z where the effective medium and hence L, has a discontinuity. Let $L^{\pm}, \mathbf{y}^{\pm}$ represent values of $L, \mathbf{y}$ at $z^{\pm}$, below and above such an interface. Equating $L^{\pm}\mathbf{y}^{\pm}$ then yields the relation

$$\mathbf{y}^{+} = J\mathbf{y}^{-} \tag{3.13}$$

where the jump matrix J is given by

$$J = (L^{+})^{-1}L^{-} \ . \tag{3.14}$$

From (3.14), (3.1), (3.4) (3.7) we obtain

$$J = \begin{bmatrix} J_A & J_B \\ J_B & J_A \end{bmatrix} \tag{3.15}$$

where the 2×2 matrices J_A, J_B are

$$\begin{aligned} J_A &= \frac{1}{2}(L_2^{+T}L_1^{-} + L_1^{+T}L_2^{-}) \\ J_B &= \frac{1}{2}(L_2^{+T}L_1^{-} - L_1^{+T}L_2^{-}) \ . \end{aligned} \tag{3.16}$$

To jump the other way, $\mathbf{y}^{-} = J^{-1}\mathbf{y}^{+}$, where

$$J^{-1} = \begin{bmatrix} J_A^{-} & J_B^{-} \\ J_B^{-} & J_A^{-} \end{bmatrix} \tag{3.17}$$

and J_A^{-}, J_B^{-} are obtained by interchanging $\pm$ in (3.16). By transposing the resulting identities, we obtain

$$J_A^{-} = J_A^{T} \ , \quad J_B^{-} = -J_B^{T} \ . \tag{3.18}$$

Now at $z = 0^{-}$ we have

$$\mathbf{y}(0^{-}) = \begin{bmatrix} \mathbf{U}^{\mathrm{refl}} \\ \mathbf{D}^{\mathrm{inc}} \end{bmatrix} \tag{3.19}$$

where $\mathbf{U}^{\mathrm{refl}}$ is the reflected, and $\mathbf{D}^{\mathrm{inc}}$ the incident waves. From (3.13), (3.17), (3.18) we obtain the boundary condition at $z = 0^{+}$

$$-J_B^{T}(0)\mathbf{U}(0^{+}) + J_A^{T}(0)\mathbf{D}(0^{+}) = \mathbf{D}^{\mathrm{inc}} \tag{3.20}$$

and the expression for the reflected wave

$$\mathbf{U}^{\mathrm{refl}} = J_A^{T}(0)\mathbf{U}(0^{+}) - J_B^{T}(0)\mathbf{D}(0^{+}) \ . \tag{3.21}$$

Below the slab, at $z = \overline{z}^{+}$, we have no upgoing wave, so that

$$\mathbf{y}(\overline{z}^{+}) = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}^{\mathrm{trans}} \end{bmatrix} \ . \tag{3.22}$$

Thus

$$J_A(\overline{z})\mathbf{U}(\overline{z}^{-}) + J_B(\overline{z})\mathbf{D}(\overline{z}^{-}) = \mathbf{0} \tag{3.23}$$

and

$$\mathbf{D}^{\text{trans}} = J_B(\overline{z})\mathbf{U}(\overline{z}^-) + J_A(\overline{z})\mathbf{D}(\overline{z}^-) \ . \tag{3.24}$$

Equations (3.20), (3.23) are boundary conditions for $\mathbf{y}$, which satisfies the differential equation

$$\frac{d\mathbf{y}}{dz} = -\frac{i\omega}{\epsilon}\{\Lambda_1 + \nu(z/\epsilon^2)\}\mathbf{y} \ . \tag{3.25}$$

Here the mean zero random matrix ν is determined from the fluctuation matrix

$$\widehat{M} = M - \overline{M} \tag{3.26}$$

by

$$\nu = L^{-1}\widehat{M}\mathrm{L} \ . \tag{3.27}$$

Writing

$$\widehat{M} = \begin{bmatrix} 0 & \widehat{M}_1 \\ \widehat{M}_2 & 0 \end{bmatrix} \tag{3.28}$$

where $\widehat{M}_j = M_j - \overline{M}_j$, $j = 1,2$, we obtain that

$$\nu = \begin{bmatrix} \nu_1 & -\nu_2 \\ \nu_2 & -\nu_1 \end{bmatrix} \tag{3.29}$$

where

$$\begin{aligned} \nu_1 &= \frac{1}{2}\left[L_2^T\widehat{M}_1 L_2 + L_1^T\widehat{M}_2 L_1\right] \\ \nu_2 &= \frac{1}{2}\left[L_2^T\widehat{M}_1 L_2 - L_1^T\widehat{M}_2 L_1\right] \end{aligned} \tag{3.30}$$

are real symmetric 2×2 matrices. Let $\delta_j(z/\epsilon^2)$, $j = 1,\ldots 6$, be scalar mean zero stationary stochastic processes, such that

$$\begin{aligned} \nu_1 &= \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{bmatrix} \\ \nu_2 &= \begin{bmatrix} \delta_4 & \delta_5 \\ \delta_5 & \delta_6 \end{bmatrix} \ . \end{aligned} \tag{3.31}$$

In view of (3.24), (3.28), the δ_j represent coupling processes for the various modes. Thus, for instance, δ_2 produces scattering from a down-going P to a down-going S, or vice-versa. Similarly, δ_4 produces scattering from a down-going P to an up-going P and vice-versa. A summary of these interpretations is contained in Table I. Note that the first column, containing $\delta_1, \delta_2, \delta_3$ scatters in the same direction, while the second, with $\delta_4, \delta_5, \delta_6$ scatters in the opposite direction. Thus, essentially ν_1 is a local transmission coefficient matrix, while ν_2 is a local reflection coefficient matrix.

COUPLING	*SAME DIRECTION*	*OPPOSITE DIRECTION*
P–P	δ_1	δ_4
P–S or S–P	δ_2	δ_5
S–S	δ_3	δ_6

Table I. Role of the scattering processes, δ_j

4 Transfer (Propagator) Matrices. Reflection and Transmission Matrices

We will solve the linear boundary value problem (3.25), (3.20) and (3.23) in terms of an initial value problem for an appropriate 4×4 fundamental solution, the transfer or propagator matrix. For this purpose, it is convenient to start at the bottom of the slab, and also to remove the deterministic part of the coefficient matrix in (3.25), represented by Λ_1, with a rotation. We thus define

$$\zeta = -z \tag{4.1}$$

so that $\zeta \in (-\overline{z}, 0)$, and let

$$\mathbf{w}(\zeta) = e^{-i\frac{\omega}{\epsilon}\zeta\Lambda_1}\mathbf{y}(-\zeta) \ . \tag{4.2}$$

Then $\mathbf{w}$ satisfies

$$\frac{d\mathbf{w}}{d\zeta} = \frac{i\omega}{\epsilon}\eta(\zeta/\epsilon^2, \zeta/\epsilon)\mathbf{w} \tag{4.3}$$

where

$$\eta = e^{-i\frac{\omega}{\epsilon}\zeta\Lambda_1}\nu e^{i\frac{\omega}{\epsilon}\zeta\Lambda_1} = \begin{bmatrix} \eta_1 & -\eta_2^* \\ \eta_2 & -\eta_1^* \end{bmatrix} \tag{4.4}$$

and

$$\begin{aligned} \eta_1 &= e^{-i\frac{\omega}{\epsilon}\zeta\Lambda}\ \nu_1\ e^{i\frac{\omega}{\epsilon}\zeta\Lambda} \\ \eta_2 &= e^{i\frac{\omega}{\epsilon}\zeta\Lambda}\ \nu_2\ e^{i\frac{\omega}{\epsilon}\zeta\Lambda} \end{aligned} \tag{4.5}$$

Note that η_1, η_2 have mean zero, and that

$$\begin{aligned} \eta_1^{*T} &= \eta_1 && \text{(Hermitian)} \\ \eta_2^T &= \eta_2 && \text{(Complex symmetric)} \ . \end{aligned} \tag{4.6}$$

We define the transfer matrix Q as the following 4×4 fundamental solution of (4.3)

$$\begin{aligned} \frac{dQ}{d\zeta} &= \frac{i\omega}{\epsilon}\eta(\zeta/\epsilon^2, \zeta/\epsilon)Q \\ Q|_{\zeta=-\overline{z}} &= I \ , \end{aligned} \tag{4.7}$$

where I is the 4×4 identity.

Then for $-\overline{z} < \zeta < 0$

$$\mathbf{w}(\zeta) = Q(\zeta)\mathbf{w}(-\overline{z}^+) \ . \tag{4.8}$$

From (4.8), (4.2), (3.19) and (3.22), we have, on setting $\zeta = 0^-$

$$Q(0)e^{i\frac{\omega}{\epsilon}\overline{z}\Lambda_1}J^{-1}(\overline{z})\begin{bmatrix} \mathbf{0} \\ \mathbf{D}^{\text{trans}} \end{bmatrix} = J(0)\begin{bmatrix} \mathbf{U}^{\text{refl}} \\ \mathbf{D}^{\text{inc}} \end{bmatrix} \ . \tag{4.9}$$

Thus once $Q(0)$ has been determined, we may solve for the reflected and transmitted waves from (4.9). However, before doing so, we will investigate the symmetry properties of Q.

First, note that there exist 2×2 matrices A, B such that

$$Q = \begin{bmatrix} A & B^* \\ B & A^* \end{bmatrix} \tag{4.10}$$

where * denotes complex conjugate.

To see this, first note that $Q(0) = I$ is of this form. Now let $\mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2]^T$ be a vector solution of (4.3), where $\mathbf{w}_1, \mathbf{w}_2$ are 2-vectors. Then it may be verified using (4.4), (4.6) that $[\mathbf{w}_2^*, \mathbf{w}_1^*]$ is also a solution. Thus if the first two columns of (4.10) are as given, the second two columns must also be of the form indicated there. From (4.4), (4.6), (4.7) and (3.18) we obtain

$$\begin{aligned} \frac{dA}{d\zeta} &= \frac{i\omega}{\epsilon}[\eta_1 A - \eta_2^* B], \quad A(-\overline{z}) = I \\ \frac{dB}{d\zeta} &= \frac{i\omega}{\epsilon}[\eta_2 A - \eta_1^* B], \quad B(-\overline{z}) = 0 \ . \end{aligned} \tag{4.11}$$

Let

$$\mathcal{O} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{4.12}$$

where I is the 2×2 identity. Note that $\mathcal{O}^T = \mathcal{O}^{-1} = \mathcal{O}$. Another important symmetry relation is that

$$Q^{-1} = \mathcal{O}Q^{*T}\mathcal{O} \ . \tag{4.13}$$

To see this, first note from (4.7) that

$$Q^{-1}|_{\zeta=-\overline{z}} = \mathcal{O}Q^{*T}\mathcal{O}|_{\zeta=-\overline{z}} = I \ , \tag{4.14}$$

so that (4.13) is satisfied at $\zeta = -\overline{z}$. But now

$$\begin{aligned} \frac{d}{d\zeta}(Q^{-1})^T &= -\frac{i\omega}{\epsilon}\eta^T(Q^{-1})^T \\ \frac{d}{d\zeta}\mathcal{O}Q^*\mathcal{O} &= -\frac{i\omega}{\epsilon}\mathcal{O}\eta^*\mathcal{O}(\mathcal{O}Q^*\mathcal{O}) \ . \end{aligned} \tag{4.15}$$

However,

$$\eta^T = \mathcal{O}\eta^*\mathcal{O} \tag{4.16}$$

by (4.12), (4.4), and (4.6). Thus $(Q^{-1})^T = \mathcal{O}Q^*\mathcal{O}$ since they are equal initially, and satisfy the same linear differential equation. Transposing this relation yields (4.13).

Substitution of (4.10) into (4.13) yields

$$Q^{-1} = \begin{bmatrix} A^{*T} & -B^{*T} \\ -B^T & A^T \end{bmatrix} \ . \tag{4.17}$$

The identity $Q^{-1}Q = I$ implies that

$$A^{*T}A - B^{*T}B = I \tag{4.18}$$

$$A^T B = B^T A \tag{4.19}$$

while $QQ^{-1} = I$ implies

$$AA^{*T} - B^*B^T = I \tag{4.20}$$

$$AB^{*T} = B^*A^T \ . \tag{4.21}$$

Thus (4.18) – (4.21) are equivalent to (4.13) (or (4.17)) on use of (4.10). Also, J_A, J_B satisfy (4.18) – (4.21) because of (3.15), (3.17), and (3.18).

We may now revisit the boundary conditions (4.9). Writing Q in the form (4.10), and J in the form (3.15), (4.9) yields expressions for the waves reflected and transmitted from the random slab, in terms of the initial value problem (4.11). Using (4.18) – (4.21) the solution may be written as the following four step process:

First, we define a local reflection matrix, Γ_T, and a local transmission matrix, τ_T, which describe the effects of the mismatch in effective properties at the transmission end of the slab, $\zeta = -\overline{z}$

$$\Gamma_T = -e^{i\frac{\omega}{\epsilon}\overline{z}\Lambda} J_A^{-1} J_B e^{i\frac{\omega}{\epsilon}\overline{z}\Lambda} \tag{4.22}$$

$$\tau_T = (J_A^T)^{-1} e^{i\frac{\omega}{\epsilon}\overline{z}\Lambda} \; . \tag{4.23}$$

We may verify that Γ_T is symmetric

$$\Gamma_T = \Gamma_T^T \tag{4.24}$$

and that Γ_T, τ_T satisfy the conservation of energy relation for reflected and transmitted waves (see appendix C for further discussion)

$$\Gamma_T^{*T} \Gamma_T + \tau_T^{*T} \tau_T = I \; . \tag{4.25}$$

Second, we define a reflection matrix, $\Gamma(\zeta)$ and a transmission matrix, $\tau(\zeta)$, for propagation throughout the interior of the slab

$$\Gamma(\zeta) = (B^*(\zeta) + A(\zeta)\Gamma_T)(A^*(\zeta) + B(\zeta)\Gamma_T)^{-1} \tag{4.26}$$

$$\tau(\zeta) = \tau_T (A^*(\zeta) + B(\zeta)\Gamma_T)^{-1} \; . \tag{4.27}$$

Similarly Γ is symmetric and Γ, τ satisfy conservation of energy

$$\Gamma^T = \Gamma = (A^{*T} + \Gamma_T B^T)^{-1}(B^{*T} + \Gamma_T A^T) \tag{4.28}$$

$$\Gamma^{*T}(\zeta)\Gamma(\zeta) + \tau^{*T}(\zeta)\tau(\zeta) = I \; . \tag{4.29}$$

Third, we incorporate the effects of the mismatch in effective medium properties at the incidence end of the slab by defining the complete reflection and transmission matrices

$$\overline{\Gamma} = (J_A(0) - \Gamma(0)J_B(0))^{-1}(\Gamma(0)J_A(0) - J_B(0)) \tag{4.30}$$

$$\overline{\tau} = \tau(0)(J_A^T(0) - J_B^T(0)\Gamma(0))^{-1} \; , \tag{4.31}$$

which have symmetry and energy conservation laws

$$\overline{\Gamma}^T = \overline{\Gamma} = (J_A^T(0)\Gamma(0) - J_B^T(0))(J_A^T(0) - J_B^T(0)\Gamma(0))^{-1} \; . \tag{4.32}$$

$$\overline{\Gamma}^{*T}\overline{\Gamma} + \overline{\tau}^{*T}\overline{\tau} = I \; . \tag{4.33}$$

Finally, for any incident wave, the reflected and transmitted waves may be computed by

$$\mathbf{U}^{\mathrm{refl}} = \overline{\Gamma}\mathbf{D}^{\mathrm{inc}} \tag{4.34}$$

$$\mathbf{D}^{\mathrm{trans}} = \overline{\tau}\mathbf{D}^{\mathrm{inc}} \; . \tag{4.35}$$

A special role is played by the "matched medium" case, that is , when the homogeneous medium, either above or below the slab, has properties that are "matched", or identical to, the effective medium properties of the random slab. In that case, J is the identity, and so $J_A = I, J_B = 0$.

When the medium below the slab is matched, we have that $\Gamma_T = 0$ and τ_T is a matrix of pure phases, giving the (deterministic) travel-times through the slab for P and S waves in the effective medium.

When the medium above the slab is matched, we have that $\overline{\Gamma} = \Gamma(0), \overline{\tau} = \tau(0)$, and so $\Gamma(0), \tau(0)$ need no modification to represent reflection and transmission. This observation leads to an interpretation of $\Gamma(\zeta), \tau(\zeta)$ for $\zeta \neq 0$, in terms of "invariant embedding": We conceive of a fictitious slab occupying $(-\overline{z}, \zeta) \subset (-\overline{z}, 0)$, i.e., embedded in the real slab. $\Gamma(\zeta), \tau(\zeta)$ for $\zeta \neq 0$ then give reflection and transmission through the embedded slab, when a wave is incident from a matched medium above.

While the concept of a matched medium is useful for separating the effects of random propagation from those of scattering by macroscopic interfaces, we recall from section 2 that in general the matched medium will not be isotropic. An exception discussed there is when there are no fluctuations in shear modulus μ.

If we assume that waves are localized, we may derive some properties of waves reflected from a random half space. Letting the slab length $\overline{z} \to \infty$, no waves may penetrate, whatever the incident field, and so $\overline{\tau} = 0$. From (4.32), (4.33) $\overline{\Gamma}$ is then a symmetric unitary matrix. Moreover, using the concept of invariant embedding, no wave may penetrate the embedded medium either, so that $\tau(\zeta) = 0$ for all finite ζ, and hence $\Gamma(\zeta)$ is symmetric and unitary for all finite ζ by (4.28), (4.29). We will use these facts in the next section to compute reflection statistics. They will be shown in more detail in sections 6–8 when we compute the localization length and use it in representations for Γ, τ.

For finite $\overline{z}$, we can, by differentiating (4.26) and using (4.11), derive the matrix Riccati equation for Γ

$$
\begin{aligned}
\frac{d\Gamma}{d\zeta} &= -\frac{i\omega}{\epsilon}\left\{\eta_2^* - \eta_1\Gamma - \Gamma\eta_1^* + \Gamma\eta_2\Gamma\right\} \\
\Gamma|_{\zeta=-\overline{z}} &= \Gamma_T \,.
\end{aligned} \tag{4.36}
$$

While we will primarily use the propagator matrices, and reflection and transmission matrices as defined above, we will also need to consider some variants. One such variant is the propagator

$$
\overline{Q}(\zeta) = e^{i\frac{\omega}{\epsilon}\Lambda_1\zeta} Q(\zeta) e^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}} \,. \tag{4.37}
$$

$\overline{Q}(\zeta)$ satisfies the equation

$$
\begin{aligned}
\frac{d\overline{Q}}{d\zeta} &= \frac{i\omega}{\epsilon}\left\{\Lambda_1 + \nu(-\zeta/\epsilon^2)\right\}\overline{Q} \\
\overline{Q}|_{\zeta=-\overline{z}} &= I
\end{aligned} \tag{4.38}
$$

and so is a fundamental solution matrix for the vector of up and down-going waves, but in the upward coordinate system $\zeta = -z$. Each of the propagators, $Q, \overline{Q}$, has a separate use, as follows:

Q satisfies an equation, (4.7) which is "centered", that is, the coefficients have zero mean. We are thus able to find a useful limit, as $\epsilon \downarrow 0$, even though the factor $1/\epsilon$, which is becoming infinite, multiplies the right hand side. Thus the transformation, equation (4.2), that "centers" the equation is directly analogous to "subtracting the mean", when deriving the central limit theorem for sums of independent random variables.

A complication incurred in centering the equation is that the coefficients in (4.7) involve rapidly-varying, but deterministic, trigonometric factors on the ζ/ϵ scale. In contrast, the coefficients for $\overline{Q}$, equation (4.38), are stationary random processes. $\overline{Q}$ is therefore of the form needed for the Oseledec theorem, which is used in section 6 to show the existence of Lyapunov exponents, and other structural facts about $\overline{Q}$. With the relation (4.37), we can then easily infer the analogous facts for Q, which is used in the detailed calculations.

From (4.37) and the symmetry relations for Q, we find that Q and $\overline{Q}$ have the same symmetries. Thus

$$\overline{Q} = \begin{bmatrix} \overline{A} & \overline{B}^* \\ \overline{B} & \overline{A}^* \end{bmatrix} , \tag{4.39}$$

where

$$\begin{aligned} \overline{A} &= e^{i\frac{\omega}{\epsilon}\Lambda\zeta} A e^{i\frac{\omega}{\epsilon}\Lambda\overline{z}} \\ \overline{B} &= e^{-i\frac{\omega}{\epsilon}\Lambda\zeta} B e^{i\frac{\omega}{\epsilon}\Lambda\overline{z}} . \end{aligned} \tag{4.40}$$

Also

$$\overline{Q}^{-1} = \mathcal{O}\overline{Q}^{*T}\mathcal{O} \tag{4.41}$$

so that $\overline{A}, \overline{B}$ satisfy the symmetry relations (4.18) – (4.21).

Another variant of the propagator equations, that will be useful for certain aspects of the transmission problem, arises from a formulation in terms of the downward coordinate system $z = -\zeta$. This formulation my be interpreted as another kind of "invariant embedding", where we add material continuously to the *transmission* end of the slab, rather than the incidence end, as was done previously. Let $\overline{Q}'(z)$ be a fundamental solution matrix for (3.24)

$$\begin{aligned} \frac{d\overline{Q}'(z)}{dz} &= -\frac{i\omega}{\epsilon}\left\{\Lambda_1 + \nu(z/\epsilon^2)\right\}\overline{Q}'(z) \\ \overline{Q}'|_{z=0} &= I . \end{aligned} \tag{4.42}$$

We will solve for the matched medium case only, so that

$$\overline{Q}'(\overline{z}) \begin{bmatrix} \mathbf{U}^{\text{refl}} \\ \mathbf{D}^{\text{inc}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}^{\text{trans}} \end{bmatrix} . \tag{4.43}$$

Note that if $\nu(z/\epsilon^2)$ is statistically reversible, that is, isotropic in one dimension, then $\overline{Q}'(z)$ has the same distribution as $\overline{Q}^*(z - \overline{z})$.

The centered version is

$$Q'(z) = e^{i\frac{\omega}{\epsilon}\Lambda_1 z}\overline{Q}'(z) \tag{4.44}$$

which satisfies

$$\begin{aligned} \frac{dQ'(z)}{dz} &= -\frac{i\omega}{\epsilon}\eta(-z/\epsilon^2, -z/\epsilon)Q'(z) \\ Q'|_{z=0} &= I . \end{aligned} \tag{4.45}$$

Then Q' has the same symmetry properties as Q, i.e.

$$Q' = \begin{bmatrix} A' & (B')^* \\ B' & (A')^* \end{bmatrix} \tag{4.46}$$

$$(Q')^{-1} = \mathcal{O}(Q')^{*T}\mathcal{O} \tag{4.47}$$

Putting (4.46), (4.44) into (4.43) then yields

$$\begin{aligned} \mathbf{U}^{\text{refl}} &= -(A'(\overline{z}))^{-1}(B'(\overline{z}))^*\mathbf{D}^{\text{inc}} \\ \mathbf{D}^{\text{trans}} &= e^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}\left[(A'(\overline{z}))^* - B'(\overline{z})(A'(\overline{z}))^{-1}(B'(\overline{z}))^*\right]\mathbf{D}^{\text{inc}} \end{aligned} \tag{4.48}$$

From the statistical relations between $\overline{Q}, \overline{Q}'$ we have that

$$Q'(\overline{z}) \stackrel{\mathcal{D}}{=} e^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}} Q^*(0) e^{-i\frac{\omega}{\epsilon}\Lambda_1\overline{z}} \tag{4.49}$$

That is, the matrices have the same probability distribution.

In Table II, we have summarized the notation for the four different propagator matrices. Note that primed variables propagate downward, i.e., in z-coordinates, while unprimed variables propagate upward, i.e., in ζ-coordinates. For upward propagating variables, we will alternatively need to keep the transmission end fixed at $\overline{\zeta} = 0$, by using the coordinate $\overline{\zeta} = \zeta + \overline{z}$. Also note that barred propagators have stationary coefficients in their defining equations, while for unbarred propagators these coefficients have mean zero.

PROPAGATOR	*DIRECTION*	*COEFFICIENTS*	*EQUAL TO*
Q	up (ζ or $\overline{\zeta}$)	mean zero	$e^{-i\frac{\omega}{\epsilon}\Lambda_1\zeta}\overline{Q}e^{-i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}$
$\overline{Q}$	up (ζ or $\overline{\zeta}$)	stationary	$e^{i\frac{\omega}{\epsilon}\Lambda_1\zeta}Qe^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}$
Q'	down (z)	mean zero	$e^{i\frac{\omega}{\epsilon}\Lambda_1 z}\overline{Q}'$
$\overline{Q}'$	down (z)	stationary	$e^{-i\frac{\omega}{\epsilon}\Lambda_1 z}Q'$

Table II. Summary of propagators. Coordinates are $\zeta = -z$, $\overline{\zeta} = \zeta + \overline{z}$.

5 Mode conversion of backscatter from a random half-space

Let $,_{ij}, i,j = 1,2$ with $,_{12} = ,_{21}$ be the elements of , . From (4.36), (4.5), (3.30) and (3.3) we obtain

$$\begin{aligned}
\frac{d,_{11}}{d\zeta} &= -\frac{i\omega}{\epsilon}\Big\{\delta_4 e^{-2i\frac{\omega}{\epsilon}\alpha_p\zeta} - 2\delta_1 ,_{11} - 2\delta_2 e^{-i\frac{\omega}{\epsilon}(\alpha_p-\alpha_s)\zeta} ,_{12} \\
&\quad + \delta_4 e^{2i\frac{\omega}{\epsilon}\alpha_p\zeta} ,_{11}^2 + 2\delta_5 e^{i\frac{\omega}{\epsilon}(\alpha_p+\alpha_s)\zeta} ,_{11} ,_{12} + \delta_6 e^{2i\frac{\omega}{\epsilon}\alpha_s\zeta} ,_{12}^2\Big\} \\
\frac{d,_{12}}{d\zeta} &= -i\frac{\omega}{\epsilon}\Big\{\delta_5 e^{-i\frac{\omega}{\epsilon}(\alpha_p+\alpha_s)\zeta} - (\delta_1+\delta_3) ,_{12} - \delta_2 e^{-i\frac{\omega}{\epsilon}(\alpha_p-\alpha_s)\zeta} ,_{22} \\
&\quad - \delta_2 e^{i\frac{\omega}{\epsilon}(\alpha_p-\alpha_s)\zeta} ,_{11} + \delta_4 e^{2i\frac{\omega}{\epsilon}\alpha_p\zeta} ,_{11} ,_{12} \\
&\quad + \delta_5 e^{i\frac{\omega}{\epsilon}(\alpha_p+\alpha_s)\zeta}\Big[,_{11} ,_{22} + ,_{12}^2\Big] + \delta_6 e^{2i\frac{\omega}{\epsilon}\alpha_s\zeta} ,_{12} ,_{22}\Big\} \\
\frac{d,_{22}}{d\zeta} &= -i\frac{\omega}{\epsilon}\Big\{\delta_6 e^{-2i\frac{\omega}{\epsilon}\alpha_s\zeta} - 2\delta_2 e^{i\frac{\omega}{\epsilon}(\alpha_p-\alpha_s)\zeta} ,_{12} - 2\delta_3 ,_{22} + \delta_4 e^{2i\frac{\omega}{\epsilon}\alpha_p\zeta} ,_{12}^2 \\
&\quad + 2\delta_5 e^{i\frac{\omega}{\epsilon}(\alpha_p+\alpha_s)\zeta} ,_{12} ,_{22} + \delta_6 e^{2i\frac{\omega}{\epsilon}\alpha_s\zeta} ,_{22}^2\Big\}
\end{aligned} \tag{5.1}$$

For reflection from a random half-space we have, from the discussion of the last section, that , is unitary. Thus , can be characterized by the three real variables θ, ϕ, ψ, where

$$,_{11} \quad = \quad \tanh\frac{\theta}{2} e^{i(\phi+\psi)}$$

$$
\begin{aligned}
,_{12} &= i\mathrm{sech}\frac{\theta}{2}e^{i\phi} \\
,_{22} &= \tanh\frac{\theta}{2}e^{i(\phi-\psi)}
\end{aligned}
\tag{5.2}
$$

Substitution of (5.2) into (5.1) yields

$$
\begin{aligned}
\frac{d\theta}{d\zeta} &= \frac{2\omega}{\epsilon}\Big\{-2\delta_2\cosh\frac{\theta}{2}\cos\left(\frac{\omega}{\epsilon}\left[\alpha_p-\alpha_s\right]\zeta+\psi\right)-\delta_4\sin\left(\frac{2\omega}{\epsilon}\alpha_p\zeta+\theta+\phi\right) \\
&\quad +2\delta_5\sinh\frac{\theta}{2}\cos\left(\frac{\omega}{\epsilon}\left[\alpha_p+\alpha_s\right]\zeta+\phi\right)-\delta_6\sin\left(\frac{2\omega}{\epsilon}\alpha_s\zeta+\phi-\psi\right)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\phi}{d\zeta} &= \frac{2\omega}{\epsilon}\Big\{\frac{1}{2}\left(\delta_1+\delta_3\right)-\frac{1}{2}\delta_4\tanh\frac{\theta}{2}\cos\left(\frac{2\omega}{\epsilon}\alpha_p\zeta+\phi+\psi\right) \\
&\quad +\delta_5\mathrm{sech}\frac{\theta}{2}\sin\left(\frac{\omega}{\epsilon}\left[\alpha_p+\alpha_s\right]\zeta+\phi\right)-\frac{1}{2}\delta_6\tanh\frac{\theta}{2}\cos\left(\frac{2\omega}{\epsilon}\alpha_s\zeta+\phi-\psi\right)\Big\}
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\frac{d\psi}{d\zeta} &= \frac{2\omega}{\epsilon}\Big\{\frac{1}{2}(\delta_1-\delta_3)+\delta_2\mathrm{csch}\frac{\theta}{2}\sin\left(\frac{\omega}{\epsilon}[\alpha_p-\alpha_s]\zeta+\psi\right)-\frac{1}{2}\delta_4\coth\frac{\theta}{2}\cos\left(\frac{2\omega}{\epsilon}\alpha_p\zeta+\phi+\psi\right) \\
&\quad +\frac{1}{2}\delta_6\coth\frac{\theta}{2}\cos\left(\frac{2\omega}{\epsilon}\alpha_s\zeta+\phi-\psi\right)\Big\}
\end{aligned}
$$

Recall that $\delta_j = \delta_j(-\zeta/\epsilon^2)$ are stationary random processes with mean zero. Thus, the vector $\boldsymbol{\psi} = (\theta, \phi, \psi)^T$ satisfies a nonlinear ordinary differential equation of the form

$$
\frac{d\boldsymbol{\psi}}{d\zeta} = \frac{1}{\epsilon}\mathbf{F}(\zeta/\epsilon^2, \zeta/\epsilon, \boldsymbol{\psi})
\tag{5.4}
$$

where the function $\mathbf{F}(\zeta, \xi, \boldsymbol{\psi})$ is, for fixed values of ξ and $\boldsymbol{\psi}$, a vector of stationary random processes in ζ, with mean zero.

Equations similar to (5.4), and variants thereof have been the subject of extensive study [16], [17]. The essential features are: (i) that the random part varies rapidly when ζ is varied, i.e. $\zeta \to \zeta/\epsilon^2$ in the random coefficients of $\mathbf{F}(\zeta, \xi, \boldsymbol{\psi})$, and (ii) that $\mathbf{F}$ has mean zero, and a large amplitude factor, $1/\epsilon$, multiplying it in the equation. A nonstandard feature of (5.4) is the second scale of rapid variation, $\xi \to \zeta/\epsilon$, introduced by the trigonometric functions in (5.3). However, we can use here the general limit theorem of appendix A in Burridge et. al. [15], which was used in that paper to study the reflection of acoustic pulses. The result is that as $\epsilon \downarrow 0, \boldsymbol{\psi}$ converges (weakly) to a diffusion Markov process. Thus, roughly speaking, the fast fluctuations in (5.4) may be approximated by appropriate "white noises" for small ϵ

The limiting diffusion process in characterized by a linear differential operator $\mathcal{L}$, the "infinitesimal generator", in which $\boldsymbol{\psi}$ plays the role of an independent variable. $\mathcal{L}$ can then be used to construct deterministic "diffusion-like" partial differential equations, the Kolmogorov or Fokker-Planck, equations, for various statistical quantities associated with the process.

To construct $\mathcal{L}$, we define the averaging operator for the rapid trigonometric scale

$$
\langle\cdot\rangle_\xi = \lim_{\xi_0\to\infty}\frac{1}{\xi_0}\int_0^{\xi_0} d\xi\ .
\tag{5.5}
$$

Then the infinitesimal generator can be written as

$$
\mathcal{L} = \int_0^\infty d\zeta' E\left[\langle\mathbf{F}(\zeta, \xi, \boldsymbol{\psi})\cdot\nabla_{\boldsymbol{\psi}}\mathbf{F}(\zeta+\zeta', \xi, \boldsymbol{\psi})\cdot\nabla_{\boldsymbol{\psi}}\rangle_\xi\right]\ .
\tag{5.6}
$$

Let

$$\sigma_{ij} = \int_0^\infty E\left[\delta_j(\zeta)\delta_j(\zeta+\zeta')\right] d\zeta' \qquad i,j = 1,\ldots,6\ . \tag{5.7}$$

Then application of (5.6) to the specific system (5.3) yields

$$\begin{aligned}
\overline{\mathcal{L}} &= 2\omega^2 \left\{ \frac{1}{2}\sigma_{11}\left(\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\psi}\right)^2 + \sigma_{13}\left(\frac{\partial^2}{\partial\phi^2}-\frac{\partial^2}{\partial\psi^2}\right)\right. \\
&+ \sigma_{33}\left(\frac{\partial}{\partial\phi}-\frac{\partial}{\partial\psi}\right)^2 + \sigma_{22}\left[4\cosh\frac{\theta}{2}\frac{\partial}{\partial\theta}\left(\cosh\frac{\theta}{2}\frac{\partial}{\partial\theta}\right)\right. \\
&+ \left. 2\coth\frac{\theta}{2}\frac{\partial}{\partial\theta} + \operatorname{csch}^2\frac{\theta}{2}\frac{\partial^2}{\partial\psi^2}\right] \\
&+ \sigma_{55}\left[4\sinh\frac{\theta}{2}\frac{\partial}{\partial\theta}\left(\sinh\frac{\theta}{2}\frac{\partial}{\partial\theta}\right) - 2\tanh\frac{\theta}{2}\frac{\partial}{\partial\theta}\right. \\
&+ \left. \operatorname{sech}^2\frac{\theta}{2}\frac{\partial^2}{\partial\phi^2}\right] \\
&+ \sigma_{44}\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}\left(\tanh\frac{\theta}{2}+\coth\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}\right. \\
&+ \left. \frac{1}{4}\left(\tanh\frac{\theta}{2}\frac{\partial}{\partial\phi}+\coth\frac{\theta}{2}\frac{\partial}{\partial\psi}\right)^2\right] \\
&+ \sigma_{66}\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}\left(\tanh\frac{\theta}{2}+\coth\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}\right. \\
&+ \left.\left. \frac{1}{4}\left(\tanh\frac{\theta}{2}\frac{\partial}{\partial\phi}-\coth\frac{\theta}{2}\frac{\partial}{\partial\psi}\right)^2\right]\right\}
\end{aligned} \tag{5.8}$$

In deriving (5.8) we have assumed that

$$\alpha_s \neq 3\alpha_p \tag{5.9}$$

When (5.9) is violated a special resonance is created, since then $\alpha_s - \alpha_p = 2\alpha_p$, and products of some of the trigonometric terms, which averaged to zero in computing (5.8), would then have nonzero averages. Thus more terms occur in the generator $\overline{\mathcal{L}}$. Physically, this resonance occurs, because when (5.9) is violated, the faster P-wave can resonate, in the effective medium, with the slower S, in the following way: The waves start together at $z = z_1$, and the S-wave travels directly to $z = z_2 > z_1$. Meanwhile, the P-wave travels from z_1 to z_2, is reflected back to z_1 and then back again to z_2 to arrive exactly in phase with the S- wave.

Let $P(\zeta,\theta,\phi,\psi)$ be the joint probability density of (θ,ϕ,ψ) at position ζ. Then P satisfies the Forward Kolmogorov, or Fokker-Planck equation

$$\frac{\partial P}{\partial\zeta} = \overline{\mathcal{L}}^+ P \tag{5.10}$$

where $\overline{\mathcal{L}}^+$ is the adjoint of $\overline{\mathcal{L}}$. Furthermore, we may seek a steady-state, ζ-independent solution of (5.10) since the initial condition is at $\zeta = -\infty$. Note also that none of the coefficients in (5.8) depend on ϕ or ψ, so that we may obtain a solution independent of ϕ and ψ as well. Therefore

$$P(\theta,\phi,\psi) = \frac{1}{4\pi^2}\,\overline{P}(\theta) \tag{5.11}$$

where the marginal density of $\theta, \overline{P}(\theta)$ satisfies

$$\begin{aligned}\frac{\partial \overline{P}}{\partial \zeta} = 0 &= 4\omega^2 \frac{\partial}{\partial \theta}\left\{\left[\sigma_{22} - \sigma_{55} + \frac{1}{2}(\sigma_{44} + \sigma_{66})\right]\overline{P}\right. \\ &+ \left.\frac{\partial}{\partial \theta}\left([\sigma_{22} + \sigma_{55}]\cosh\theta + \sigma_{22} - \sigma_{55} + \frac{1}{2}[\sigma_{44} + \sigma_{66}]\right)\overline{P}\right\}\end{aligned} \tag{5.12}$$

We take $\theta \geq 0$, so that $\int_0^\infty \overline{P}(\theta)d\theta = 1$. Let

$$\Sigma = \frac{(\sigma_{22} - \sigma_{55})}{(\sigma_{22} + \sigma_{55})} + \frac{1}{2}\frac{(\sigma_{44} + \sigma_{66})}{(\sigma_{22} + \sigma_{55})} \ . \tag{5.13}$$

Then $\overline{P}(\theta)$ can be written in terms of the single parameter Σ

$$\overline{P}(\theta) = \begin{cases} \sqrt{\frac{1+\Sigma}{2}}\frac{(e^{2\theta}-1)}{(e^{2\theta}+2\Sigma e^{\theta}+1)^{3/2}}, & \Sigma \neq 1 \\ \\ \frac{(e^{\theta}-1)}{(e^{\theta}+1)^2}, & \Sigma = 1 \end{cases} \tag{5.14}$$

Note $\Sigma > -1$ is necessary to avoid the singularity in (5.14).

Since $\overline{\Gamma}$ is symmetric unitary,

$$|\overline{\Gamma}_{11}|^2 + |\overline{\Gamma}_{12}|^2 = 1 \tag{5.15}$$

$$|\overline{\Gamma}_{11}|^2 = |\overline{\Gamma}_{22}|^2 \ . \tag{5.16}$$

Equation (5.15) may be interpreted as saying that for an incident P-wave, the sum of reflected P-wave and S-wave energies is conserved. $|\overline{\Gamma}_{12}|^2$ is the proportion that is mode-converted to S. Because of (5.16), there is complete symmetry, in terms of energies, for incident P converted to S and incident S converted to P. That is, the fraction of energy which returns in the same mode as the incident wave is $|\overline{\Gamma}_{11}|^2$.

Moments of $|\overline{\Gamma}_{11}|^2$ can be computed from

$$E\left[|\overline{\Gamma}_{11}|^{2n}\right] = \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} |\overline{\Gamma}_{11}|^{2n} P(\theta, \phi, \psi)d\phi d\psi d\theta \ . \tag{5.17}$$

Here P is given by (5.11), (5.14), Γ by (5.2), and $\overline{\Gamma}$ by (4.30) or (4.32). Thus the problem is reduced to quadratures. In the special case of a matched medium we have $\overline{\Gamma} = \Gamma$ and hence

$$m_n = E\left[|\overline{\Gamma}_{11}|^{2n}\right] = \int_0^\infty \left(\tanh\frac{\theta}{2}\right)^{2n} \overline{P}(\theta)d\theta \ . \tag{5.18}$$

From elementary integration, we find

$$m_1 = \begin{cases} 1 + \frac{2}{(\Sigma-1)}\left[1 - \frac{1}{2}\sqrt{\frac{(\Sigma+1)}{(\Sigma-1)}}\ln\left(\Sigma + \sqrt{\Sigma^2-1}\right)\right], & \Sigma > 1 \\ \\ \frac{2}{3}, & \Sigma = 1 \\ \\ 1 + \frac{2}{(1-\Sigma)}\left[\sqrt{\frac{(1+\Sigma)}{(1-\Sigma)}}\arctan\left(\sqrt{\frac{(1-\Sigma)}{(1+\Sigma}}\right) - 1\right], & \Sigma < 1 \end{cases} \tag{5.19}$$

Similarly, m_2 may be calculated by the following formulas for $\Sigma > 1$:

$$\begin{aligned}
C_1(\Sigma) &= \frac{1}{\sqrt{\Sigma-1}} \ln(\Sigma + \sqrt{\Sigma^2-1}) \\
C_1'(\Sigma) &= \frac{1}{(\Sigma-1)}\left[-\frac{\ln(\Sigma+\sqrt{\Sigma^2-1})}{2\sqrt{\Sigma-1}} + \frac{1}{\sqrt{\Sigma+1}}\right] \\
C_2(\Sigma) &= \frac{1}{2(\Sigma-1)}\left[\sqrt{\Sigma+1} - C_1(\Sigma)\right] \\
C_2'(\Sigma) &= -\frac{1}{2(\Sigma-1)^2}\left[\sqrt{\Sigma+1} - C_1(\Sigma)\right] \\
&+ \frac{1}{2(\Sigma-1)}\left[\frac{1}{2\sqrt{\Sigma+1}} - C_1'(\Sigma)\right] , \qquad (5.20)
\end{aligned}$$

whence

$$m_2 = 1 + 4\sqrt{\Sigma+1}\,(C_1'(\Sigma) - C_2'(\Sigma)) \qquad \text{for } \Sigma > 1 \ . \qquad (5.21)$$

Also

$$m_2 = \frac{8}{15}, \quad \Sigma = 1 \ , \qquad (5.22)$$

while for $\Sigma < 1$, we compute

$$\begin{aligned}
C_3(\Sigma) &= \frac{2}{\sqrt{1-\Sigma}}\left[\frac{\pi}{2} - \arctan\left(\sqrt{\frac{1+\Sigma}{1-\Sigma}}\right)\right] \\
C_3'(\Sigma) &= \frac{1}{(1-\Sigma)^{3/2}}\left[\frac{\pi}{2} - \arctan\left(\sqrt{\frac{1+\Sigma}{1-\Sigma}}\right) - \sqrt{\frac{1-\Sigma}{1+\Sigma}}\right] \\
C_4(\Sigma) &= \frac{1}{2(\Sigma-1)}\left[\sqrt{\Sigma+1} - C_3(\Sigma)\right] \\
C_4'(\Sigma) &= -\frac{1}{2(\Sigma-1)^2}\left[\sqrt{\Sigma+1} - C_3\right] + \frac{1}{2(\Sigma-1)}\left[\frac{1}{2\sqrt{\Sigma+1}} - C_3'(\Sigma)\right] \qquad (5.23)
\end{aligned}$$

and then

$$m_2 = 1 + 4\sqrt{1+\Sigma}\,(C_3'(\Sigma) - C_4'(\Sigma)) \ , \qquad \text{for } -1 < \Sigma < 1 \ . \qquad (5.24)$$

To gain some insight into the physical content of these results, we first consider the parameter Σ, defined by (5.13). The σ_{jj}, defined by (5.7), are nonnegative power spectra evaluations. Noting Table I, these parameters can be subdivided into two subgroups on each of two issues. The two issues are forward scattering vs. backscattering and same-mode coupling vs. cross-mode coupling. Thus σ_{22} and σ_{55} are measures of cross-mode forward scattering and backscattering strengths, respectively; σ_{44} and σ_{66}, on the other hand, are two measures of same-mode backscattering strength. As written in (5.13), Σ is a sum of two terms. The first term, $(\sigma_{22} - \sigma_{55})/(\sigma_{22} + \sigma_{55})$, is a measure of the persistence of direction in cross-mode scattering; it is a ratio of the net cross-mode forward scattering strength to the total cross-mode scattering strength. The second term in the sum (5.13) that forms Σ, i.e. $\frac{1}{2}(\sigma_{44} + \sigma_{66})/(\sigma_{22} + \sigma_{55})$, can be viewed as a ratio of the average same-mode backscattering strength to the total cross-mode scattering strength.

The parameter Σ can take values on $(-1, \infty)$. From (5.13) it is clear that values close to -1 are attained only if $\sigma_{55} \gg \sigma_{22} + \sigma_{44} + \sigma_{66}$, i.e. if cross-mode backscattering is the dominant coupling mechanism. Values of Σ close to -1 will therefore correspond to a high degree of mode conversion. On the other hand, $\Sigma \gg 1$ is attained when $\sigma_{44} + \sigma_{66} \gg \sigma_{22} + \sigma_{55}$. In this case, backscattering of

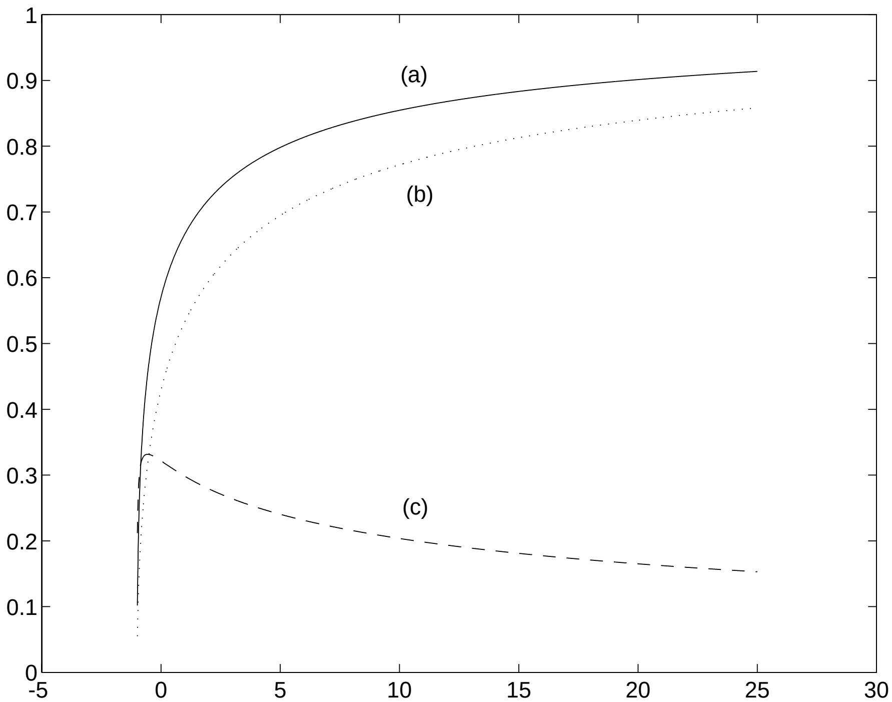

Figure 1: (a) Second moment m_2, (b) Fourth moment m_4, and (c) Fluctuations $\sqrt{m_4 - m_2^2}$ vs. Mode conversion parameter Σ

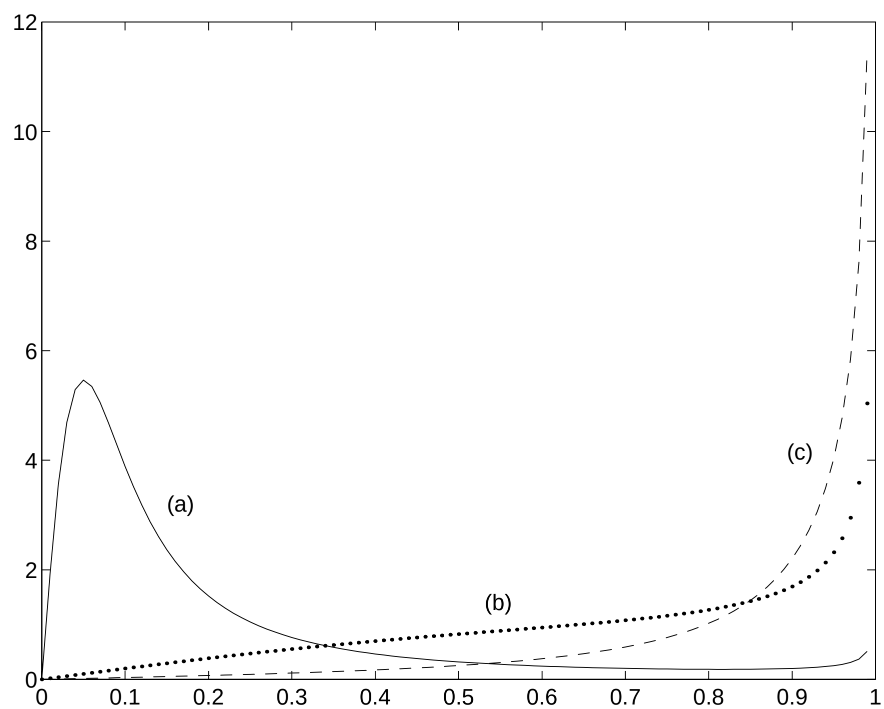

Figure 2: Invariant $P(r)$ vs. Reflection coefficient modulus r for different values of Mode conversion parameter Σ (a) $\Sigma = -.99$ (b) $\Sigma = 0$ (c) $\Sigma = 5.0$

energy within the same mode is the dominant coupling mechanism, overshadowing total cross-mode coupling. At this extreme, very little mode conversion will occur.

This behavior is illustrated in Figures 1 and 2. The variations of of m_2, m_4 and fluctuations $\sqrt{m_4 - m_2^2}$ as functions of Σ are plotted in Figure 1. Recall that m_2, in particular, represents the mean $P(S)$ wave energy reflected by the randomly-layered half-space, given that a unit of $P(S)$ wave energy is incident. As Figure 1 indicates, $m_2 \searrow 0$ as $\Sigma \searrow -1$; in this limit, therefore, the random half-space reflection properties tend toward total mode conversion. On the other hand, $m_2 \nearrow 1$ as $\Sigma \nearrow \infty$ and this limit corresponds to the absence of mode conversion.

Note, however, the vertical asymptote in Figure 1 at $\Sigma = -1$. Significant mode conversion thus occurs for values of Σ quite close to -1. This fact is illustrated in a different way in Figure 2, which shows how the invariant probability density itself varies as the parameter Σ is changed. For these plots, we define:

$$r = |,_{11}| = |,_{22}| = \tanh\frac{\theta}{2} \ , \quad P(r) = 2\overline{P}(\theta)\cosh^2\frac{\theta}{2}\Big|_{\tanh\frac{\theta}{2}=r} \tag{5.25}$$

Figure 2 shows the variation of invariant density $P(r)$ vs. reflection coefficient modulus r for several values of Σ. It is clear that even for $\Sigma = -0.99$, the coalesence of probability mass near $r = 0$ is not particularly pronounced.

To further evaluate the results obtained, a specific random layering model was adopted. In this model, layering was controlled by a zero mean, piecewise constant process $\chi(\cdot)$. The distance between layer switching was assumed to be exponentially distributed, with unit rate parameter; the random variables defining χ on the intervals between switches were assumed to be independent and identically distributed on the interval $[-\delta, \delta\,]$. In terms of this process, the material parameters in (2.9) were taken to be

$$\begin{aligned} \lambda &= \lambda(z/\varepsilon^2) = \lambda_0\left(1 + \chi(z/\varepsilon^2)\right) \\ \mu &= \overline{\mu} \\ \rho &= \rho(z/\varepsilon^2) = \overline{\rho}\left(1 + \chi(z/\varepsilon^2)\right) \quad , \quad z \geq 0 \ . \end{aligned} \tag{5.26}$$

In the model, therefore, the mean layer thickness was ultimately $O(\varepsilon^2)$. The Lamé parameter μ was chosen constant so that, as noted in section 2, the effective medium could be made isotropic and matched to a deterministic elastic half-space in $z < 0$. Consequently, in the region $z < 0$ the material parameters were chosen to be:

$$\begin{aligned} \lambda &= \overline{\lambda} = \left[E\left\{\frac{1}{\lambda_0(1+\chi) + 2\overline{\mu}}\right\}\right]^{-1} - 2\overline{\mu} \\ \mu &= \overline{\mu} \\ \rho &= \overline{\rho} \quad , \quad z < 0 \ . \end{aligned} \tag{5.27}$$

Material parameter values in the region $z \geq 0$, i.e. λ_0, $\overline{\mu}$, $\overline{\rho}$, were chosen so that, in the absence of random fluctuations, the half-space would correspond to the first layer of the Gutenberg Earth model [18]. This layer has a compressional wave velocity of 6.14 km./sec., a shear wave velocity of 3.55 km./sec. and a density of 2.74 gm./cm^3. Then, adopting the length, velocity, and density scales mentioned in section 2, we obtained the following scaled, nondimensionalized material parameter values:

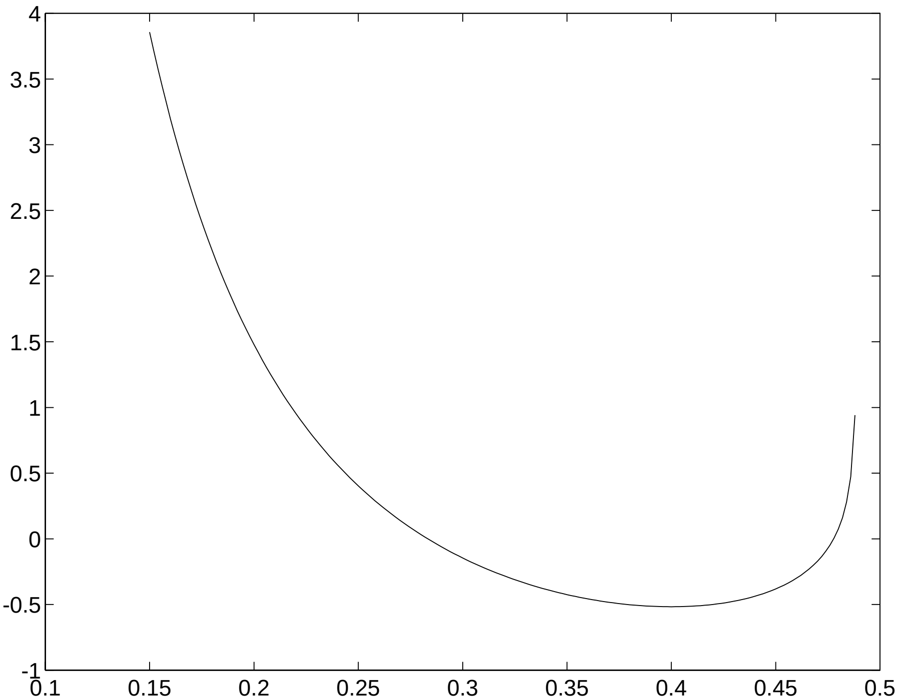

Figure 3: Mode conversion parameter Σ vs. Slowness p

$$\lambda_0 = 1.39 \quad , \quad \overline{\mu} = 1.40 \quad , \quad \overline{\rho} = 1.00 \ . \tag{5.28}$$

Specification of the model was completed by assuming $\delta = 0.3$, i.e. a 30% peak fluctuation level, and $\varepsilon^2 = 10^{-3}$. The parameter $\overline{\lambda}$ could then be computed using (5.27).

Figure 3 shows the variation of the mode parameter Σ as a function of (nondimensionalized) slowness p. Recall that mode conversion decreases as Σ increases. Therefore, it is clear that Σ should become infinite as p approaches zero. At normal incidence ($p = 0$), the compressional and shear waves decouple and in the absence of coupling there can be no mode conversion. As the figure indicates, mode conversion also tends to decrease rather sharply as p approaches the compressional wave cutoff value ($\approx .488$).

Figures 4 - 6 show the variation of second moment m_2, fourth moment m_4 and fluctuations $\sqrt{m_4 - m_2^2}$ as functions of slowness p not only for the matched medium case but for varying degrees of interface mismatch. The curves labelled (a) correspond to the case of a matched medium, i.e. to the case where the material parameters of the deterministic medium in $z < 0$ equal those of the effective medium in $z \geq 0$. In this case, in particular, the shear velocity was 3.55 km./sec. (or 1.183 in nondimensional units). To introduce mismatch, the shear velocity in the deterministic half-space was subsequently reduced by factors of $1/2, 1/4$ and $1/8$. These results are presented in Figures 4 - 6 by the curves labelled (b), (c), and (d), respectively.

In the case of mismatch, the interface jump matrix $J(0)$ given by (3.14) and (3.15) is no longer the 4×4 identity matrix. The moments recorded by curves (b), (c), and (d) in Figures 4 - 6 therefore correspond to the moments of $|\overline{\Gamma}_{11}| = |\overline{\Gamma}_{22}|$, where the complete reflection matrix $\overline{\Gamma}$ is defined by (4.30). These mismatch curves basically correspond to an evaluation of integral (5.17) for the slowness values of interest.

The behavior exhibited in Figures 4 - 6 is qualitatively consistent with what one would expect

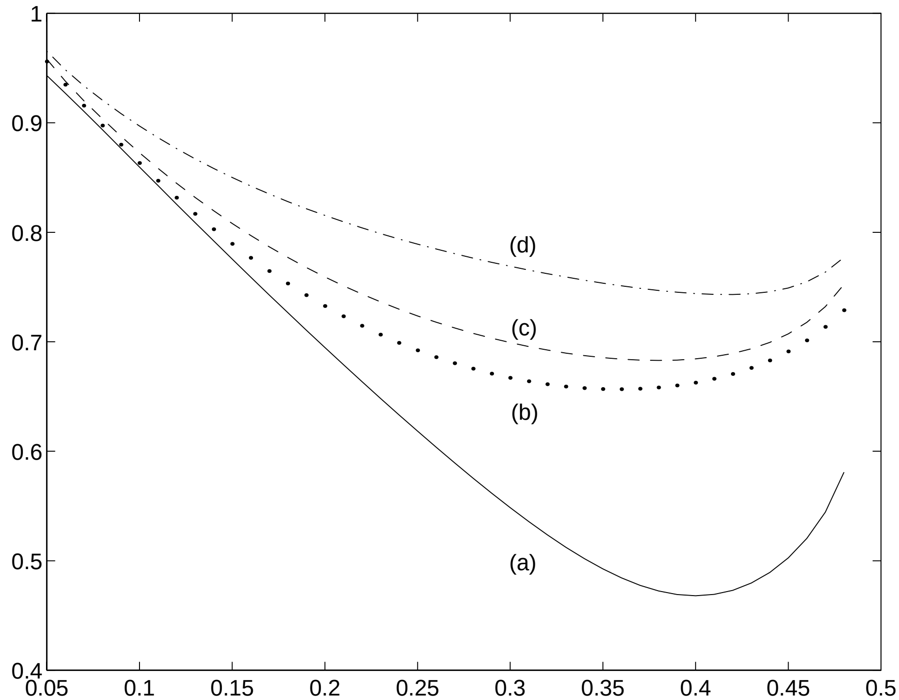

Figure 4: Second moment m_2 vs. Slowness p (a) matched medium (b) shear velocity reduced by 1/2 in $z < 0$. (c) shear velocity reduced by 1/4 in $z < 0$. (d) shear velocity reduced by 1/8 in $z < 0$.

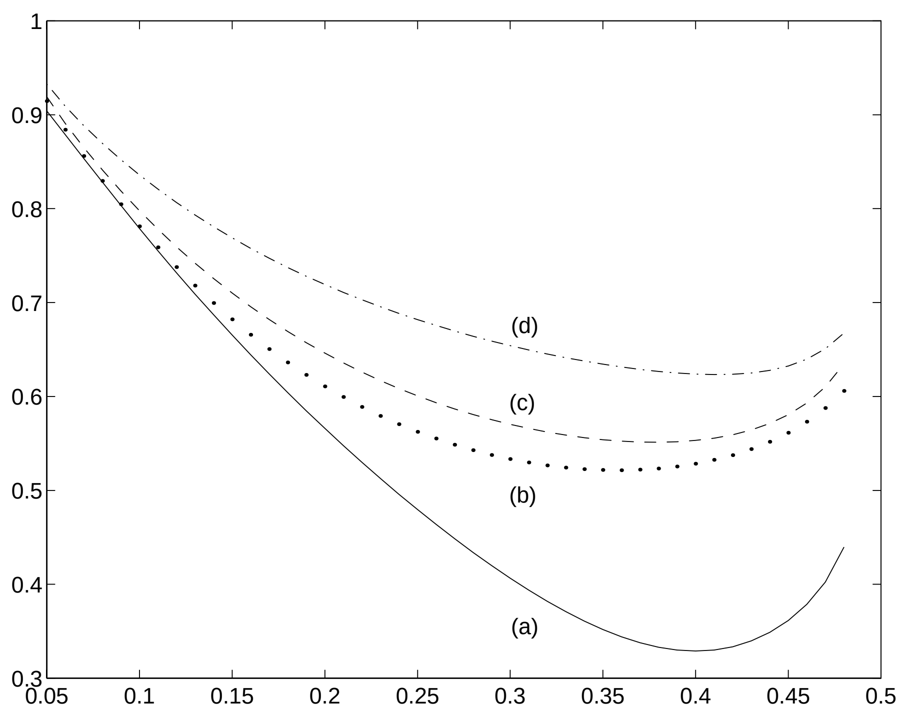

Figure 5: Fourth moment m_4 vs. Slowness p (a) matched medium (b) shear velocity reduced by 1/2 in $z < 0$. (c) shear velocity reduced by 1/4 in $z < 0$. (d) shear velocity reduced by 1/8 in $z < 0$.

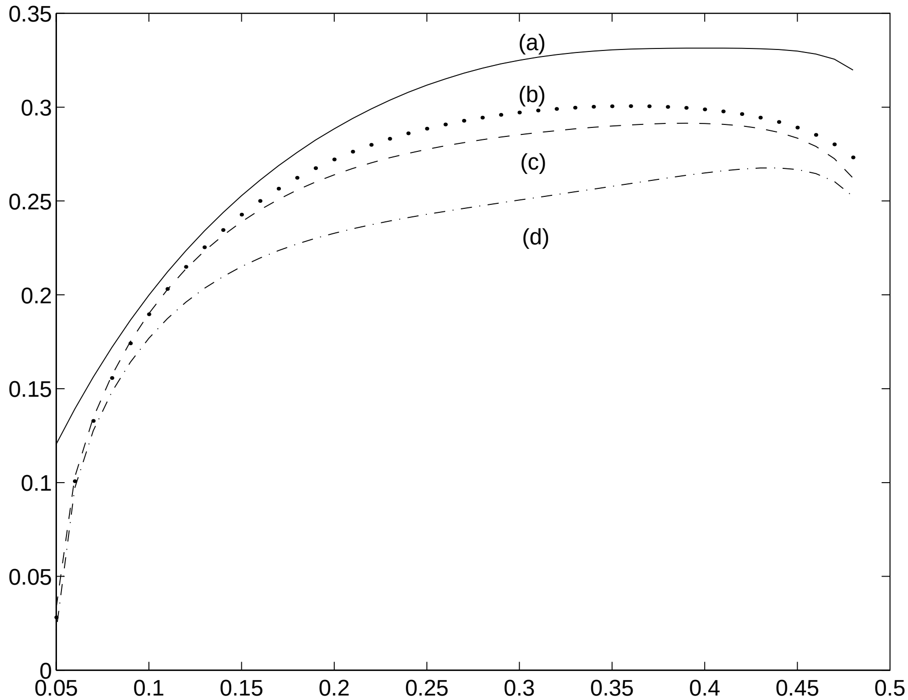

Figure 6: Fluctuations $\sqrt{m_4 - m_2^2}$ vs. Slowness p. (a) matched medium (b) shear velocity reduced by 1/2 in $z < 0$. (c) shear velocity reduced by 1/4 in $z < 0$. (d) shear velocity reduced by 1/8 in $z < 0$.

on physical grounds. We had noted that as the incident wave approaches normal incidence, mode coupling (and hence mode conversion) tends to zero. Therefore, the curves m_2 and m_4 should in all cases approach unity as slowness p tends to zero. Mode conversion should likewise tend to decrease as p approaches the cutoff value for the compressional wave in the effective medium; thus the curves m_2 and m_4 all increase as p approaches this limit. The curves show that increasing the degree of mismatch increases the moment values (thus decreasing the amount of mode conversion). In the matched medium case, the maximum amount of mode conversion occurs. In this case, no interface discontinuity exists at $z = 0$ between the deterministic and effective medium half-spaces. The incident wave can thus penetrate the random half-space without interface reflection, subsequently undergoing multiple scattering induced by the random fluctuations. The introduction of mismatch brings with it interface reflection. As the degree of mismatch in increased, the incident wave is increasingly unable to penetrate the random half-space and interact with the random inhomogeneities. The suppression of fluctuations with increasing mismatch (shown in Figure 6) is consistent with this interpretation.

6 The Oseledec Theorem and singular value decomposition

In sections 6-8 we will determine how a wave transmitted through a large slab decreases as the width of the slab increases to infinity; we will also study other quantities related to localization. For this purpose it will be convenient to keep the output end of the slab fixed. We therefore define

$$\bar{\zeta} = \zeta + \overline{z} \ , \tag{6.1}$$

so that the output end is fixed at $\bar{\zeta} = 0$. We also take $\bar{\zeta} = 0$ as the origin of coordinates for the random processes.

For each $\bar{\zeta}$ the matrix $\overline{Q}^{*T}\overline{Q}$ is Hermitian and positive definite. The Oseledec theorem [19], [20], [21] guarantees that under wide hypotheses there exists a random Hermitian positive semi-definite matrix E such that, with probability one

$$\lim_{\bar{\zeta}\to\infty}\left[\overline{Q}^{*T}(\bar{\zeta})\overline{Q}(\bar{\zeta})\right]^{\frac{1}{2\bar{\zeta}}} = E\ . \tag{6.2}$$

While the eigenspaces of E are, in general, random, the eigenvalues are not random. We denote the (real, non-negative) eigenvalues of E as

$$e^{\Omega_1} \geq e^{\Omega_2} \geq e^{\Omega_3} \geq e^{\Omega_4}\ . \tag{6.3}$$

Thus, Ω_j, called the Lyapunov exponents, give (deterministic) exponential growth and decay rates of $\overline{Q}^{*T}\overline{Q}$. We will calculate them explicitly, as $\epsilon \downarrow 0$, in the next section.

As explained in section 4, and illustrated by the calculations in section 5, it is Q rather that $\overline{Q}$ which is most useful for the detailed calculations. In this section, we will use the Oseledec theorem for $\overline{Q}$ to derive structural facts which can then be directly translated, via equation (4.37) into the corresponding facts for Q. To avoid writing these facts twice, we will drop bars for the remainder of this section, indicating at the end of the section how to interpret the equations so that they are indeed true for Q rather than $\overline{Q}$.

For $\overline{\zeta}$ fixed, let $\ell_j > 0$ be the eigenvalues, and $\mathbf{w}_j$ the eigenvectors of $Q^{*T}Q$

$$\begin{aligned} Q^{*T}Q\mathbf{w}_j &= \ell_j\mathbf{w}_j, \quad j = 1,\ldots,4\ . \\ |\mathbf{w}_j| &= 1 \end{aligned} \tag{6.4}$$

We assume that the ℓ_j are distinct

$$\ell_1 > \ell_2 > \ell_3 > \ell_4 > 0\ . \tag{6.5}$$

From (6.3)

$$\Omega_j = \lim_{\bar{\zeta}\to\infty}\frac{1}{2\bar{\zeta}}\ln\ell_j\ . \tag{6.6}$$

Now $\{\mathbf{w}_j\}$ forms a complete orthonormal basis of $\mathbb{C}^4$ as does $\{\widetilde{\mathbf{w}}_j\}$, where

$$\widetilde{\mathbf{w}}_j = \frac{Q\mathbf{w}_j}{|Q\mathbf{w}_j|}\ . \tag{6.7}$$

since multiplication of (6.7) by QQ^{*T} yields

$$\begin{aligned} QQ^{*T}\widetilde{\mathbf{w}}_j &= \ell_j\widetilde{\mathbf{w}}_j\ . \\ |\widetilde{\mathbf{w}}_j| &= 1 \end{aligned} \tag{6.8}$$

That is, the $\widetilde{\mathbf{w}}_j$ are the corresponding eigenvectors of QQ^{*T}. Since

$$|Q\mathbf{w}_j| = \left[\mathbf{w}_j^{*T}QQ^{*T}\mathbf{w}_j\right]^{1/2} = \sqrt{\ell_j}\ . \tag{6.9}$$

we obtain the following singular value decomposition of Q [22]

$$Q = \sum_{j=1}^{4}\sqrt{\ell_j}\,\widetilde{\mathbf{w}}_j\mathbf{w}_j^{*T}\ . \tag{6.10}$$

To verify (6.10), note that for all j, (6.7) implies $Q\mathbf{w}_j = |Q\mathbf{w}_j|\widetilde{\mathbf{w}}_j$, and that this relationship is duplicated by the matrix in (6.10).

Now using the form (4.39) for Q, one may verify the following fact. If $\mathbf{w}_j = [\mathbf{v}_j, \widehat{\mathbf{v}}_j]^T$ where $\mathbf{v}_j, \widehat{\mathbf{v}}_j$ are 2-vectors, then $[\widehat{\mathbf{v}}_j^*, \mathbf{v}_j^*]^T$ is also an eigenvector of $Q^{*T}Q$ corresponding to ℓ_j. Adding this to $\mathbf{w}_j$ then produces an eigenvector of the form $[\mathbf{v}_j + \widehat{\mathbf{v}}_j^*, (\mathbf{v}_j + \widehat{\mathbf{v}}_j^*)^*]^T$. Thus we may choose eigenvectors of the form

$$\begin{aligned} \mathbf{w}_j &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{v}_j^* \end{bmatrix} \\ |\mathbf{v}_j| &= 1 \; . \end{aligned} \tag{6.11}$$

Insertion of (6.11) into (6.7) yields that

$$\begin{aligned} \widetilde{\mathbf{w}}_j &= \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{\mathbf{v}}_j \\ \widetilde{\mathbf{v}}_j^* \end{bmatrix} \\ |\widetilde{\mathbf{v}}_j| &= 1 \end{aligned} \tag{6.12}$$

where

$$\widetilde{\mathbf{v}}_j = (A\mathbf{v}_j + B^*\mathbf{v}_j^*)/\sqrt{\ell_j} \; . \tag{6.13}$$

We next multiply (6.4) by $i\mathcal{O}(Q^{*T}Q)^{-1}$ (c.f. (4.12)) and use (4.41) to obtain that

$$Q^{*T}Q(i\mathcal{O}\mathbf{w}_j) = \frac{1}{\ell_j}(i\mathcal{O}\mathbf{w}_j) \; . \tag{6.14}$$

Therefore

$$i\mathcal{O}\mathbf{w}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} i\mathbf{v}_j \\ (i\mathbf{v}_j)^* \end{bmatrix} \tag{6.15}$$

is an eigenvector of $Q^{*T}Q$ corresponding to $\frac{1}{\ell_j}$. In particular, (6.14) shows that the eigenvalues occur in reciprocal pairs

$$\begin{aligned} \ell_3 &= \frac{1}{\ell_2} \\ \ell_4 &= \frac{1}{\ell_1} \; . \end{aligned} \tag{6.16}$$

Furthermore, we may choose the eigenvectors so that

$$\begin{aligned} \mathbf{v}_3 &= i\mathbf{v}_2 \\ \mathbf{v}_4 &= i\mathbf{v}_1 \; . \end{aligned} \tag{6.17}$$

Similarly, multiplication of (6.8) by $i\mathcal{O}(QQ^{*T})^{-1}$ and use of (4.41) yields that

$$QQ^{*T}(i\mathcal{O}\widetilde{\mathbf{w}}_j) = \frac{1}{\ell_j}(i\mathcal{O}\widetilde{\mathbf{w}}_j) \; . \tag{6.18}$$

We are not, however, free to choose $\widetilde{\mathbf{v}}_3, \widetilde{\mathbf{v}}_4$ arbitrarily as in (6.17), since they are constrained by (6.13). However, since the eigenspaces are one dimensional, we have

$$i\mathcal{O}\widetilde{\mathbf{w}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i\widetilde{\mathbf{v}}_2 \\ (i\widetilde{\mathbf{v}}_2)^* \end{bmatrix} = \frac{e^{i\kappa_2}}{\sqrt{2}} \begin{bmatrix} \widetilde{\mathbf{v}}_3 \\ \widetilde{\mathbf{v}}_3^* \end{bmatrix} \; , \tag{6.19}$$

where κ_2 is real. Then equating the two components implies that $\kappa_2 = 0$. Repeating the argument for $\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_4$ then yields that

$$\begin{aligned} \tilde{\mathbf{v}}_3 &= i\tilde{\mathbf{v}}_2 \\ \tilde{\mathbf{v}}_4 &= i\tilde{\mathbf{v}}_1 \ . \end{aligned} \tag{6.20}$$

Now insertion of (6.16), (6.17), (6.20), (6.11) and (6.12) into (6.10) yields a representation for Q of the form (4.10) with

$$A = \frac{1}{2}\left(\sqrt{\ell_1} + \frac{1}{\sqrt{\ell_1}}\right)\tilde{\mathbf{v}}_1\mathbf{v}_1^{*T} + \frac{1}{2}\left(\sqrt{\ell_2} + \frac{1}{\sqrt{\ell_2}}\right)\tilde{\mathbf{v}}_2\mathbf{v}_2^{*T} \tag{6.21}$$

$$B = \frac{1}{2}\left(\sqrt{\ell_1} - \frac{1}{\sqrt{\ell_1}}\right)\tilde{\mathbf{v}}_1^{*}\mathbf{v}_1^{*T} + \frac{1}{2}\left(\sqrt{\ell_2} - \frac{1}{\sqrt{\ell_2}}\right)\tilde{\mathbf{v}}_2^{*}\mathbf{v}_2^{*T} \ . \tag{6.22}$$

Next we show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for $\mathbb{C}^2$, as is $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2\}$. To check this, we use the fact that $\mathbf{w}_1$ is orthogonal to both $\mathbf{w}_2$ and $\mathbf{w}_3$, yielding a system of two homogeneous linear equations for $\mathbf{v}_1^{*T}\mathbf{v}_2$ and $(\mathbf{v}_1^{*T}\mathbf{v}_2)^*$, which implies that these quantities are zero. A similar argument shows that $\tilde{\mathbf{v}}_1$ is orthogonal to $\tilde{\mathbf{v}}_2$.

Using the orthonormality of the basis vectors, we can derive the following two expressions.

$$A^{-1} = \frac{2}{\left(\sqrt{\ell_1} + \frac{1}{(\sqrt{\ell_1})}\right)}\mathbf{v}_1\tilde{\mathbf{v}}_1^{*T} + \frac{2}{\left(\sqrt{\ell_2} + \frac{1}{\sqrt{\ell_2}}\right)}\mathbf{v}_2\tilde{\mathbf{v}}_2^{*T} \tag{6.23}$$

$$A^{*T}A = \frac{1}{4}\left(\ell_1 + \frac{1}{\ell_1} + 2\right)\mathbf{v}_1\mathbf{v}_1^{*T} + \frac{1}{4}\left(\ell_2 + \frac{1}{\ell_2} + 2\right)\mathbf{v}_2\mathbf{v}_2^{*T} \ . \tag{6.24}$$

Furthermore from (6.24) we obtain

$$|\det A|^2 = \det(A^{*T}A) = \frac{1}{16}(\ell_1 + \frac{1}{\ell_1} + 2)(\ell_2 + \frac{1}{\ell_2} + 2) \ . \tag{6.25}$$

We may now reintroduce Q, rather than $\overline{Q}$. From (4.37), the singular value decomposition (6.10) holds for Q, as written, provided we replace

$$\begin{aligned} \mathbf{w}_j &\Rightarrow e^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}\mathbf{w}_j \\ \tilde{\mathbf{w}}_j &\Rightarrow e^{-i\frac{\omega}{\epsilon}\Lambda_1\zeta}\tilde{\mathbf{w}}_j \end{aligned}$$

From (6.11), (6.12), and (3.6) this amounts to the substitution

$$\begin{aligned} \mathbf{v}_j &\Rightarrow e^{i\frac{\omega}{\epsilon}\Lambda\overline{z}}\mathbf{v}_j \\ \tilde{\mathbf{v}}_j &\Rightarrow e^{-i\frac{\omega}{\epsilon}\Lambda\zeta}\tilde{\mathbf{v}}_j \ . \end{aligned}$$

Thus, by redefining $\mathbf{w}_j, \tilde{\mathbf{w}}_j, \mathbf{v}_j, \tilde{\mathbf{v}}_j$ all the formulas (6.4) – (6.25) hold for Q. However, the convergent vectors become $e^{-i\frac{\omega}{\epsilon}\Lambda\overline{z}}\mathbf{v}_j$, $e^{-i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}\mathbf{w}_j$, converging as $\overline{\zeta} = \overline{z} \to \infty$.

The result of this section can also be adapted to apply to $Q'(z)$, since by (4.44)

$$Q'^{*T}Q' = \overline{Q}'^{*T}\overline{Q}' \ . \tag{6.26}$$

and the Oseledec theorem applies to $\overline{Q}'$. Thus formulas (6.4) – (6.25) apply if all variables are primed, and $\bar{\zeta}$ is replaced by z. From (4.49) we can relate the probability distributions of the primed and unprimed variables

$$
\begin{aligned}
\ell'_j &\stackrel{\mathcal{D}}{=} \ell_j \\
\widetilde{\mathbf{w}}'_j &\stackrel{\mathcal{D}}{=} e^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}\widetilde{\mathbf{w}}^*_j \\
\mathbf{w}_j &\stackrel{\mathcal{D}}{=} e^{i\frac{\omega}{\epsilon}\Lambda_1\overline{z}}\mathbf{w}^*_j \\
\widetilde{\mathbf{v}}'_j &\stackrel{\mathcal{D}}{=} e^{i\frac{\omega}{\epsilon}\Lambda\overline{z}}\widetilde{\mathbf{v}}^*_j \\
\mathbf{v}'_j &\stackrel{\mathcal{D}}{=} e^{i\frac{\omega}{\epsilon}\Lambda\overline{z}}\mathbf{v}^*_j \ .
\end{aligned} \tag{6.27}
$$

In (6.27) the primed variables are evaluated at $z = \overline{z}$, the unprimed at $\bar{\zeta} = \overline{z}$, ie., $\zeta = 0$.

7 Calculation of the Lyapunov exponents

In this section we derive explicit expressions for the two nonnegative Lyapunov exponents $\Omega_1 > \Omega_2$, under the assumption that they are positive. From (6.6), (6.16) the other two exponents are

$$
\Omega_3 = -\Omega_2, \quad \Omega_4 = -\Omega_1 \ . \tag{7.1}
$$

We will first determine Ω_1. Let $\mathbf{w}_0 \,\epsilon\, \mathbb{C}^4$ be a nonrandom vector. From (6.10), (6.6)

$$
\lim_{\bar{\zeta}=\overline{z}\to\infty} \frac{1}{\bar{\zeta}} \ln|Q\mathbf{w}_0| = \lim_{\bar{\zeta}=\overline{z}\to\infty} \frac{1}{2\bar{\zeta}} \ln \sum_{j=1}^{4} |\mathbf{w}_j^{*T}\mathbf{w}_0|^2 \ell_j = \Omega_1 \tag{7.2}
$$

since $\ell_j/\ell_1 \to 0$ as $\overline{\zeta} \to \infty$ for $j \neq 1$. That is, Ω_1, since it is the largest growth rate, can be determined by calculating the growth rate of $|Q\mathbf{w}_0|$ for arbitrary $\mathbf{w}_0$, provided $\mathbf{w}_0$ is not asymptotically orthogonal to $\mathbf{w}_1$. The probability of orthogonality is zero if the convergent vector $e^{-i\frac{\omega}{\epsilon}\Lambda\overline{z}}\mathbf{v}_1(\overline{\zeta} = \overline{z})$ has asymptotically a continuous probability density in $\mathbb{C}^2$.

The vector $\mathbf{w} = Q\mathbf{w}_0$ is a vector solution of (4.7) with initial condition $\mathbf{w}(0) = \mathbf{w}_0$. Taking $\mathbf{w}_0$ of the form $\mathbf{w}_0 = [\mathbf{v}_0, \mathbf{v}_0^*]^T$, yields a solution also of this form, i.e., $\mathbf{w} = [\mathbf{v}, \mathbf{v}^*]^T$, where $\mathbf{v} = A\mathbf{v}_0 + B^*\mathbf{v}_0$. Substitution of this form into (4.7) then yields

$$
\begin{aligned}
\frac{d\mathbf{v}}{d\bar{\zeta}} &= \frac{i\omega}{\epsilon}\left[\eta_1\mathbf{v} - \eta_2^*\mathbf{v}^*\right] \\
\mathbf{v}|_{\bar{\zeta}=0} &= \mathbf{v}_0
\end{aligned} \tag{7.3}
$$

From (7.2)

$$
\Omega_1 = \lim_{\bar{\zeta}=\overline{z}\to\infty} \frac{1}{2\bar{\zeta}} \ln|\mathbf{v}(\bar{\zeta})|^2 \ . \tag{7.4}
$$

Let

$$
\mathbf{v} = \begin{bmatrix} r_1 e^{i\phi_1} \\ r_2 e^{i\phi_2} \end{bmatrix} \tag{7.5}
$$

where r_1, r_2, ϕ_1, ϕ_2 are real, and let

$$
\begin{aligned}
r &= \frac{1}{2}\ln(r_1^2 + r_2^2) \\
R &= \ln\left(\frac{r_1}{r_2}\right)
\end{aligned} \tag{7.6}
$$

so that

$$\Omega_1 = \lim_{\bar\zeta=\bar z\to\infty} \frac{r(\overline{\zeta})}{\overline{\zeta}} \ . \tag{7.7}$$

Substitution of (7.5), (7.6) into (7.3) and use of (4.5), (3.31) yields

$$\begin{aligned}
\frac{\partial r}{\partial \zeta} &= \frac{\omega}{\epsilon}(1+e^{2R})^{-1}\Big\{ -\delta_4 e^{2R}\sin(2\omega\alpha_p\frac{\zeta}{\epsilon}+2\phi_1) \\
&\quad -2\delta_5 e^R \sin\left(\omega\left[\alpha_p+\alpha_s\right]\frac{\zeta}{\epsilon}+\phi_1+\phi_2\right) - \delta_6\sin\left(2\omega\alpha_s\frac{\zeta}{\epsilon}+2\phi_2\right)\Big\} \\
\frac{\partial R}{\partial \zeta} &= \frac{\omega}{\epsilon}\Big\{2\delta_2\cosh R\sin\left(\omega\left[\alpha_p-\alpha_s\right]\frac{\zeta}{\epsilon}+\phi_1-\phi_2\right) \\
&\quad -\delta_4\sin(2\omega\alpha_p\frac{\zeta}{\epsilon}+2\phi_1) \\
&\quad +2\delta_5\sinh R\sin(\omega[\alpha_p+\alpha_s]\frac{\zeta}{\epsilon}+\phi_1+\phi_2) \\
&\quad +\delta_6\sin(2\omega\alpha_s\frac{\zeta}{\epsilon}+2\phi_2)\Big\} \\
\frac{\partial \phi_1}{\partial \zeta} &= \frac{\omega}{\epsilon}\Big\{\delta_1+\delta_2 e^{-R}\cos(\omega[\alpha_p-\alpha_s]\frac{\zeta}{\epsilon}+\phi_1-\phi_2) \\
&\quad -\delta_4\cos(2\omega\alpha_p\frac{\zeta}{\epsilon}+2\phi_1)-\delta_5 e^{-R}\cos(\omega[\alpha_p+\alpha_s]\frac{\zeta}{\epsilon}+\phi_+\phi_2)\Big\} \\
\frac{\partial \phi_2}{\partial \zeta} &= \frac{\omega}{\epsilon}\Big\{\delta_2 e^R\cos(\omega[\alpha_p-\alpha_s]\frac{\zeta}{\epsilon}+\phi_1-\phi_2)+\delta_3 \\
&\quad -\delta_5 e^R\cos(\omega[\alpha_p+\alpha_s]\frac{\zeta}{\epsilon}+\phi_1+\phi_2)-\delta_6\cos(2\omega\alpha_s\frac{\zeta}{\epsilon}+2\phi_2)\Big\} \ .
\end{aligned} \tag{7.8}$$

In (7.8) we have reverted to ζ, not $\overline{\zeta}$, coordinates. We will return to $\bar\zeta = \zeta + \overline{z}$ after taking the limit $\epsilon \downarrow 0$.

Equation (7.8) is a stochastic equation of the form (5.4), with a small parameter ϵ, for the vector process $\boldsymbol{\psi} = [r, R, \phi_1, \phi_2]^T$. As discussed in section 5, $\boldsymbol{\psi}$ is well approximated (in distribution) as $\epsilon \downarrow 0$ by a limit which is a Markov diffusion process. The limit process may be characterized by its infinitesimal generator, a second order elliptic differential operator, call it $\widetilde{\mathcal{L}}$, which is given by equation (5.6). In the abstract, this is the same procedure that produced equation (5.8) from (5.4).

We will not write out in full the operator $\widetilde{\mathcal{L}}$, which has fourteen terms, corresponding to the ten coefficients of $\partial^2/\partial\psi_j\partial\psi_k, j,k = 1,\ldots,4$, and the four coefficients of $\partial/\partial\psi_j, j = 1,\ldots,4$. It suffices that none of these coefficients depends on ϕ_1 or ϕ_2 so that the real two-vector (r, R) is a Markov diffusion process, without the need to include ϕ_1, ϕ_2. This is analogous to the way that ϕ and ψ were eliminated from the calculation of reflection statistics in section 5, since they did not appear in the coefficients of $\overline{\mathcal{L}}$, equation (5.8). Dropping all derivatives with respect to ϕ_1, ϕ_2 in $\widetilde{\mathcal{L}}$ then produces the infinitesimal generator, $\mathcal{L}_{r,R}$, of (r, R)

$$\begin{aligned}
\mathcal{L}_{r,R} &= \frac{\omega^2}{2}\Big\{\Big[\left(\sigma_{55}-\frac{1}{4}\left[\sigma_{44}+\sigma_{66}\right]\right)\operatorname{sech}^2 R+\frac{1}{2}\left(\sigma_{44}-\sigma_{66}\right)\tanh R \\
&+ \frac{1}{2}\left(\sigma_{44}+\sigma_{66}\right)\Big]\frac{\partial^2}{\partial r^2}+\left[-4\left(\sigma_{55}-\frac{1}{4}\left[\sigma_{44}-\sigma_{66}\right]\right)\tanh R+\left(\sigma_{44}-\sigma_{66}\right)\right]\frac{\partial^2}{\partial r\partial R} \\
&+ \left[2\left(\sigma_{22}+\sigma_{55}\right)\cosh(2R)+\left(2\sigma_{22}-2\sigma_{55}+\sigma_{44}+\sigma_{66}\right)\right]\frac{\partial^2}{\partial R^2} \\
&+ \left[-2\left(\sigma_{55}-\frac{1}{4}\left[\sigma_{44}+\sigma_{66}\right]\right)\operatorname{sech}^2 R+\left(\sigma_{44}-\sigma_{66}\right)\tanh R+4\left(\sigma_{55}-\frac{1}{4}\left[\sigma_{44}+\sigma_{66}\right]\right)\right]\frac{\partial}{\partial r}
\end{aligned}$$

$$
+ \quad (2\sigma_{44} - 2\sigma_{66}) \frac{\partial}{\partial R} \Big\} \; . \tag{7.9}
$$

In writing (7.9) we have assumed the no-resonance condition (5.9).

Inspection of equation (7.9) shows that none of the coefficients depends on r, either, so that R is Markovian by itself. The probability density $P_R\,(\bar{\zeta}, R)$ of $R(\bar{\zeta})$ satisfies the forward Kolmogorov, or Fokker-Planck, equation

$$
\begin{aligned}
\frac{\partial}{\partial \bar{\zeta}} P_R(\bar{\zeta}, R) \;&=\; \omega^2(\sigma_{22} + \sigma_{55}) \left\{ - \frac{\partial}{\partial R} [\gamma P_R] \right. \\
&+ \; \left. \frac{\partial^2}{\partial R^2} [(\cosh 2R + \Sigma) P_R] \right\}
\end{aligned} \tag{7.10}
$$

where Σ is given by (5.13), and

$$
\gamma = \frac{(\sigma_{44} - \sigma_{66})}{(\sigma_{22} + \sigma_{55})} \; . \tag{7.11}
$$

The operator on the right in (7.10) is the adjoint of $\mathcal{L}_{r,R}$ restricted to functions of R. Note from Table I and (5.7) that the parameter γ is a ratio of differential same-mode backscatter to total cross-mode scattering.

The steady-state, $\bar{\zeta}$-independent solution of (7.10) is the invariant measure, $\overline{P}_R(R)$ for R. Setting the $\bar{\zeta}$-derivative to zero then gives the following expressions

$$
\overline{P}_R(R) = \frac{\gamma\left[\Sigma+\sqrt{\Sigma^2-1}\right]^{\frac{\gamma}{\sqrt{\Sigma^2-1}}}}{\left(\left[\Sigma+\sqrt{\Sigma^2-1}\right]^{\frac{\gamma}{\sqrt{\Sigma^2-1}}}-1\right)} \cdot \left[\frac{e^{2R}+\Sigma-\sqrt{\Sigma^2-1}}{e^{2R}+\Sigma+\sqrt{\Sigma^2-1}}\right]^{\frac{\gamma}{2\sqrt{\Sigma^2-1}}} \frac{1}{(\cosh(2R)+\Sigma)}, \quad \Sigma^2 > 1\,, \gamma \neq 0 \tag{7.12a}
$$

$$
\overline{P}_R(R) = \frac{\sqrt{\Sigma^2-1}}{\ln(\Sigma+\sqrt{\Sigma^2-1})} \frac{1}{(\cosh(2R)+\Sigma)}\;, \quad \Sigma^2 > 1, \gamma = 0 \tag{7.12b}
$$

$$
\overline{P}_R(R) = \frac{\gamma}{(e^{\gamma}-1)} \cdot e^{\left[\frac{\gamma e^{2R}}{e^{2R}+1}\right]} \cdot \frac{1}{(\cosh(2R)+1)}\;, \quad \Sigma = 1, \gamma \neq 0 \tag{7.12c}
$$

$$
\overline{P}_R(R) = \frac{1}{(\cosh(2R)+1)}\;, \quad \Sigma = 1, \gamma = 0 \tag{7.12d}
$$

$$
\begin{aligned}
\overline{P}_R(R) = &\frac{\gamma}{\left[e^{\left\{\frac{2\gamma}{\sqrt{1-\Sigma^2}}\left(\frac{\pi}{2}-\arctan\left(\sqrt{\frac{1+\Sigma}{1-\Sigma}}\right)\right)\right\}}-1\right]} \cdot \frac{1}{(\cosh(2R)+\Sigma)} \\
&\cdot\, e^{\left\{\frac{\gamma}{\sqrt{1-\Sigma^2}}\left[\arctan\left(\frac{e^{2R}+\Sigma}{\sqrt{1-\Sigma^2}}\right)-\arctan\left(\frac{\Sigma}{\sqrt{1-\Sigma^2}}\right)\right]\right\}}, \quad \Sigma^2 < 1, \gamma \neq 0
\end{aligned} \tag{7.12e}
$$

$$
\overline{P}_R(R) = \frac{\sqrt{1-\Sigma^2}}{2\left[\frac{\pi}{2}-\arctan\left(\sqrt{\frac{1+\Sigma}{1-\Sigma}}\right)\right]} \frac{1}{(\cosh(2R)+\Sigma)}\;, \quad \Sigma^2 < 1, \gamma = 0\;. \tag{7.12f}
$$

To determine Ω_1, we will use the interpretation of any Markov diffusion process in terms of stochastic differential equation of Itô type (a nonlinear "white noise" equation) [23]. In general, let $\boldsymbol{\psi} \in \mathbb{R}^d$ be a Markov diffusion process with infinitesimal generator

$$
\mathcal{L}_{\boldsymbol{\psi}} = \frac{1}{2} a_{j,k}(\boldsymbol{\psi}) \frac{\partial^2}{\partial \psi_j \partial \psi_k} + b_j(\boldsymbol{\psi}) \frac{\partial}{\partial \psi_j}, \tag{7.13}
$$

where for each $\boldsymbol{\psi}$, $(a_{j,k})$ is a symmetric, positive definite matrix. Let $\boldsymbol{\beta}(\bar{\zeta}) = (\beta,(\bar{\zeta}),\ldots,\beta_d(\bar{\zeta}))^T$ be a vector of independent Brownian motions. Let $\hat{a}_{j,k}$ satisfy

$$\hat{a}_{j,\ell}\hat{a}_{k,\ell} = a_{j,k} \tag{7.14}$$

Then $\boldsymbol{\psi}$ has the same distribution as the solution of the Itô equation

$$d\psi_j = b_j(\boldsymbol{\psi})d\bar{\zeta} + \hat{a}_{j,k}(\boldsymbol{\psi})d\beta_k(\bar{\zeta}) \ . \tag{7.15}$$

In particular, for $d=2$, we may take $\hat{a}_{1,2}=0, \hat{a}_{1,1}=\sqrt{a_{1,1}}$, $\hat{a}_{2,2} = \sqrt{a_{2,2} - a_{2,1}^2/a_{1,1}}$, producing, for our system (7.9)

$$\begin{aligned} dr &= b_1(R)d\bar{\zeta} + \hat{a}_{1,1}(R)d\beta_1 \\ dR &= b_2(R)d\bar{\zeta} + \hat{a}_{2,1}(R)d\beta_1 + \hat{a}_{2,2}(R)d\beta_2 \end{aligned} \tag{7.16}$$

where, in particular

$$\begin{aligned} b_1(R) &= \frac{\omega^2}{2}\left[-2\left(\sigma_{55} - \frac{1}{4}\left[\sigma_{44}+\sigma_{66}\right]\right)\operatorname{sech}^2 R + (\sigma_{44}-\sigma_{66})\tanh R \right. \\ &\quad \left. +4\left(\sigma_{55} - \frac{1}{4}\left[\sigma_{44}+\sigma_{66}\right]\right)\right] \\ b_2(R) &= \omega^2\left(\sigma_{44}-\sigma_{66}\right) \ . \end{aligned} \tag{7.17}$$

The coefficients $\hat{a}_{j,\ell}(R)$ can also be explicitly determined; and they are bounded functions of R only.

From (7.16), (7.7)

$$\begin{aligned} \Omega_1 &= \lim_{\zeta_0\to\infty}\frac{1}{\zeta_0}\int_0^{\zeta_0} \hat{b}_{1,1}(R(\bar{\zeta}))d\bar{\zeta} \\ &\quad + \lim_{\zeta_0\to\infty}\frac{1}{\zeta_0}\int_0^{\zeta_0} \hat{a}_{1,1}(R(\bar{\zeta}))d\beta(\bar{\zeta}) \ . \end{aligned} \tag{7.18}$$

The second limit on the right in (7.18) vanishes with probability one. To see that it vanishes in probability, we may check the limiting variance

$$\begin{aligned} &\lim_{\zeta_0\to\infty}\frac{1}{\zeta_0^2}\int_0^{\zeta_0} E\left[(\hat{a}_{1,1}(R(\bar{\zeta}))^2\right]d\bar{\zeta} \\ &= \lim_{\zeta_0\to\infty}\frac{1}{\zeta_0^2}\int_0^{\zeta_0} E\left[a_{1,1}\left(R(\bar{\zeta})\right)\right]d\bar{\zeta} = 0 \end{aligned} \tag{7.19}$$

where we have used the Itô calculus, the fact that $d\beta(\bar{\zeta}_1)d\beta(\bar{\zeta}_2) = \delta(\bar{\zeta}_1-\bar{\zeta}_2)d\bar{\zeta}_1$, i.e., the "white noise" property, and the boundedness of $a_{1,1}$.

The first term on the right side in (7.18) represents a spatial average of $b_1(R(\bar{\zeta}))$. Using the ergodic theorem, this may be replaced by a probability average of $b_1(R)$, so that

$$\Omega_1 = E_\infty\left[b_1(R)\right] \tag{7.20}$$

where E_∞ is expectation with respect to the invariant measure of the process $R(\bar{\zeta})$. Thus, the limit depends only on the steady-state distribution of R, and b_1, which is the coefficient of $\frac{\partial}{\partial r}$ in the generator, (7.9). From (7.17)

$$\begin{aligned}\Omega_1 &= \frac{\omega^2}{2}\left[-2\left(\sigma_{55}-\frac{1}{4}\left[\sigma_{44}+\sigma_{66}\right]\right)\int_{-\infty}^{\infty}\operatorname{sech}^2(R)\overline{P}_R(R)dR\right.\\ &\quad +(\sigma_{44}-\sigma_{66})\int_{-\infty}^{\infty}\tanh(R)\overline{P}_R(R)dR\\ &\quad \left.+4\left(\sigma_{55}+\frac{1}{4}[\sigma_{44}+\sigma_{66}]\right)\right]\ . \qquad (7.21)\end{aligned}$$

$\overline{P}_R(R)$ is given in (7.12).

We next determine Ω_2. From (6.25)

$$\begin{aligned}\lim_{\bar{\zeta}=\overline{z}\to\infty}\frac{1}{\bar{\zeta}}2\mathrm{Re}\left[\ln\det A\right] &= \lim_{\bar{\zeta}=\overline{z}\to\infty}\frac{1}{\bar{\zeta}}\ln\left(\ell_1(\bar{\zeta})+\frac{1}{\ell_1(\bar{\zeta})}+2\right)+\\ &\quad \lim_{\bar{\zeta}=\overline{z}\to\infty}\frac{1}{\bar{\zeta}}\ln\left(\ell_2(\bar{\zeta})+\frac{1}{\ell_2(\bar{\zeta})}+2\right)\ . \qquad (7.22)\end{aligned}$$

Inserting (6.6) into (7.22) then gives

$$\Omega_1+\Omega_2=\lim_{\bar{\zeta}=\overline{z}\to\infty}\frac{1}{\bar{\zeta}}\mathrm{Re}\left[\ln\det A\right]\ . \qquad (7.23)$$

Let

$$,_M(\bar{\zeta})=B^*(\bar{\zeta})A^{*-1}(\bar{\zeta})\ . \qquad (7.24)$$

From (4.26), $,_M(\bar{\zeta})$ may be interpreted as the reflection matrix, when the medium is matched at both transmission and incidence ends of the slab. $,_M(\bar{\zeta})$ therefore satisfies the Riccati equation (4.36) with $,_T=0$. Using $,_M$, we may write an equation for A, from (4.11)

$$\frac{dA}{d\bar{\zeta}}=\frac{i\omega}{\epsilon}\left[\eta_1-\eta_2^*,^*_M\right]A\ , \qquad (7.25)$$

so that

$$\frac{d}{d\bar{\zeta}}\ln\det A=\frac{i\omega}{\epsilon}\mathrm{tr}\left\{\eta_1-\eta_2^*,^*_M\right\}\ . \qquad (7.26)$$

Substitution of (7.26) into (7.23) gives

$$\Omega_1+\Omega_2=\lim_{\zeta_0\to\infty}\frac{1}{\zeta_0}\left[-\frac{\omega}{\epsilon}\int_0^{\zeta_0}\mathrm{trIm}(\eta_1-\eta_2^*,^*_M)d\bar{\zeta}\right]\ . \qquad (7.27)$$

Note trIm $(\eta_1)=0$, since η_1 is Hermitian, (4.6).

Let $H(\bar{\zeta})$ be defined by

$$\frac{dH}{d\bar{\zeta}}=-(\frac{\omega}{\epsilon})\mathrm{trIm}\eta_2(\bar{\zeta}),_M(\bar{\zeta})\ ,\qquad H(0)=0\ . \qquad (7.28)$$

Then

$$\Omega_1+\Omega_2=\lim_{\bar{\zeta}=\overline{z}\to\infty}\frac{1}{\bar{\zeta}}H(\bar{\zeta})\ . \qquad (7.29)$$

Now as $\bar{\zeta} \to \infty$, , $_M(\bar{\zeta})$ becomes symmetric and unitary, so it may be represented in polar form (5.2) in terms of θ, ϕ, ψ. Then equation (7.28) becomes

$$\begin{aligned}\frac{dH}{d\zeta} &= -\frac{\omega}{\epsilon}\left\{\delta_4 \tanh\left(\frac{\theta}{2}\right)\sin(2\omega\alpha_p\frac{\zeta}{\epsilon}+\phi+\psi)\right. \\ &\quad +2\delta_5 \mathrm{sech}\left(\frac{\theta}{2}\right)\cos(\omega[\alpha_p+\alpha_s]\frac{\zeta}{\epsilon}+\phi) \\ &\quad \left.+\delta_6\tanh\left(\frac{\theta}{2}\right)\sin(2\omega\alpha_s\frac{\zeta}{\epsilon}+\phi-\psi)\right\} \; . \end{aligned} \tag{7.30}$$

Again, we have reverted to ζ coordinates to take the $\epsilon \downarrow 0$ limit.

Equation (7.30) may be appended to the three equations in (5.3) to give an equation of the form (5.4) for the four-dimensional vector $\boldsymbol{\psi} = [r, \theta, \phi, H]^T$. Again, we may apply the limit theorem, as $\epsilon \downarrow 0$, and use equation (5.6) to compute the infinitesimal generator, $\widehat{\mathcal{L}}$ of the limit process. Then assuming (5.9)

$$\widehat{\mathcal{L}} = \overline{\mathcal{L}} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22} \tag{7.31}$$

where $\overline{\mathcal{L}}$ is given by (5.8) and

$$\begin{aligned}\mathcal{L}_{12} &= \omega^2\left\{\sigma_{44}+2\sigma_{55}+\sigma_{66}\right\}\frac{\partial}{\partial H} \\ \mathcal{L}_{21} &= \omega^2\left\{\sigma_{44}+\sigma_{66}-4\sigma_{55}\right\}\tanh\frac{\theta}{2}\frac{\partial^2}{\partial\theta\partial H} \\ \mathcal{L}_{22} &= \frac{\omega^2}{2}\left\{(\sigma_{44}+\sigma_{66})\tanh^2\frac{\theta}{2}+4\sigma_{55}\mathrm{sech}^4\frac{\theta}{2}\right\}\frac{\partial^2}{\partial H^2} \; . \end{aligned} \tag{7.32}$$

From (7.29), (7.31), (7.32) we may argue from the Itô interpretation of these calculations, as in the discussion of the calculation of Ω_1. Only the coefficient of $\partial/\partial H$ in (7.31) contributes to the growth of H, and this coefficient must be averaged with respect to the invariant density of (θ, ϕ, ψ). However, in this case, the coefficient is constant, and so

$$\Omega_1 + \Omega_2 = \omega^2\left\{2\sigma_{55}+\sigma_{44}+\sigma_{66}\right\} \; . \tag{7.33}$$

Thus, Ω_1, Ω_2 are determined by (7.21), (7.33).

Finally, recall the dual orthonormal bases of $\mathbb{C}^2$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2\}$, from section 6. While each $e^{-i\frac{\omega}{\epsilon}\Lambda\bar{z}}\mathbf{v}_j$ converges as $\bar{\zeta} = \bar{z} \to \infty$, the probability distribution of the limit has not been determined. On the other hand, even though $\widetilde{\mathbf{v}}_1$ does not converge at the level of realizations, it does have a convergent probability distribution, which can be determined as a byproduct of our calculation of Ω_1. Then $\widetilde{\mathbf{v}}_2$ can be determined by orthogonality to $\widetilde{\mathbf{v}}_1$.

From (6.10), as $\bar{\zeta} = \bar{z} \to \infty$

$$\mathbf{w} = Q\mathbf{w}_0 \sim \sqrt{\ell_1}\widetilde{\mathbf{w}}_1(\mathbf{w}_1^{*T}\mathbf{w}_0) \tag{7.34}$$

and so for $\mathbf{w}_0 = [\mathbf{v}_0, \mathbf{v}_0^*]^T$

$$\mathbf{v} \sim \left(\sqrt{\ell_1}\mathbf{w}_1^{*T}\mathbf{w}_0\right)\widetilde{\mathbf{v}}_1 \tag{7.35}$$

where $\mathbf{v}$ satisfies (7.3). Since $\widetilde{\mathbf{v}}_1$ is a unit vector, we have from (7.5), (7.6)

$$\widetilde{\mathbf{v}}_1 \sim \begin{bmatrix} \frac{e^R}{\sqrt{1+e^{2R}}} & e^{i\phi_1} \\ \frac{1}{\sqrt{1+e^{2R}}} & e^{i\phi_2} \end{bmatrix} \tag{7.36}$$

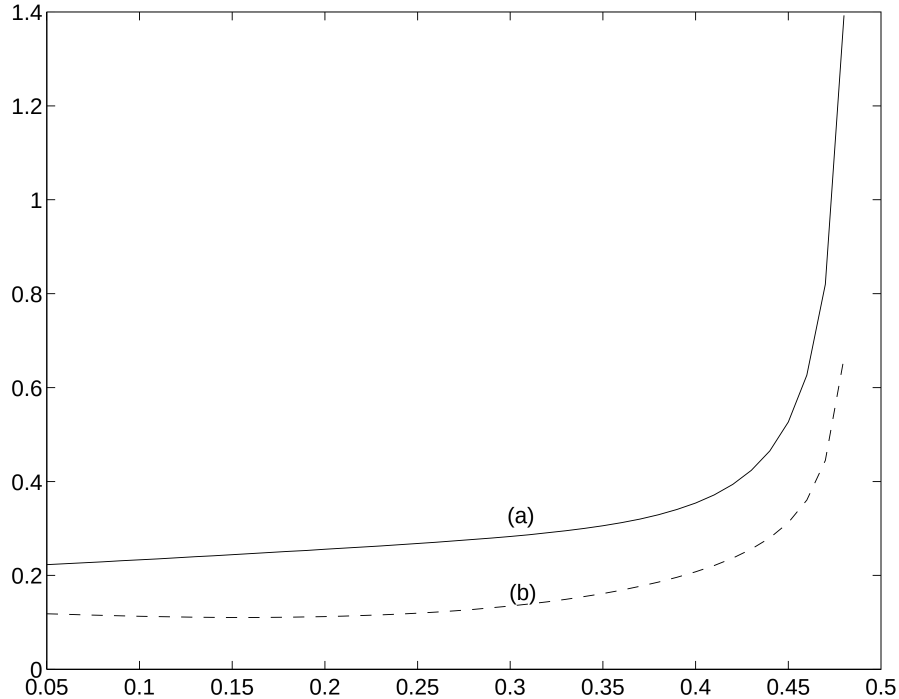

Figure 7: Lyapunov exponents (a) Ω_1 and (b) Ω_2 vs. Slowness p

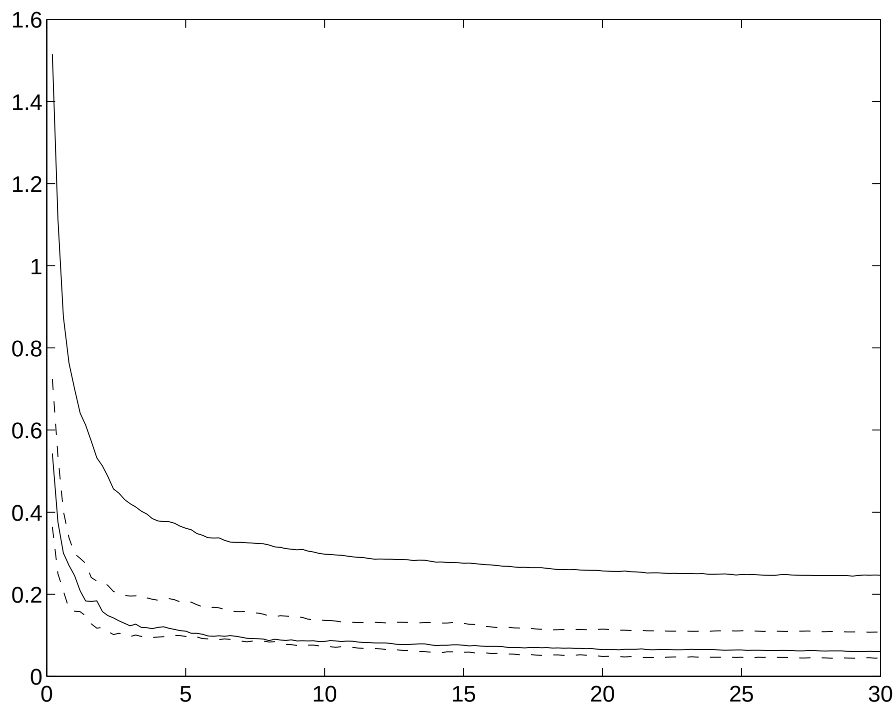

Figure 8: Lyapunov exponent simulations ($p = 0.10$). Mean and fluctuations vs. slab thickness $\overline{\zeta}$ The upper and lower solid curves show the mean and fluctuations of $\frac{1}{2\overline{\zeta}} \ln \overline{\ell}_1$, respectively. The dashed curves present similar information for $\frac{1}{2\overline{\zeta}} \ln \overline{\ell}_2$.

where R has probability density $\overline{P}_R(R)$.

Figure 7 shows the variation of Lyapunov exponents Ω_1 and Ω_2 with respect to slowness p. The exponents were computed using (7.21) and (7.33), where the layering model discussed in section 5 was used to compute the parameters σ_{jj}. As Figure 7 shows, the two Lyapunov exponents are slowly increasing, relatively insensitive, functions of slowness over most of its range; both exponents increase rapidly, however, as slowness approaches the P wave cutoff value.

SLOWNESS	Ω_1	$E\{\overline{\Omega}_1(30)\}$	Ω_2	$E\{\overline{\Omega}_2(30)\}$
0.10	.233	.247	.113	.108
0.25	.268	.278	.120	.114

Table III. Comparison of theoretical Lyapunov exponent values with simulation results.

Simulations were generated to test the validity of these theoretical predictions. The layering model of section 5 was used to generate 100 realizations of a random elastic slab. As the layer thickness $\bar{\zeta}$ of each realization was built up to a maximum value of 30, the fundamental matrix $Q(\bar{\zeta})$ was computed for slowness values of $p = 0.10$ and $p = 0.25$ and the eigenvalues of $Q^{*T}(\bar{\zeta})Q(\bar{\zeta})$ were determined. Let these (random) eigenvalues be denoted by $\bar{\ell}_1(\bar{\zeta}) > \bar{\ell}_2(\bar{\zeta}) > \bar{\ell}_3(\bar{\zeta}) > \bar{\ell}_4(\bar{\zeta})$. We again have $\bar{\ell}_3 = 1/\bar{\ell}_2, \bar{\ell}_4 = 1/\bar{\ell}_2$. In figures 8 and 9, the mean and fluctuations of $(1/2\bar{\zeta})\ln \bar{\ell}_j, j = 1, 2$ are plotted as functions of $\bar{\zeta}$ for the two slowness values, $p = 0.10$ and 0.25. As Table III indicates, the theoretical predictions Ω_1 and Ω_2 agree quite well with the mean values of $(1/2\bar{\zeta})\ln \bar{\ell}_j(\bar{\zeta})$, denoted by $E\{\overline{\Omega}_j(\bar{\zeta})\}$, $j = 1, 2$, evaluated at $\bar{\zeta} = 30$. (In section 8, $1/\Omega_2$ will be identified as the localization length. Since $\Omega_2 \approx 0.1$ in both cases, $\bar{\zeta} = 30$ corresponds to roughly 3 localization lengths.)

8 Localization and mode conversion on transmission

In this section we show that the existence of positive Lyapunov exponents implies localization, and examine the consequences for transmission through a large slab. From (4.27), (6.22), (6.23)

$$\tau = \tau_T(I + G,_T)^{-1}A^{*-1} \tag{8.1}$$

where

$$\begin{aligned} G &= A^{*-1}B = \sum_{j=1}^{2} \frac{(\ell_j - 1)}{(\ell_j + 1)} \mathbf{v}_j^* \mathbf{v}_j^{*T} \\ &\sim \sum_{j=1}^{2} \mathbf{v}_j^* \mathbf{v}_j^{*T} \end{aligned} \tag{8.2}$$

for large slabs, since $\ell_j \rightarrow \infty$. From (8.2) G is symmetric and is asymptotically unitary for large slabs. Therefore, using (4.25) one can show that asymptotically, all eigenvalues of $G,_T$ have absolute values bounded by $\sqrt{1 - ||\tau_T||^2}$. It follows that $I + G,_T$ has a bounded inverse. Therefore, from (8.1), (6.23), $\tau \rightarrow 0$ as $\ell_1, \ell_2 \rightarrow \infty$. Also, from (4.29) we conclude that , is asymptotically unitary as was assumed previously.

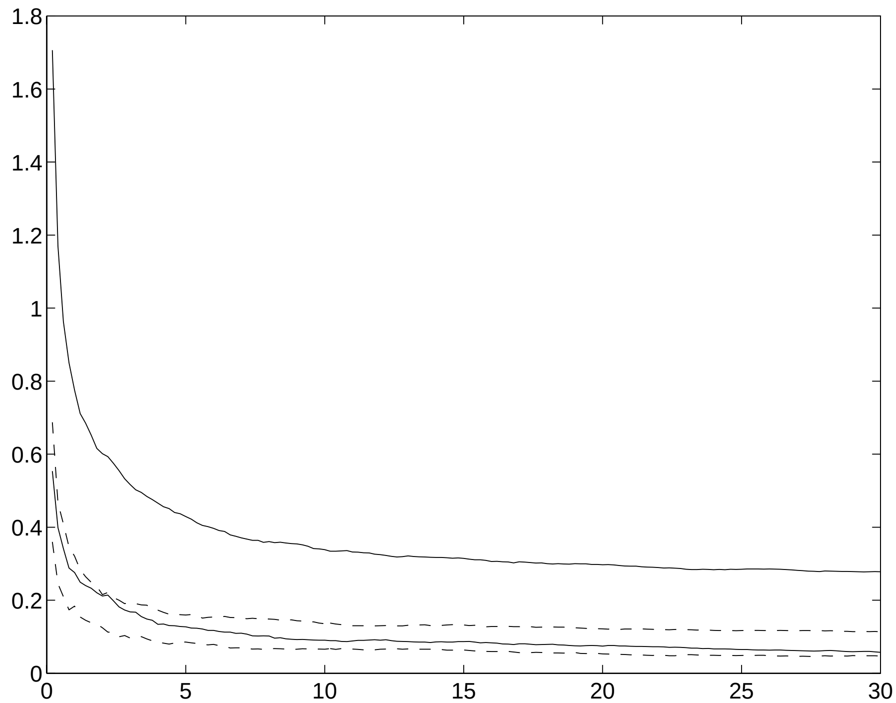

Figure 9: Lyapunov exponent simulations ($p = 0.25$). Mean and fluctuations vs. slab thickness $\overline{\zeta}$. The upper and lower solid curves show the mean and fluctuations of $\frac{1}{2\overline{\zeta}} \ln \overline{\ell}_1$, respectively. The dashed curves present similar information for $\frac{1}{2\overline{\zeta}} \ln \overline{\zeta}_2$.

From (4.31) we can show that $\overline{\tau} \to 0$ if $\left(J_A^T(0) - J_B^T(0),\ \right)$ has a bounded universe. But

$$J_A^T - J_B^T, \ = \left\{ \left(I - ,\ J_B J_A^{-1} \right) J_A \right\}^T \ . \tag{8.3}$$

Using (4.18) we can show that for , symmetric unitary, all eigenvalues of , $J_B J_A^{-1}$ have absolute values bounded by $\sqrt{1 - ||J_A^{-1}||^2}$, so that the matrix in (8.3) has a bounded universe.

Noting (4.31), (8.1) and (6.23), we have for any incident wave $\mathbf{D}^{\mathrm{inc}}$

$$\begin{aligned} \mathbf{D}^{\mathrm{trans}} &= \overline{\tau}\mathbf{D}^{\mathrm{inc}} = \tau_T(I + G,\ _T)^{-1} \sum_{j=1}^{2} \frac{2}{(\sqrt{\ell_j} + \frac{1}{\sqrt{\ell_j}})} \mathbf{v}_j^* \tilde{\mathbf{v}}_j^T \cdot (J_A^T(0) - J_B^T(0),\ )^{-1} \mathbf{D}^{\mathrm{inc}} \\ &\sim \frac{1}{\sqrt{\ell_2}} \left(2\tilde{\mathbf{v}}_2^T (J_A^T(0) - J_B^T(0),\ )^{-1} \mathbf{D}^{\mathrm{inc}} \right) \left[\tau_T (I + G,\ _T)^{-1} \mathbf{v}_2^* \right] \end{aligned} \tag{8.4}$$

since $\ell_1, \ell_2 \to \infty, \ell_1 \gg \ell_2$. Thus $\mathbf{D}^{\mathrm{trans}} \to \mathbf{0}$. In particular

$$\lim_{\zeta \to \infty} \frac{1}{\overline{\zeta}} \ln |\mathbf{D}^{\mathrm{trans}}| = - \lim_{\zeta \to \infty} \frac{1}{2\overline{\zeta}} \ln \ell_2 = -\Omega_2 \tag{8.5}$$

Thus the transmitted wave decays exponentially with the slab width, with an exponential decay length, or localization length, z_{loc}

$$z_{\mathrm{loc}} = \frac{1}{\Omega_2} \ . \tag{8.6}$$

Further note from (8.4) that $\mathbf{D}^{\mathrm{trans}}$ is asymptotically proportional to the vector $\tau_T(I + G,\ _T)^{-1}\mathbf{v}_2^*$, which does not depend on the incident wave. Moreover, this vector can be seen to converge for large slab widths, using (4.22), (4.23) and the observations made before (6.26). Thus, *as the slab width increases to infinity, the ratio of shear energy to compressional energy in the transmitted wave converges to a (random) constant, which does not depend on the incident wave.*

We will determine explicitly the probability distribution for the ratio of S energy to P energy for the case of matched media above and below the random slab. In this case

$$\begin{aligned} \mathbf{D}^{\mathrm{trans}} &= e^{i\frac{\omega}{\epsilon}\bar{\zeta}\Lambda} \sum_{j=1}^{2} \frac{2}{(\sqrt{\ell_j} + \frac{1}{\sqrt{\ell_j}})} \mathbf{v}_j^* \tilde{\mathbf{v}}_j^T \mathbf{D}^{\mathbf{inc}} \\ &\sim \frac{2}{\sqrt{\ell_2} + \frac{1}{\sqrt{\ell_2}}} \left(\tilde{\mathbf{v}}_2^T \mathbf{D}^{\mathrm{inc}} \right) \left(e^{-i\frac{\omega}{\epsilon}\bar{\zeta}\Lambda} \mathbf{v}_2 \right)^* \ , \end{aligned} \tag{8.7}$$

provided ℓ_1 is large. Therefore the S-to-P energy ratio is constant for slab widths much greater than the equilibrium length z_{equil}, where

$$z_{\mathrm{equil}} = \frac{1}{\Omega_1} \ . \tag{8.8}$$

Unfortunately, we cannot obtain the desired energy ratio from (8.7), since the asymptotic distribution of the convergent vector $\exp\left\{-i\frac{\omega}{\epsilon}\bar{\zeta}\Lambda\right\} \mathbf{v}_2$ is unknown. It would have been more convenient had we obtained an expression involving $\tilde{\mathbf{v}}_2$, which is orthogonal to $\tilde{\mathbf{v}}_1$, (7.36).

A physical argument from invariant embedding suggests how we can bring this about: we have set up the embedding problem so that new random material is continuously being added *at* *the incidence end* of the slab, which is increasingly remote from the transmission end. It is in this formulation that the S-to-P transmission energy ratio converges as the slab length increases. Alternatively, we may embed in the opposite direction, by continuously adding material *at the transmission end*. In this reformulation, the energy ratio cannot converge as the slab length is increased, since the material being added has a direct effect on the transmitted wave. We must therefore obtain the tilde-variables, which do not converge. Since either embedding formulation analyzes the same random slab and thus solves the same problem, we are free to choose whichever is more convenient.

We therefore return to z coordinates, and embed with added layers at the transmission end. We use the propagator matrix $Q'(z)$, so that the transmitted wave is given by equation (4.48). Using (6.21) – (6.23) we may write the transmitted wave in the primed variables

$$\begin{aligned} \mathbf{D}^{\text{trans}} &= e^{i\frac{\omega}{\epsilon}\bar{\zeta}\Lambda} \sum_{j=1}^{2} \frac{2}{(\sqrt{\ell'_j} + \frac{1}{\sqrt{\ell'_j}})} (\tilde{\mathbf{v}}'_j)^* (\mathbf{v}'_j)^T \mathbf{D}^{\text{inc}} \\ &\sim \frac{2}{\sqrt{\ell'_2}} \left((\mathbf{v}'_2)^T \mathbf{D}^{\text{inc}} \right) e^{i\frac{\omega}{\epsilon}\bar{\zeta}\Lambda} (\tilde{\mathbf{v}}'_2)^* \; . \end{aligned} \tag{8.9}$$

$e^{i\frac{\omega}{\epsilon}\bar{\zeta}\lambda}(\tilde{\mathbf{v}}'_2)^*$ has the same distribution as $\tilde{\mathbf{v}}_2$ (by (6.27) and $\tilde{\mathbf{v}}_2$ is orthogonal to $\tilde{\mathbf{v}}_1$, (7.36). Thus the ratio of S energy to P energy on transmission is

$$\frac{\varepsilon_s}{\varepsilon_p} = e^{2R} \tag{8.10}$$

where R has probability density $\overline{P}_R(R)$, (7.12).

9 Appendices

A Small fluctuation diffusion limit

The scaling introduced in section 2 and used throughout was adopted because of its particular relevance to the exploration geophysics problem. The purpose of this appendix is to point out, however, that the results obtained apply with trivial modification to another scaling of interest, commonly called the diffusion scaling [16], [17].

In this diffusion scaling, wavelength and the correlation length of the random layering are assumed to be of comparable size. This differs from the scaling of section 2, where wavelength was assumed to be large relative to correlation length. For the sake of discussion, then, let wavelength and correlation length now both be $O(1)$ and let ε denote the small scaling parameter. In the diffusion scaling one is again interested in long propagation paths, specifically in paths that are $O(\varepsilon^{-2})$ in length. The amplitude of the material parameter fluctuations cannot now, however, be taken to be $O(1)$. In the diffusion scaling, these fluctuations (e.g. the peak value of $|\chi|$ in (5.25)) must be assumed small, i.e. $O(\varepsilon)$. A comparison of the two scalings, which we shall refer to as the exploration and diffusion scalings, is given in Table IV.

As an example of the diffusion scaling, consider a layering model where (as in section 2) the velocity scale $\bar{c} = 3$km/sec, the correlation length is 3 m and the propagation path is on the order of 3 km. In the diffusion scaling, the ratio of correlation length to propagation path is again $O(\varepsilon^2)$ (i.e. $O(1)/O(\varepsilon^{-2})$) so that $\varepsilon^2 \approx 10^{-3}$ or $\varepsilon \approx 0.03$ as in section 2. Now, however, wavelength would be on the order of 3 m, so that frequency $f \approx 1$kHz. Moreover, the material parameter fluctuation level, the peak level of χ in (5.25), would be $O(\varepsilon)$, or roughly 0.03.

PARAMETER	*EXPLORATION SCALING*	*DIFFUSION SCALING*
Wavelength	$O(\varepsilon)$	$O(1)$
Layer correlation length	$O(\varepsilon^2)$	$O(1)$
Propagation path length	$O(1)$	$O(\varepsilon^{-2})$
Peak value of material parameter fluctuations	$O(1)$	$O(\varepsilon)$

Table IV. Comparison of the two scalings.

The layered elastic problem, formulated in the diffusion scaling, would initially involve the following counterpart of (2.5):

$$\frac{d}{d\overline{z}}\mathbf{X} = -i\omega M\mathbf{X} \tag{a.1}$$

where the matrix M possesses the structure of (2.6) – (2.8), ω is $O(1)$ and $\overline{z}$ is a depth variable that is $O(1)$ on the correlation length scale. In contrast to (2.9), we would have:

$$\lambda = E[\lambda] + \varepsilon\widehat{\lambda}(\overline{z})\ ,\ \mu = E[\mu] + \varepsilon\widehat{\mu}(\overline{z})\ ,\quad \rho = E[\rho] + \varepsilon\widehat{\rho}(\overline{z}) \tag{a.2}$$

where the expected values are constants and the carated quantities are zero mean, $O(1)$ random fluctuations. The matrix M thus assumes the form:

$$M(\overline{z},\varepsilon) = \overline{M}(\varepsilon) + \varepsilon \widehat{M}(\overline{z}) \tag{a.3}$$

where $\widehat{M}$ is an $O(1)$ zero mean fluctuation matrix and we have explicitly noted the regular dependence of $\overline{M}$ upon ε. The dependence of $\widehat{M}$ upon ε vanishes in the asymptotic limit and has thus been ignored. To obtain an equation similar in structure to (2.15), we introduce the variable z by defining

$$z = \varepsilon^2 \overline{z} \ . \tag{a.4}$$

Note that z takes on $O(1)$ values on the propagation path length scale, i.e. when $\overline{z} \sim O(\varepsilon^{-2})$. In terms of this new variable, (a.1) becomes:

$$\frac{d}{dz}\mathbf{X} = -i\omega \left[\frac{1}{\varepsilon^2}\overline{M} + \frac{1}{\varepsilon}\widehat{M}\left(\frac{z}{\varepsilon^2}\right)\right]\mathbf{X} \ . \tag{a.5}$$

Recall that in (2.15), $M = \overline{M} + \widehat{M}$, where $\overline{M} = E[M]$ is a deterministic constant matrix and $\widehat{M}$ is a $O(1)$ zero mean fluctuation matrix. Therefore, the basic difference between (2.15) and (a.5) lies in the scaling of $\overline{M}$. This matrix appears in (2.15) with a ε^{-1} multiplier and in (a.5) with a ε^{-2} multiplier.

This difference ultimately manifests itself in the fact that the diagonal matrix Λ_1 (3.6) must be rescaled as $\varepsilon^{-1}\Lambda_1$ to obtain the correct corresponding equation in the diffusion scaling. Thus, for example, the diffusion scaling counterpart of (4.4) is

$$\eta = e^{-i\frac{\omega}{\varepsilon^2}\zeta\Lambda_1} \nu e^{i\frac{\omega}{\varepsilon^2}\zeta\Lambda_1} \ . \tag{a.6}$$

Once this change is made, however, the diffusion scaling analysis closely parallels that of the text. In particular, when considering the problem of mode conversion, the governing equations again become (5.3), with the exception that $\alpha_p \to \varepsilon^{-1}\alpha_p, \alpha_s \to \varepsilon^{-1}\alpha_s$. When the limit theorem of references [16], [17], is in turn applied, this rescaling of velocities initially leads to a Markov diffusion process whose generator, call it $\widetilde{\mathcal{L}}$, has somewhat more structure than its counterpart $\overline{\mathcal{L}}$ (5.8). Let σ_{11}, σ_{13} and σ_{33} be defined again as in (5.7). Now, however, we shall define

$$\begin{aligned}
\sigma_{22} + i\tilde{\sigma}_{22} &= \int_0^\infty E\left[\delta_2(\zeta)\delta_2(\zeta+\zeta')\right] e^{i\omega(\alpha_p-\alpha_s)\zeta'} d\zeta' \\
\sigma_{44} + i\tilde{\sigma}_{44} &= \int_0^\infty E\left[\delta_4(\zeta)\delta_4(\zeta+\zeta')\right] e^{i2\omega\alpha_p\zeta'} d\zeta' \\
\sigma_{55} + i\tilde{\sigma}_{55} &= \int_0^\infty E\left[\delta_5(\zeta)\delta_5(\zeta+\zeta')\right] e^{i\omega(\alpha_p+\alpha_s)\zeta'} d\zeta' \\
\sigma_{66} + i\tilde{\sigma}_{66} &= \int_0^\infty E\left[\delta_6(\zeta)\delta_6(\zeta+\zeta')\right] e^{i2\omega\alpha_s\zeta'} d\zeta'
\end{aligned} \tag{a.7}$$

Then the generator $\widetilde{\mathcal{L}}$ obtained in the diffusion scaling becomes:

$$\widetilde{\mathcal{L}} = \overline{\mathcal{L}} + 2\omega^2\left[\tilde{\sigma}_{22}\frac{\partial}{\partial\psi} - \tilde{\sigma}_{55}\frac{\partial}{\partial\phi} - \frac{1}{2}\tilde{\sigma}_{44}\left(\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\right) - \frac{1}{2}\tilde{\sigma}_{66}\left(\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}\right)\right] \tag{a.8}$$

where $\overline{\mathcal{L}}$ is generator (5.8) with $\sigma_{jj}, j = 2, \ldots, 6$, however, defined by (a.7).

This result is entirely consistent with the physical differences between the two scalings. Note that if α_p and α_s are set equal to zero in (a.7), the $\tilde{\sigma}_{jj}$ vanish, the σ_{jj} revert back to the parameters defined in (5.7) and generator $\widetilde{\mathcal{L}}$ reduces exactly to $\overline{\mathcal{L}}$ as defined in (5.8). In the exploration scaling, wavelength is large relative to correlation length. Only the zero-wavenumber power spectrum content of the fluctuations, as measure by parameters (5.7), plays a role in defining the asymptotic diffusion process. In the diffusion scaling, on the other hand, wavelength is comparable to correlation length; the waves can interact, so to speak, with the fine structure of the random layering. The σ_{jj}, as defined by (a.7), now measure the fluctuation power spectrum content at the sum and difference wavenumbers of $\omega\alpha_p$ and $\omega\alpha_s$.

When restricted to action upon functions of θ only, $\widetilde{\mathcal{L}}f(\theta) = \overline{\mathcal{L}}f(\theta)$. Therefore, the mode conversion results discussed in section 5 apply as well to the diffusion scaling. Again, the only difference is that the parameters $\sigma_{jj}, j = 2, \ldots, 6$, are defined by (a.7) rather than (5.7). One can further show that the Lyapunov exponent and transmission results of sections 7 and 8 likewise carry over to the diffusion scaling with this modification.

B Matched layered elastic medium without density fluctuation

Throughout the discussion, we have assumed generic random layering and excitation conditions, conditions which, in particular, lead to distinct nonzero Lyapunov exponents (c.f. (6.5)). We will now show that if density fluctuations are suppressed in the matched medium layering model of section 5, the situation is no longer achieved.

Accordingly, we shall consider the layering model (5.25), (5.26), modified so that $\rho = \overline{\rho}$ in $z \geq 0$. In this case, the matrices M_1, M_2 (2.7), (2.8) can be written as

$$\begin{aligned} M_1 &= \begin{bmatrix} (\lambda + 2\overline{\mu})^{-1} & p[1 - 2\overline{\mu}(\lambda + 2\overline{\mu})^{-1}] \\ p[1 - 2\overline{\mu}(\lambda + 2\overline{\mu})^{-1}] & \overline{\rho} - 4p^2\overline{\mu} + 4p^2\overline{\mu}^2(\lambda + 2\overline{\mu})^{-1} \end{bmatrix} \\ M_2 &= \begin{bmatrix} \overline{\rho} & p \\ p & \overline{\mu}^{-1} \end{bmatrix} \end{aligned} \tag{b.1}$$

where $\lambda = \lambda_0(1 + \chi)$. M_2 in particular is a deterministic matrix. The fluctuation matrices $\widehat{M}_j, j = 1, 2$, in (3.28) thus become:

$$\begin{aligned} \widehat{M}_1 &= m \begin{bmatrix} 1 & -2\overline{\mu}p \\ -2\overline{\mu}p & 4\overline{\mu}^2p^2 \end{bmatrix} \\ \widehat{M}_2 &= 0 \\ m &= (\lambda + 2\overline{\mu})^{-1} - E\{(\lambda + 2\overline{\mu})^{-1}\} = (\lambda + 2\overline{\mu})^{-1} - (\overline{\lambda} + 2\overline{\mu})^{-1} \end{aligned} \tag{b.2}$$

Note that $\widehat{M}_1$ is a zero mean, rank one fluctuation matrix. Moreover, it follows from (3.30) that:

$$\nu_1 = \nu_2 = \frac{1}{2} L_2^T \widehat{M}_2 L_2 \ . \tag{b.3}$$

Since the effective medium is an isotropic elastic medium with material parameters $\overline{\lambda}, \overline{\mu}, \overline{\rho}$, we infer from (3.9), (2.16), and (2.17) that:

$$L_2 = \begin{bmatrix} (\overline{\rho} - 2\overline{\mu}p^2)(\overline{\rho\alpha}_p)^{-1/2} & 2\overline{\mu}p(\overline{\alpha}_s/\overline{\rho})^{1/2} \\ -p(\overline{\rho\alpha}_p)^{-1/2} & (\overline{\alpha}_s/\overline{\rho})^{1/2} \end{bmatrix} \tag{b.4}$$

$$\overline{\alpha}_p = \sqrt{\frac{\overline{\rho}}{\overline{\lambda} + 2\overline{\mu}} - p^2} \quad , \quad \overline{\alpha}_s = \sqrt{\frac{\overline{\rho}}{\overline{\mu}} - p^2} \; . \tag{b.5}$$

Using (b.5) in (b.3) leads to:

$$\nu_1 = \nu_2 = \frac{m\overline{\rho}}{2\overline{\alpha}_p} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; . \tag{b.6}$$

From (3.31), then:

$$\delta_1 = \delta_4 = \frac{m\overline{\rho}}{2\overline{\alpha}_p} \quad , \quad \delta_2 = \delta_3 = \delta_5 = \delta_6 = 0 \; . \tag{b.7}$$

Equations (b.7), together with Table I, imply that the random scattering process will only involve the P-wave. There is no modal coupling; S-wave energy will propagate through the elastic slab totally unaffected by the random layering. S-wave energy will not be localized.

To develop these observations quantitatively, note from (4.38), (6.1), (3.29), (3.6), and (3.3) that:

$$\begin{aligned} \frac{d}{d\overline{\zeta}}\overline{Q} &= \frac{i\omega}{\epsilon} \begin{bmatrix} \overline{\alpha}_p + \delta_1 & 0 & -\delta_1 & 0 \\ 0 & \overline{\alpha}_s & 0 & 0 \\ \delta_1 & 0 & -\overline{\alpha}_p - \delta_1 & 0 \\ 0 & 0 & 0 & -\overline{\alpha}_s \end{bmatrix} \overline{Q} \\ \overline{Q}|_{\overline{\zeta}=0} &= I \end{aligned} \tag{b.8}$$

One can check by direct calculation that $\overline{Q}$ has the structure:

$$\overline{Q}(\overline{\zeta}) = e^{i\frac{\omega}{\epsilon}\Lambda\overline{\zeta}} \begin{bmatrix} a & 0 & b^* & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & a^* & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = e^{i\frac{\omega}{\epsilon}\Lambda\overline{\zeta}} Q(\overline{\zeta}) \tag{b.9}$$

where:

$$\begin{aligned} \frac{d}{d\overline{\zeta}}a &= i\frac{\omega}{\epsilon}\delta_1[a - e^{i2\frac{\omega}{\epsilon}\overline{\alpha}_p\overline{\zeta}}b] \quad , \quad a|_{\overline{\zeta}=0} = 1 \\ \frac{d}{d\overline{\zeta}}b &= i\frac{\omega}{\epsilon}\delta_1[e^{i2\frac{\omega}{\epsilon}\overline{\alpha}_p\overline{\zeta}}a - b] \quad , \quad b|_{\overline{\zeta}=0} = 0 \end{aligned} \tag{b.10}$$

It follows from (b.9) that the eigenvalues of $\overline{Q}^{*T}(\overline{\zeta})\overline{Q}(\overline{\zeta})$ are 1,1, $(|a| - |b|)^2, (|a| + |b|)^2$. It is also readily seen from (b.10) that $\frac{d}{d\overline{\zeta}}[|a|^2 - |b|^2] = 0$, so that $(|a| + |b|)(|a| - |b|) = 1$. Therefore (6.6):

$$\Omega_1 = \lim_{\overline{\zeta}\to\infty} \frac{1}{\overline{\zeta}} \ln[|a| + |b|], \quad \Omega_2 = \Omega_3 = 0, \quad \Omega_4 = -\Omega_1 \tag{b.11}$$

Since the smaller Lyapunov exponent Ω_2 is zero, the localization length $z_{\rm loc}$ (8.6) is undefined. For the matched medium, $J_A = I, J_B = 0$. Therefore, from (4.22), (4.23), (8.1), and (b.9):

$$\tau^{*T}\tau = (A^{-1})A^{*-1} = \begin{bmatrix} |a|^{-2} & 0 \\ 0 & 1 \end{bmatrix} \tag{b.12}$$

The fraction of transmitted P-wave energy, $|a(\bar{\zeta})|^{-2}$, thus tends to zero as $\bar{\zeta} \to \infty$ while the S-wave energy propagates unattenuated.

C Energy considerations

The elastic power flux vector $\mathbf{s}$ [24] has components

$$s_j = -\left(\frac{\partial}{\partial t} u_i\right) \tau_{ij} \quad , \quad j = 1, 2, 3 \tag{c.1}$$

For time-harmonic excitation (2.3), then, the time-averaged power flux in the (downward) z-direction is:

$$\hat{s}_3 = -\frac{1}{2} Re \left[-i\varpi \left(\hat{u}_1 \hat{\tau}_{13}^* + \hat{u}_3 \hat{\tau}_{33}^*\right)\right] \tag{c.2}$$

where we have used the fact that $\hat{u}_2 = 0$. Let the vector $\mathbf{X}$ (2.4) be written as:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ ---- \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 \\ ---- \\ \mathbf{X}_2 \end{bmatrix} \tag{c.3}$$

Then, the downward, time-averaged flux becomes

$$\hat{s}_3 = -\frac{1}{2} Re \left[x_4 x_2^* + x_1 x_3^*\right] = -\frac{1}{2} Re \left[\mathbf{X}_1^{*T} \mathbf{X}_2\right] \tag{c.4}$$

since $Re\left[x_1 x_3^*\right] = Re\left[x_1^* x_3\right]$. On the other hand, from (3.10), (3.7):

$$\begin{aligned} \mathbf{U} &= \frac{1}{\sqrt{2}} \left[L_2^T \mathbf{X}_1 + L_1^T \mathbf{X}_2\right] \\ \mathbf{D} &= \frac{1}{\sqrt{2}} \left[L_2^T \mathbf{X}_1 - L_1^T \mathbf{X}_2\right] \end{aligned} \tag{c.5}$$

Using (4.26), (3.1) we obtain:

$$\frac{1}{4}\left[\mathbf{D}^{*T}\mathbf{D} - \mathbf{U}^{*T}\mathbf{U}\right] = \frac{1}{4}\mathbf{D}^{*T}\left[I - \,,^{*T} ,\right]\mathbf{D} = -\frac{1}{2} Re \left[\mathbf{X}_1^{*T} \mathbf{X}_2\right] = \hat{s}_3 \tag{c.6}$$

Therefore, at any depth z within the slab, we can interpret $|\frac{1}{2}\mathbf{D}|^2, |\frac{1}{2},\, \mathbf{D}|^2$ and $|\frac{1}{2}\tau\mathbf{D}|^2$ as incident, reflected and transmitted downward power flux. Moreover, by direct calculation, using (2.5) – (2.8)

$$\frac{d}{dz} Re \left[\mathbf{X}_1^{*T} \mathbf{X}_2\right] = 0 \tag{c.7}$$

reflecting the fact that the power flux transmitted through the lossless slab remains constant with depth.

D Evanescent P wave, propagating S-wave

Assume that a slab of randomly layered elastic material occupies the region $0 \le z \le \overline{z}$ (or $-\overline{z} \le \zeta \le 0$). The half-spaces $z < 0$ and $z > \overline{z}$ are assumed to be composed of homogeneous, isotropic elastic media. The plane wave excitation incident upon the slab from the region $z < 0$, is assumed to have a slowness value such that in the slab effective medium the faster (P) wave is evanescent while the slower (S) wave is propagating. We assume that P and S waves are both propagating in the two homogeneous half-spaces.

Our assumption is tantamount to assuming that α_p in (3.3) is imaginary, i.e.,

$$\Lambda = \begin{bmatrix} i\beta_p & 0 \\ 0 & \alpha_s \end{bmatrix} \tag{d.1}$$

where β_p, α_s are positive real parameters. Because of this, the matrix L (3.4), as well as the jump matrices $J(0)$ and $J(\overline{z})$, will also be complex. In particular, one can show that

$$L_1 = \widehat{L}_1\Phi \quad , \quad L_2 = \widehat{L}_2\Phi^* \tag{d.2}$$

where $\widehat{L}_1, \widehat{L}_2$ are real-valued matrices and $\Phi = \mathrm{diag}(e^{i\pi/4}, 1)$. Since the fluctuation matrices $\widehat{M}_1, \widehat{M}_2$ (3.28) remain real-valued, the matrices ν_1, ν_2 (3.30) become complex-valued. In fact, we obtain:

$$\delta_j = i\hat{\delta}_j \ , \ j = 1,4 \ ; \ \delta_j = \hat{\delta}_j^{(1)} + i\hat{\delta}_j^{(2)} \ , \ j = 2,5 \ ; \ \delta_j = \hat{\delta}_j \ , \ j = 3,6 \tag{d.3}$$

where the $\hat{\delta}$'s are real-valued processes. Note from Table I that the scattering processes that involve only the propagating S wave remain real, those involving only the evanescent P wave become purely imaginary, while those generating cross-mode coupling possess real and imaginary parts.

We next introduce ups and downs, $\mathbf{U}$ and $\mathbf{D}$, as in (3.10), and introduce a reflection matrix Γ by

$$\mathbf{U}(\zeta) = \Gamma(\zeta)\mathbf{D}(\zeta) \quad , \quad -\overline{z} \le \zeta \le 0 \ . \tag{d.4}$$

The reflection matrix Γ is found to satisfy the following Riccati equation and initial condition

$$\begin{aligned} \frac{d\Gamma}{d\zeta} &= i\frac{\omega}{\epsilon}\left[\Lambda\Gamma + \Gamma\Lambda + \nu_1\Gamma + \Gamma\nu_1 - \nu_2 - \Gamma\nu_2\Gamma\right] \\ \Gamma|_{\zeta=-\overline{z}^+} &= \Gamma_T \end{aligned} \tag{d.5}$$

where Λ is given by (d.1), ν_1 and ν_2 are defined by (3.31) with the δ_j given by (d.3), and Γ_T is the termination matrix arising from the mismatch between the effective medium and the homogeneous half-space $z > \overline{z}(\zeta < -\overline{z})$. Γ_T is a symmetric matrix and this symmetry is clearly preserved by the structure of the Riccati equation; $\Gamma(\zeta)$ is a symmetric matrix, $-\overline{z} \le \zeta \le 0$.

Let $\Gamma_{ij}, i,j = 1,2$ with $\Gamma_{21} = \Gamma_{12}$ be the elements of Γ. The component equations of (d.5) become

$$\begin{aligned} \frac{d}{d\zeta}\Gamma_{11} = &- 2\frac{\omega}{\epsilon}\beta_p\Gamma_{11} - i\frac{\omega}{\epsilon}\left[-2\delta_1\Gamma_{11} - 2\delta_2\Gamma_{12} + \delta_4 + \delta_4\Gamma_{11}^2 + 2\delta_5\Gamma_{11}\Gamma_{12} + \delta_6\Gamma_{12}^2\right] \\ \frac{d}{d\zeta}\Gamma_{12} = &- \frac{\omega}{\epsilon}\beta_p\Gamma_{12} + i\frac{\omega}{\epsilon}\alpha_s\Gamma_{12} - i\frac{\omega}{\epsilon}\left[-\delta_2\Gamma_{11} - (\delta_1 + \delta_3)\Gamma_{12} - \delta_2\Gamma_{22} + \delta_5 + \delta_4\Gamma_{11}\Gamma_{12}\right. \\ &+ \left.\delta_5(\Gamma_{11}\Gamma_{22} + \Gamma_{12}^2) + \delta_6\Gamma_{12}\Gamma_{22}\right] \\ \frac{d}{d\zeta}\Gamma_{22} = &\ i2\frac{\omega}{\epsilon}\alpha_s\Gamma_{22} - \frac{i\omega}{\epsilon}\left[-2\delta_2\Gamma_{12} - 2\delta_3\Gamma_{22} + \delta_6 + \delta_4\Gamma_{12}^2 + 2\delta_5\Gamma_{12}\Gamma_{22} + \delta_6\Gamma_{22}^2\right] \end{aligned} \tag{d.6}$$

If equations (d.6) are compared with equations (5.1), one notes that the equations of section 5 correspond to a centered problem. Rapid deterministic phase oscillations were removed by transformation (4.2). The resulting Riccati equation (4.36), (5.1) has a right hand side having only zero-mean, random coefficients. In equations (d.6), on the other hand, such centering has not been done. Note that , $_{11}$ and , $_{12}$ in (d.6) undergo rapid, deterministic attenuation caused by the right hand terms $-2\frac{\omega}{\epsilon}\beta_p$, $_{11}$ and $-\frac{\omega}{\epsilon}\beta_p$, $_{12}$, respectively. Only , $_{22}$ has a deterministic right hand coefficient that generates only rapid phase oscillations.

We recast (d.6) into a system of equations having the desired structure by the following change of variables; let

$$, _{11} = x_1 + ix_2 \ , \ , _{12} = e^{i\frac{\omega}{\epsilon}\alpha_s(\zeta+\bar{z})}(x_3 + ix_4) \ , \ , _{22} = e^{i2\frac{\omega}{\epsilon}\alpha_s(\zeta+\overline{z})}(re^{i\phi}) \tag{d.7}$$

In (d.7) we have "removed" the rapid phase oscillations in a manner analogous to transformation (4.2) and have introduced polar coordinates into the , $_{22}$ description. To further compare with section 5, let

$$\begin{aligned} \boldsymbol{\psi}_1 &= (r, \phi)^T \ , \ \boldsymbol{\psi}_2 = (x_1, x_2, x_3, x_4)^T \ , \\ D_p &= \mathrm{diag}(2\omega\beta_p \ , \ 2\omega\beta_p \ , \ \omega\beta_p \ , \ \omega\beta_p) \end{aligned} \tag{d.8}$$

Then, equations (d.6) ultimately transform into a system of nonlinear stochastic ordinary differential equations having the following form

$$\begin{aligned} \frac{d}{d\zeta}\boldsymbol{\psi}_1 &= \frac{1}{\varepsilon}\mathbf{F}_1(\frac{\zeta}{\varepsilon^2}, \frac{\zeta}{\varepsilon}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \\ \frac{d}{d\zeta}\boldsymbol{\psi}_2 &= -\frac{1}{\varepsilon}D_p\boldsymbol{\psi}_2 + \frac{1}{\varepsilon}\mathbf{F}_2(\frac{\zeta}{\varepsilon^2}, \frac{\zeta}{\varepsilon}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \end{aligned} \tag{d.9}$$

where the functions $\mathbf{F}_j(\zeta, \xi, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2), j = 1, 2$, for fixed $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2$, are random processes in ζ, with mean zero. In contrast to (5.4), system (d.9) has a component, $\boldsymbol{\psi}_2$, which undergoes rapid exponential attenuation. Systems similar to (d.9) have been analyzed [25]. The basic behavior of the $\boldsymbol{\psi}$-process in the asymptotic limit as ε tends to zero involves two stages. As ζ increases from the initial value of $-\overline{z}^+$, there is an $O(\varepsilon)$ boundary layer in which $\boldsymbol{\psi}_2$ (essentially , $_{11}$ and , $_{12}$) collapses from its initial value (determined from , $_T$) to $\mathbf{0}$. This collapse is essentially deterministic since random effects are $o(1)$ over an $O(\varepsilon)$ ζ- interval. The $\boldsymbol{\psi}_1$ process (basically , $_{22}$), on the other hand, does not undergo exponential attenuation. Rather, as ζ increases from the initial value of $-\overline{z}^+$, $\boldsymbol{\psi}_1$ has an asymptotic description in terms of a Markov diffusion process, much the same as the $\boldsymbol{\psi}$-process in section 5. What is perhaps surprising is the fact that the generator of this diffusion is not simply that of (5.6), with $\mathbf{F}(\zeta, \xi, \boldsymbol{\psi})$ replaced by $\mathbf{F}_1(\zeta, \xi, \boldsymbol{\psi}_1, \mathbf{0})$. Some coupling of the $\boldsymbol{\psi}_1$-process to the exponentially-damped $\boldsymbol{\psi}_2$-process persists in the asymptotic limit, specifically influencing the phase drift of the diffusion process.

To describe the generator of this diffusion process, assume that the σ_{jj} are again defined by (5.7). Now, however, it follows from (d.3) that σ_{11} and σ_{44} are negative real parameters, σ_{33} and σ_{66} are positive real, σ_{13} is imaginary and σ_{22}, σ_{55} have both real and imaginary parts. One can show, in particular, that $\delta_5 = -i\delta_2^*$, so that:

$$Re[\sigma_{22} + \sigma_{55}] = 0, \quad \mathrm{Im}[\sigma_{22} + \sigma_{55}] = 2\mathrm{Im}[\sigma_{22}] = 2\mathrm{Im}[\sigma_{55}] \tag{d.10}$$

In terms of these parameters, the generator of the asymptotic $\boldsymbol{\psi}_1 = (r, \phi)^T$ diffusion process becomes:

$$
\begin{aligned}
\mathcal{L} &= 2\omega^2 \left\{ \frac{\sigma_{66}}{4} (1-r^2)^2 \left(\frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} \right) + \left[2\sigma_{33} + \frac{\sigma_{66}}{4} (r^{-1} + r)^2 \right] \frac{\partial^2}{\partial \phi^2} \right. \\
&\quad \left. - \mathrm{Im}[\sigma_{22} + \sigma_{55}] \frac{\partial}{\partial \phi} \right\} \qquad \text{(d.11)}
\end{aligned}
$$

Note that the $\frac{\partial}{\partial \phi}$ term arises because of scattering between the propagating S and evanescent P waves. Let $u \equiv (1+r^2)/(1-r^2)$. Assume that $(,\, _T)_{22} = r_T e^{i\phi_T}$ and let $u_T \equiv (1+r_T^2)/(1-r_T^2)$. Then, recalling that $\bar{\zeta} = \zeta + \bar{z}$ and noting (5.10), the joint probability density of the (u, ϕ) process at position $\bar{\zeta}$, call it $P(\bar{\zeta}, u, \phi)$ is a solution of:

$$
\begin{aligned}
\partial_{\bar{\zeta}} P &= 2\omega^2 \left\{ \sigma_{66} \frac{\partial}{\partial u} \left((u^2-1) \frac{\partial}{\partial u} \right) + \left[2\sigma_{33} + \sigma_{66} \frac{u^2}{u^2-1} \right] \frac{\partial^2}{\partial \phi^2} \right. \\
&\quad \left. + \mathrm{Im}[\sigma_{22} + \sigma_{55}] \frac{\partial}{\partial \phi} \right\} P \quad , \quad 0 < \bar{\zeta} < \bar{z} \\
P(0, u, \phi) &= \delta(u - u_T)\delta(\phi - \phi_T) \qquad \text{(d.12)}
\end{aligned}
$$

Problem (d.12) can be solved in terms of Fourier series and Mehler transforms [26], [27], and [28]. Let

$$
P(\bar{\zeta}, u, \phi) = \sum_{m=-\infty}^{\infty} P_m(\bar{\zeta}, u) e^{im\phi} \qquad \text{(d.13)}
$$

Then, P_m is a solution of the initial value problem

$$
\begin{aligned}
\partial_{\bar{\zeta}} P_m &= 2\omega^2 \left\{ \sigma_{66} \left[\frac{\partial}{\partial u} \left((u^2-1) \frac{\partial}{\partial u} \right) - \frac{m^2}{u^2-1} \right] P_m - m^2 [2\sigma_{33} + \sigma_{66}] P_m \right. \\
&\quad \left. + im \mathrm{Im}[\sigma_{22} + \sigma_{55}] P_m \right\} \\
P_m(0, u) &= \frac{1}{2\pi} \delta(u - u_T) e^{-im\phi_T} \qquad \text{(d.14)}
\end{aligned}
$$

The solution of (d.14) has the representation:

$$
\begin{aligned}
P_m(\bar{\zeta}, u) &= \frac{1}{2\pi^2} \exp\left\{ -2\omega^2 \left[m^2 (2\sigma_{33} + \sigma_{66}) + im \mathrm{Im}[\sigma_{22} + \sigma_{55}] \right] \bar{\zeta} - im\phi_T \right\} \int_0^\infty t \sinh \pi t \, \cdot \\
&\quad ,\, \left(\frac{1}{2} - |m| + it \right) ,\, \left(\frac{1}{2} - |m| - it \right) e^{-2\omega^2 \sigma_{66} (t^2 + 1/4) \bar{\zeta}} P^{|m|}_{-\frac{1}{2}+it}(u_T) P^{|m|}_{-\frac{1}{2}+it}(u) dt
\end{aligned}
\qquad \text{(d.15)}
$$

where , in (d.15) denotes the gamma function [28]. We shall assume that one is interested in studying the reflection problem of the scattering of waves by the randomly layered slab back into the half-space $z < 0 (\bar{\zeta} > \bar{z})$. Specifically, we shall consider the moments of the elements of $\bar{,}$ (4.30). These moments can be computed by means of the following steps

(i) The jump matrices J_A, J_B must be computed at $z = 0, \overline{z}$, using (3.16). Note that these jump matrices will be complex, since $L^{+}_{1,2}$ at $z = 0$ and $L^{-}_{1,2}$ at $z = \overline{z}$ correspond to the effective medium and have the structure (d.2).

(ii) The jump matrices at $z = \overline{z}$, in turn, determine the termination matrix $,_T$ and, therefore, the initial conditions u_T, ϕ_T. In this appendix, recall that , (and $,_T$) correspond to the "uncentered" problem. Therefore:

$$,_T = -J_A^{-1}(\overline{z}) J_B(\overline{z}) \tag{d.16}$$

(iii)At $z = 0^-(\bar{\zeta} = \bar{z}^-)$, the reflection matrix , will have asymptotic structure:

$$, \sim \begin{bmatrix} 0 & 0 \\ 0 & ,_{22} \end{bmatrix} \tag{d.17}$$

where (d.7)

$$,_{22} = e^{i2\frac{\omega}{\epsilon}\alpha_s \overline{z}} \sqrt{\frac{u+1}{u-1}} e^{i\phi} \tag{d.18}$$

(iv) The matrix $\overline{,}$, where **U**=, **D** at $z = 0^-$, is then computed using (4.30) or (4.32). The elements of $\overline{,}$ are thus obtained as functions of the (u, ϕ)-process. The moments can, in turn, be computed using $P(\bar{\zeta}, u, \phi)$ (d.13).

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