FREQUENCY CONTENT OF RANDOMLY SCATTERED SIGNALS

M. ASCH ^{*}, W. KOHLER [†], G. PAPANICOLAOU ^{*}, M. POSTEL ^{*} AND B. WHITE[‡]

Abstract. The statistical properties of acoustic signals reflected by a randomly layered medium are analyzed when a pulsed spherical wave issuing from a point source is incident upon it. The asymptotic analysis of stochastic equations and geometrical acoustics is used to arrive at a set of transport equations that characterize multiply scattered signals observed at the surface of the layered medium. The results of extensive numerical simulations are presented, illustrating the scope of the theory. A number of inverse problems for randomly layered media are also formulated where we recover large scale properties of the sound speed profile from the statistics of reflected signals.

Key words. Wave propagation, random media, stochastic equations, geometrical acoustics

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^{*} Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012,

[†] Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061

[‡] Exxon Research and Engineering Company, Route 22 East, Annandale, NJ 08801

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1. Introduction. Waves propagating in a randomly inhomogeneous medium are expected to slow down somewhat, because of the distortion of their travel paths, and to lose some of their coherent energy to small scale fluctuations. This simple picture can be confirmed easily by perturbation expansions about a homogeneous medium. But when the random inhomogeneities are strong something different happens: the waves do not propagate at all. They become localized around the region where they are generated, and all their energy is eventually converted to random fluctuations. In two- or one-dimensional environments, such as layered media, however, waves are localized even when the random inhomogeneities are weak.

These phenomena were discovered more than thirty years ago [1] in connection with electron waves in semiconductors and a substantial theory has been developed [2]-[4]. Localization for classical waves, acoustic, electromagnetic, elastic, etc., has received much less attention until recently [5], [6], even though for one dimensional waves localization phenomena were well understood in general and as mathematical problems in the seventies [7], [8]. One reason why localization of classical waves got less attention is that the scattering by the random inhomogeneities has to be very strong and dissipation rather weak in order that the distances over which waves localize may be observed. In terrestrial phenomena such as seismic wave propagation, radio wave propagation in the atmosphere or sound propagation in the ocean, where dissipation is frequently not important, localization lengths are so long that other features of the environment play a major role: the lithological structure of the earth's subsurface, the large scale structure of the refractive index of the atmosphere, or the large scale structure of the sound speed profile in the ocean. Propagation phenomena due to the random inhomogeneities cannot be understood without incorporating in the analysis the large scale features of the medium.

In this paper we present some recent work [9]-[21] on wave propagation in randomly layered media in which the large scale structure of the media is taken into account along with wave localization. Our analysis is based on one more assumption in addition to layering: that the typical wavelengths of the propagating waves are long compared to the size of the random inhomogeneities but short compared to length scales characteristic of the large scale features of the environment. The inhomogeneities need not be weak. The effects of multiple scattering due to the inhomogeneities have a form that does not depend on the details of the random microstructure, while the waves see the details of the large scale structure. The waves can therefore act as a probe revealing the large scale features of the medium while having more or less universal statistical properties, acquired through multiple scattering. This allows us to formulate and solve inverse problems where we seek to detect large

scale features of the environment from multiply scattered waves propagating through it.

FIG. 1.1A. *Velocity profile - transmitted pulse (plane wave)*

Propagation of long waves in the earth's subsurface is modeled very well in our framework. In Fig. 1.1a we show an acoustic plane wave pulse propagating in a model of a randomly layered medium. The sound speed profile is shown at the bottom. Its average is 3km/sec until a depth of 3km; then it drops to 2km/sec and finally returns to 3km/sec at a depth of 7km. The fluctuations in the sound speed are over layers of about 3 meters thick and they are independent random variables, uniformly distributed over an interval that is $\pm 30\%$ of the mean value. A plane wave pulse is normally incident on this medium and it strikes the surface at time zero. The pulse is shown to scale in Fig. 1.1a above the sound speed profile. It is the second derivative of a Gaussian and its frequency content is in the range of 20-50Hz. Thus a typical wavelength is of the order of 100 meters, which is much larger than 3 meters, the size of the layers, but much smaller than 7 kilometers, the depth of penetration of the pulse. At the top of Fig. 1.1a we show the acoustic velocity field as a function of depth, at different times, computed numerically by solving the acoustic equations (cf. (2.4)). The pulse itself is clearly seen to move with the mean speed while getting progressively attenuated. There is no dissipation in this model so, the attenuation is due to multiple scattering and is called the apparent attenuation [22]-[26]. The energy of the pulse is converted to fluctuations which form a coda following the pulse itself. All the energy is eventually reflected back to the surface and Fig. 1.1b, top, shows the very fluctuating reflected signal as a function of scaled time (travel time, cf. (2.24)). The bottom part of Fig. 1.1b is the reflected signal when the fluctuations in the sound speed profile are removed. Note that the scale for the bottom part of Fig. 1.1b, the deterministic reflection, is on the right, while the scale of the random reflection is on the left. We see that the signal to noise ratio is of the order of 10^{-3} , which means that

extracting information from signals reflected by randomly layered media is not easy.

FIG. 1.1B. *Reflected signal with and without noise (plane wave)*

The theory we have developed gives a precise characterization of the class of stochastic processes that can be reflected signals under our modeling hypotheses. It is an asymptotic theory based on the separation of scales that we described. In formulating and solving inverse problems [12], [17], we work within the restricted class of possible reflected signals which means that we incorporate into our inversion algorithms information that comes from an underlying model problem. This is a much more tractable statistical inversion problem than direct signal processing without any modeling, while the modeling itself is done from first principles, starting from the appropriate wave equations.

The parameter values that we have used in the model in Fig. 1.1a and 1.1b are typical for a variety of problems in exploration geophysics and this has been verified by a careful statistical analysis of well log data [27] (and references therein). The correlation length of the sound speed is in the range 1.5-3 meters, with a correlation function that fits reasonably well to an exponential. These properties depend on depth, but it is only their typical values that play a role in deciding whether or not our model fits at all. The asymptotic theory can deal with local nonuniformities without difficulty. In the frequency range of 20-50Hz the localization length of the waves has been estimated [27] to be about 20km. This means that pulses with frequency content in this range will not penetrate substantially beyond this depth. The localization length is frequency dependent and the higher frequencies penetrate less. There are many other problems in wave propagation to which this theory fits, such as reflection of sound waves from the sedimentary layers beneath abyssal plains in the ocean [28], [29], but the numerical calculations presented here are based on the geophysical model.

In this paper we give a rather complete account of the theory as it applies to acoustic waves reflected by a randomly layered half space when a time-limited pulse

 FIG. 1.2. *The physical problem*

generated by a point source, which is located in the homogeneous half space above the layered medium, is incident upon it. See Fig. 1.2. In §2 we formulate the problem, scale it, and put it into a form suitable for the asymptotic analysis. We also review the effective medium theory which for layered media is an elementary form of averaging or homogenization and does not account for localization phenomena. We note that the difference between propagation in a periodic versus a random medium with the same “effective” parameters is striking. In Fig. 1.3a we plot the reflected pressure as a function of time, observed at different points on the surface which are located at increasing distances from the origin. The source is one pulse width above the origin. The periodically layered medium and the associated homogeneous effective medium produce indistinguishable results. Note in particular that the head wave¹, the first arrival before the arrival of the reflected wave, is captured very well by the effective medium theory. This is somewhat surprising because this is an “interior” theory and is not very good near macroscopic interfaces or boundaries, in general. The last arrival at each observation point on the surface is due to the bottom of the stack of layers. Below it the medium is again homogeneous. In Fig. 1.3b we show the results of a similar computation except that the sound speed in the layers varies randomly. It is quite clear that random multiple scattering has a profound effect on the reflected signals. The arrival times of the waves, the head wave and the reflection from the top and bottom of the layer stack, are according to the effective medium theory. But signal fronts are weakened somewhat and pulse energy has been converted to fluctuations.

In §3 we derive transport equations for the statistics of the reflected signal. Our main tool is the asymptotic theory of stochastic equations which we review briefly in Appendix B. Much of this theory existed already in the seventies [30]-[32] and

¹ The effective sound speed in the layered region is larger than the sound speed in the region above.

FIG. 1.3A. *Reflected pressure for a periodic medium (point source)* FIG. 1.3B. *Reflected pressure for a random medium (point source)*

was used extensively in the analysis of waves in random media [33]-[43] and in other problems [44]-[48]. It acquired progressively a more refined mathematical form [49] and played a significant role in the probabilistic treatment of homogenization problems [50]. However, the complexity of the pulsed, point source reflection problem is such that we need a new formulation that allows us to handle infinite dimensional versions [10], [51] of the previous theory. Section 3 begins with the derivation of a system of moment equations for a suitably defined reflection coefficient. We then apply the theory of the Appendix in a form that leads efficiently to the hierarchy of equations for the asymptotic limit of the moments of the reflection coefficient. These equations (cf. (3.14)), which we call the W -equations, are an infinite-dimensional, linear, hyperbolic system that preserves positivity: a system of transport equations. Nearly everything one wants to know about pulse reflection can be gotten by solving these equations in one form or another.

In the rest of §3 we analyze the W -equations. First we calculate the coherently reflected pressure field and compare it with the deterministic case which is the geometrical acoustics limit (§3.3). Then we discuss the existence of weak solutions of the transport equations using a probabilistic representation. This representation is in terms of a canonical Markov chain on the nonnegative integers that is characteristic of all multiple scattering. The macroscopic background variations enter also in a canonical, geometric way by determining the intrinsic transition rate of the Markov chain and the associated displacement functionals that are needed in the representation. We introduce a duality for the Markov chain that makes its properties particularly easy to study (§3.5). In §3.7 we solve explicitly the transport equations when there are no macroscopic variations so that only the multiple scattering produces reflections. We explain the geometric meaning of the form of the local power spectral density of the reflected signals. This leads us to §3.8 in which we introduce a geometric interpretation of multiple scattering in general. In §3.9 we discuss the influence of turning points on the reflected signal statistics. We prove that the W -equations are well posed even when they become singular at turning points. This is interesting because it tells us that we need not do anything special for the turning points, as is the case in deterministic problems. The W -equations have all the necessary ingredients for determining reflected signal statistics. Multiple scattering from the random layering has brought about a simplification in pulse reflection calculations. In the rest of §3 we take up several other issues such as dissipation, generating functions, transmitted fields, etc.

In §4 we describe the code that we have developed for simulating reflection by randomly layered media. It is a “layer” code, which means that the medium properties are constant inside each layer and the reflection coefficient is calculated by repeated matrix multiplications, after Fourier transformation in time and Hankel transformation in the transverse space variables. The main difficulty is in reconstructing the field by inverting the Hankel transform. Our code is capable of accurately computing the highly fluctuating reflected signals which have been multiply scattered by hundreds or even thousands of subwavelength layers which constitute the microstructure. In addition to generating synthetic signals for our inversion algorithms, the layer code is a very useful tool for understanding the scope and robustness of the asymptotic theory. In the course of attempting to estimate local power spectral densities of reflected signals we found that there is a particularly effective way to do it via a local Fourier transform described in Appendix E.

In §5 we describe the code we have developed for solving the transport equations

that give us the statistics of the reflected signal. Except in the uniform background case (§3.7), it is not possible to solve them explicitly, so a numerical solution is necessary. This is particularly important in implementing the inversion algorithms of §6. Although we could use the probabilistic representation of §3.5 to construct Monte Carlo solutions of the W -equations [11], we have found that finite difference methods are better in calculating the amplitude of the solutions near the singular front, which is what we need. This has to be done after an appropriate regularization of the equations using a progressing wave representation. The main objective after that is to have an efficient vectorization of the code, which we describe in §5.3. We present numerical results for several large scale sound speed profiles, including some with turning points. The well-posedness theorem of §3.9 is verified nicely by the numerical calculations, which are indeed not sensitive to the singularities in the W -equations.

In §6 we formulate several inversion problems and give algorithms for their solution. The algorithms are based on a maximum likelihood estimate [12], [17] [12,17] of the unknown, large scale sound speed profile. This requires repeated solution of the W -equations. We use here the exponential law for the estimates of the local power spectral density. This law follows if the reflected signal is Gaussian and simulation results² support both the Gaussian distribution for the signal and the exponential one for the spectral estimate. We have not succeeded in showing rigorously that the reflection process is Gaussian, although a heuristic derivation is given in [10]. Not all of the macroscopic properties of the medium are obtainable from plane wave scattering data at a single angle of incidence. We show here, however, that for acoustic problems without fluctuations in the density, data from two distinct angles of incidence are sufficient for determining sound speed as a function of depth as well as one other function, which is an effective strength of the random fluctuations. We present simulation results for this problem.

General references for waves in layered media are [52]-[56], among many others. Waves in randomly layered media are analyzed in Klyatskin's books [8], [57]. Recent work on localization and other phenomena in scattering by randomly layered media can be found in [58]-[67] and the references cited in these papers. In all these works stochastic effects are analyzed when fluctuations in the parameters are weak and their mean values uniform. We review this briefly in Appendix C. The scaling we use here (§2.3), which is particularly appropriate for inverse problems, was introduced and used in our work. The transport equations, the W -equations, appear in the form of recursion equations in some works [57], [62], [63], [64], [66]. However, their central importance in the analysis of multiple scattering and the very rich structure that they have when geometrical acoustics is included was not recognized before.

2. Formulation of the problem.

2.1. The acoustic equations. We consider acoustic wave propagation in three space dimensions using the linearized equations of momentum and mass conservation for the velocity $\mathbf{u}(t, \mathbf{x})$ and pressure $p(t, \mathbf{x})$,

$$(2.1a) \quad \rho(z) \frac{\partial \mathbf{u}}{\partial t} + \sigma \mathbf{u} + \nabla p = \mathbf{F}(t, \mathbf{x}),$$

² See Fig. E.5 and [11].

$$(2.1b) \quad \frac{1}{K(z)} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0,$$

where $\rho = \rho(z)$ and $K = K(z)$ are the material properties (density and bulk modulus, respectively) of the medium layered in the z direction, $\mathbf{x} = (x, y, z)$ is the position vector, and $\sigma = \sigma(z)$ is the dissipation. This is the simplest model of dissipation since it could in addition be time-dependent, nonlocal, or have memory properties. The local sound speed is given by

$$(2.2) \quad c(z) = (K(z)/\rho(z))^{1/2}.$$

The forcing term is due to a point source and is given by

$$(2.3) \quad \mathbf{F}(t, \mathbf{x}) = f(t)\delta(x)\delta(y)\delta(z - z_s)\mathbf{e},$$

where $\mathbf{e} = (e_1, e_2, e_3)$ is a unit directivity vector, $(0, 0, z_s)$ is the source position and $f(t)$ is a smooth pulse-shape function with compact support. In the sequel we will assume a vertical source $\mathbf{e} = (0, 0, 1)$ to simplify the formulas. The random medium occupies a slab of thickness L in the lower half space $z \leq 0$ so that ρ and K are constant for $z > 0$. Below the slab, in $-\infty < z < -L$, the constitutive parameters are again constant. The source is located in the upper homogeneous medium; hence $z_s > 0$. The system (2.1) is completed by specifying zero initial conditions for $\mathbf{u}$ and p . Note that we could have considered an embedded source, i.e., $-L < z_s < 0$. The theory extends to this case - only the complexity increases. Note also that the material properties depend on the depth z only. This is the assumption of plane layering. If we introduce slowly varying lateral dependence, we model a locally layered medium. The theory is then more involved and is not discussed here. We will assume that the dissipation $\sigma = 0$ in most of the discussion that follows. We consider it explicitly in §3.10.

2.2. Effective medium theory. We are interested in the solution of (2.1) when $\rho(z)$ and $K(z)$ are random functions. The simplest approximate theory is the effective medium theory (homogenization) according to which long waves propagate as if they were in a homogeneous medium whose density and bulk modulus, the effective parameters, are constant. This approximation is valid in general [50], not necessarily for a layered medium, but is restricted to propagation over distances of the order of, or not much larger than, a wavelength. It is very useful in the analysis of standing wave problems in confined regions but cannot account for the localization phenomena, mentioned in the introduction, that we want to analyze. Nevertheless, effective medium theory is important because it tells us how the medium properties are averaged. For example, in the layered medium, is the average of ρ the appropriate effective density? Is the average of K the appropriate effective bulk modulus?

Let $\varepsilon > 0$ be the ratio of a length scale³ characterizing the variations of ρ and K to the wavelength. Assume that macroscopic lengths are comparable to the wavelength. We want to study the behavior as $\varepsilon \rightarrow 0$ of the solution of

$$(2.4a) \quad \rho\left(\frac{\mathbf{x}}{\varepsilon}\right) \frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F}(t, \mathbf{x}),$$

³ The thickness of the layers in the layered case, for example

$$(2.4b) \quad \frac{1}{K(\mathbf{x}/\varepsilon)} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

with zero initial conditions and with $\mathbf{F}$ smooth and independent of ε . Let us assume for simplicity that ρ and K are periodic functions with period 1 in each direction. Then a simple multiple scale expansion gives the effective or homogenized equation [50] and the random case is similar but the convergence proof is more complicated. We expand $\mathbf{u}$ and p in the form

$$(2.5) \quad p = p_0(t, \mathbf{x}) + \varepsilon p_1\left(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + \cdots,$$

$$(2.6) \quad \mathbf{u} = \mathbf{u}_0(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon \mathbf{u}_1\left(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + \cdots$$

with dependence on the variable $\mathbf{y} = \mathbf{x}/\varepsilon$ assumed to be periodic. We have anticipated here the fact that p_0 does not depend on $\mathbf{y}$. Inserting these expansions in (2.4) we note first that $\nabla_{\mathbf{y}} p_0 = \mathbf{0}$ since $p_0 = p_0(t, \mathbf{x})$ in (2.5) and that

$$(2.7) \quad \nabla_{\mathbf{y}} \cdot \mathbf{u}_0(t, \mathbf{x}, \mathbf{y}) = 0,$$

so we write $\mathbf{u}_0$ in the form

$$(2.8) \quad \mathbf{u}_0 = \sum_{i=1}^3 \left(\mathbf{e}_i + \nabla_{\mathbf{y}} \times \chi^i(\mathbf{y}) \right) v_i(t, \mathbf{x}),$$

so that (2.7) is satisfied identically and the functions $\chi^i(\mathbf{y})$ and $v_i(t, \mathbf{x})$ are to be determined. We have denoted the unit vectors in the coordinate directions by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

At the next level in the expansion we get

$$(2.9) \quad \nabla_{\mathbf{y}} p_1 + \rho(\mathbf{y}) \sum_{i=1}^3 \left(\mathbf{e}_i + \nabla_{\mathbf{y}} \times \chi^i(\mathbf{y}) \right) \frac{\partial v_i}{\partial t} + \nabla_{\mathbf{x}} p_0 = \mathbf{F},$$

$$(2.10) \quad \nabla_{\mathbf{y}} \cdot \mathbf{u}_1 + \frac{\partial p_0}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{u}_0 = 0.$$

The solvability conditions for these last two equations give

$$(2.11) \quad \nabla_{\mathbf{y}} \times \left[\rho(\mathbf{y}) \left(\mathbf{e}_i + \nabla_{\mathbf{y}} \times \chi^i(\mathbf{y}) \right) \right] = \mathbf{0}, \quad i = 1, 2, 3,$$

which determine the periodic functions $\chi^i(\mathbf{y})$ uniquely up to constant factors, and the effective equations

$$(2.12) \quad \sum_{j=1}^3 \rho_{ij}^* \frac{\partial v_j}{\partial t} + \frac{\partial p_0}{\partial x_i} = F_i(t, \mathbf{x}), \quad i = 1, 2, 3$$

and

$$(2.13) \quad \frac{1}{K^*} \frac{\partial p_0}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{v} = 0.$$

Here

$$(2.14) \quad \rho_{ij}^* = \left\langle \rho \left(\delta_{ij} + \sum_{k,l=1}^3 \epsilon_{ikl} \frac{\partial \chi_l^j}{\partial y_k} \right) \right\rangle,$$

where ϵ_{ikl} is the unit antisymmetric tensor and

$$(2.15) \quad \frac{1}{K^*} = \left\langle \frac{1}{K} \right\rangle,$$

where $\langle \rangle$ denotes average over a unit period cell.

It is easy to see from (2.11) and (2.14) that if the medium is layered in the z direction then the effective equations are

$$\begin{aligned} \left\langle \frac{1}{\rho} \right\rangle^{-1} \frac{\partial v_1}{\partial t} + \frac{\partial p_0}{\partial x} &= F_1(t, \mathbf{x}), \\ \left\langle \frac{1}{\rho} \right\rangle^{-1} \frac{\partial v_2}{\partial t} + \frac{\partial p_0}{\partial y} &= F_2(t, \mathbf{x}), \\ \langle \rho \rangle \frac{\partial v_3}{\partial t} + \frac{\partial p_0}{\partial z} &= F_3(t, \mathbf{x}), \end{aligned}$$

and

$$\left\langle \frac{1}{K} \right\rangle \frac{\partial p_0}{\partial t} + \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0.$$

These are an anisotropic version of the original acoustic equations (2.1) (with $\sigma = 0$), and the effective vertical and horizontal sound speeds are

$$c_v^2 = \left\langle \frac{1}{K} \right\rangle^{-1} \langle \rho \rangle^{-1},$$

$$c_h^2 = \left\langle \frac{1}{K} \right\rangle^{-1} \left\langle \frac{1}{\rho} \right\rangle.$$

Thus the effective density in the vertical direction is the average of ρ while the effective density in the horizontal direction is the harmonic mean of ρ . The effective bulk modulus is the harmonic mean of K in both cases.

2.3. Scaling and its physical interpretation. Wave localization phenomena occur when propagation distances are large compared to both the wavelength and the size of the inhomogeneities. If ε denotes the ratio of a length scale characteristic of small scale variations to a typical wavelength, as in the previous section, then localization phenomena appear typically over propagation distances that are of order ε^{-1} . It is convenient to scale the various length parameters relative to macroscopic propagation distances. Then macroscopic scales are of order 1, typical wavelengths of order ε and typical inhomogeneities⁴ of order ε^2 . It is in this scaling that the pulse can act as a probe, as mentioned in the introduction and explained in §6. Moreover,

⁴ In the example discussed in §1 and illustrated in Fig. 1.1a and 1.1b, the parameter ε is about .02.

in this scaling the propagation of the pulse relative to the large scale structure is dealt with in a geometrical acoustics approximation while the propagation relative to the microstructure by averaging of the stochastic equations.

We denote the constant acoustic parameters in the half space, $z > 0$ by ρ_0 , K_0 , and c_0 , where $c_0 = (K_0/\rho_0)^{1/2}$ is the sound speed. In the randomly layered slab, $-L < z \leq 0$, the parameters are denoted by the subscript 1. We assume $\rho = \rho_1$ is constant here⁵ and write the scaled bulk modulus in the form

$$(2.16) \quad \frac{1}{K} = \frac{1}{K_1(z)} \left(1 + \nu(z/\varepsilon^2) \right),$$

where $\nu(\cdot)$ is a zero-mean, stationary stochastic process, bounded by a constant less than 1. We could have $\nu = \nu(z, z/\varepsilon^2)$ depend on z as well as z/ε^2 so that its statistics vary on the macroscopic scales, but we ignore this for simplicity and reintroduce it in the final form of the results. Note the two-scale character of the bulk modulus: its mean value varies on the macroscopic scale while the fluctuations vary on the microscopic scale. Below the slab, the constant parameters are ρ_2 , K_2 , and c_2 .

In nondimensional variables the acoustic equations (2.1) are

$$(2.17a) \quad \rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p = \varepsilon^{1/2} f \left(\frac{t}{\varepsilon} \right) \delta(x) \delta(y) \delta(z - z_s) \mathbf{e},$$

$$(2.17b) \quad K^{-1} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

where

$$(2.18) \quad \rho(z) = \begin{cases} \rho_0, & z > 0, \\ \rho_1, & -L < z < 0, \\ \rho_2, & z < -L \end{cases}$$

and

$$(2.19) \quad K^{-1}(z) = \begin{cases} K_0^{-1}, & z > 0, \\ K_1^{-1}(z) \left(1 + \nu\left(\frac{z}{\varepsilon^2}\right) \right), & -L < z < 0, \\ K_2^{-1}, & z < -L. \end{cases}$$

The factor $\varepsilon^{1/2}$ multiplying the source term in (2.17) makes the total energy released by the pulsed point source independent of ε . Since the problem is linear, it has no other effect.

2.4. The integral representation of the reflected pressure field. The main steps in the derivation of the integral representation of the reflected pressure field are:

- Define a scaled Fourier transform in time and the horizontal space variables and get equations for $\hat{\mathbf{u}}$, $\hat{p}$ from the dimensionless, scaled acoustic equations (2.17);
- Decompose $\hat{\mathbf{u}}$ and $\hat{p}$ into upgoing and downgoing waves;
- Derive a Riccati equation for the time-harmonic reflection coefficient , ;
- Express the reflected pressure field p_{ref} in terms of , .

⁵ For simplicity. The variable case is treated in [10], for example.

Since the acoustic speed depends only on the depth z , we can Fourier transform the horizontal spatial variables and time to convert (2.17) into a first-order, two-point boundary value problem for a system of stochastic ordinary differential equations. Let⁶

$$(2.20) \quad \hat{f}(\omega) = \int e^{i\omega t} f(t) dt$$

and

$$(2.21a) \quad \hat{p}(\boldsymbol{\kappa}, \omega, z) = \iiint e^{i\omega(t - \boldsymbol{\kappa} \cdot \mathbf{x})/\varepsilon} p(t, \mathbf{x}, z) dt d\mathbf{x},$$

$$(2.21b) \quad \hat{\mathbf{u}}(\boldsymbol{\kappa}, \omega, z) = \iiint e^{i\omega(t - \boldsymbol{\kappa} \cdot \mathbf{x})/\varepsilon} \mathbf{u}(t, \mathbf{x}, z) dt d\mathbf{x}$$

where $\mathbf{x} = (x, y)$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ is the horizontal slowness vector. Assuming that the direction of the source $\mathbf{e} = \mathbf{e}_3$ is vertical and taking Fourier transforms of the scaled acoustic equations (2.17) we obtain the following system of scalar equations for $\hat{p}$ and the velocity field $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$:

$$(2.22a) \quad \hat{u}_1 = \frac{\kappa_1}{\rho} \hat{p},$$

$$(2.22b) \quad \hat{u}_2 = \frac{\kappa_2}{\rho} \hat{p},$$

$$(2.22c) \quad -\frac{i\omega}{\varepsilon} \rho \hat{u}_3 + \frac{d\hat{p}}{dz} = \varepsilon^{3/2} \hat{f}(\omega) \delta(z - z_s),$$

$$(2.22d) \quad -\frac{i\omega}{\varepsilon} \left(\frac{1}{K} - \frac{\kappa^2}{\rho} \right) \hat{p} + \frac{d\hat{u}_3}{dz} = 0,$$

where the horizontal velocity components have been eliminated from (2.22d) using (2.22a) and (2.22b) and $\kappa \doteq |\boldsymbol{\kappa}|$.

Let $\tau = \tau(z, \kappa)$ be the travel time relative to $z = 0$ for the oblique plane waves that make up the total field. It has the form

$$(2.23) \quad \tau(z, \kappa) = \frac{(1 - c_0^2 \kappa^2)^{1/2}}{c_0} z, \quad z > 0,$$

$$(2.24) \quad \tau(z, \kappa) = \int_0^z \frac{(1 - c_1^2(s) \kappa^2)^{1/2}}{c_1(s)} ds, \quad -L < z < 0.$$

We will assume that we stay above turning points and hence that the square roots are always real. The turning point problem is considered in detail in section 3.8. Define the acoustic impedances by

$$\zeta_0 \doteq \frac{\rho_0 c_0}{(1 - \kappa^2 c_0^2)^{1/2}}, \quad z \geq 0,$$

⁶ Unless explicitly written otherwise, the interval of integration for all integrals will be from $-\infty$ to $+\infty$.

$$(2.25) \quad \zeta_1(z) \doteq \frac{\rho_1 c_1(z)}{(1 - \kappa^2 c_1^2(z))^{1/2}}, \quad -L < z < 0,$$

$$\zeta_2 \doteq \frac{\rho_2 c_2}{(1 - \kappa^2 c_2^2)^{1/2}}, \quad z < -L,$$

and, denoting $\hat{u}_3$ by $\hat{u}$, let

$$(2.26a) \quad \hat{p} = (\zeta_1)^{1/2} \left[A e^{i\omega\tau/\varepsilon} - B e^{-i\omega\tau/\varepsilon} \right],$$

$$(2.26b) \quad \hat{u} = (\zeta_1)^{-1/2} \left[A e^{i\omega\tau/\varepsilon} + B e^{-i\omega\tau/\varepsilon} \right],$$

where A and B are amplitudes of upgoing and downgoing plane waves, respectively.

Substituting the ansatz (2.26) into (2.22), we get a system of stochastic ordinary differential equations for A and B in $-L < z < 0$,

$$(2.27a) \quad \frac{dA}{dz} = \frac{i\omega}{\varepsilon} n \left[A - e^{-2i\omega\tau/\varepsilon} B \right] + \frac{1}{2\zeta_1} \frac{d\zeta_1}{dz} e^{-2i\omega\tau/\varepsilon} B,$$

$$(2.27b) \quad \frac{dB}{dz} = -\frac{i\omega}{\varepsilon} n \left[-e^{2i\omega\tau/\varepsilon} A + B \right] + \frac{1}{2\zeta_1} \frac{d\zeta_1}{dz} e^{2i\omega\tau/\varepsilon} A,$$

where

$$(2.28) \quad n \left(z, \frac{z}{\varepsilon^2}, \kappa \right) \doteq \frac{1}{2c_1(z) (1 - c_1^2(z) \kappa^2)^{1/2}} \nu \left(z/\varepsilon^2 \right).$$

In this system the downgoing wave amplitude B is given at $z = 0$ and the upgoing wave amplitude A is given at $z = -L$.

Using the Fourier transforms (2.20)-(2.21), we can now write down the full representation of u_3 and p in $-L < z \leq 0$ as

$$(2.29)$$

$$p(t, \mathbf{x}, z) = \frac{1}{(2\pi\varepsilon)^3} \iiint (\zeta_1)^{1/2} \left[A e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} + \tau - t)/\varepsilon} - B e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - \tau - t)/\varepsilon} \right] \omega^2 d\boldsymbol{\kappa} d\omega,$$

$$u_3(t, \mathbf{x}, z) = \frac{1}{(2\pi\varepsilon)^3} \iiint (\zeta_1)^{-1/2} \left[A e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} + \tau - t)/\varepsilon} + B e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - \tau - t)/\varepsilon} \right] \omega^2 d\boldsymbol{\kappa} d\omega.$$

Define interface reflection coefficients

$$(2.30) \quad , \quad I(0) \doteq \frac{\zeta_1(0^-) - \zeta_0}{\zeta_1(0^-) + \zeta_0},$$

$$, \quad I(-L) \doteq \frac{\zeta_1(-L^+) - \zeta_2}{\zeta_1(-L^+) + \zeta_2}$$

and, inside the random slab, a local reflection coefficient

$$(2.31) \quad , (z) = A(z)/B(z), \quad -L < z < 0.$$

From equations (2.27) we obtain a Riccati equation for ,

$$(2.32a) \quad \frac{d,}{dz} = -\frac{i\omega}{\varepsilon} n \left\{ e^{-2i\omega\tau/\varepsilon} - 2, + e^{2i\omega\tau/\varepsilon}, 2 \right\} \\ + \frac{1}{2\zeta_1} \frac{d\zeta_1}{dz} \left\{ e^{-2i\omega\tau/\varepsilon} - e^{2i\omega\tau/\varepsilon}, 2 \right\}, \quad -L < z < 0,$$

$$(2.32b) \quad , |_{z=-L} = e^{-i2\omega\tau(-L)/\varepsilon}, \quad , I(-L),$$

where the initial condition (2.32b) is just the continuity condition for p and u_3 at $z = -L$. At $z = 0$ the continuity condition gives

$$(2.33) \quad , (0^+) = \frac{, (0^-) - , I(0)}{1 - , (0^-), I(0)}.$$

The solution of the Riccati equation (2.32) evaluated at $z = 0^-$ and substituted in (2.33) thus gives the pressure reflection coefficient at the surface $z = 0$. If we have $\rho_1(0^-) = \rho_0$ and $c_1(0^-) = c_0$, then the interface reflection coefficient $, I$ is identically zero and we refer to this as the “matched case”. In this case we get from (2.33) simply that $, (0^+) = , (0^-)$.

The quantity of interest is the reflected pressure itself at $z = 0$. This is the upgoing component A in (2.29). But $, = A/B$, so in terms of the reflection coefficient $, ,$ we can write the reflected pressure (at $z = 0$) as

$$(2.34) \quad p_{\text{refl}}(t, \mathbf{x}, 0) = \frac{1}{(2\pi\varepsilon)^3} \iiint e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon} , (0^+, \omega, \boldsymbol{\kappa}) (\zeta_0)^{1/2} B \omega^2 d\omega d\boldsymbol{\kappa}.$$

In $z > 0$, we can use the equations (2.22) and the jump conditions at the source ($z = z_s$) to solve for the equivalent plane wave amplitude B incident at $z = 0$. This calculation can be found in Appendix A. Substituting, we obtain the following integral representation of the reflected pressure field:

$$(2.35) \quad p_{\text{refl}}(t, \mathbf{x}, 0) = \frac{-1}{2(2\pi)^3} \frac{1}{\varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 e^{i\omega \left(\boldsymbol{\kappa} \cdot \mathbf{x} + c_0^{-1} (1 - \kappa^2 c_0^2)^{1/2} z_s - t \right) / \varepsilon} , (0^+, \omega, \boldsymbol{\kappa}) d\omega d\boldsymbol{\kappa}.$$

Assuming that $|, I, (0^-)| < 1$, we expand (2.33) in a geometric series which we insert in (2.35) to get

$$(2.36) \quad p_{\text{refl}}(t, \mathbf{x}, 0) = \frac{-1}{2(2\pi)^3} \frac{1}{\varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 e^{i\omega \left(\boldsymbol{\kappa} \cdot \mathbf{x} + c_0^{-1} (1 - \kappa^2 c_0^2)^{1/2} z_s - t \right) / \varepsilon} \\ \cdot \sum_{N=0}^{\infty} \left[(, (0^-, \omega, \boldsymbol{\kappa}))^N (, I(0))^{N+1} - (, (0^-, \omega, \boldsymbol{\kappa}))^{N+1} (, I(0))^N \right] d\omega d\boldsymbol{\kappa}.$$

We will be interested in the coherently reflected field

$$(2.37) \quad \langle p_{\text{refl}}(t, \mathbf{x}, 0) \rangle = E \{ p_{\text{refl}}(t, \mathbf{x}, 0) \},$$

with angular brackets denoting expectation over the realizations of the random medium, and in the reflected two-point intensity function

$$(2.38) \quad I(t, \mathbf{x}, \bar{t}, \bar{\mathbf{x}}) = \lim_{\varepsilon \downarrow 0} \left\langle p_{\text{refl}} \left(t + \frac{\varepsilon \bar{t}}{2}, \mathbf{x} + \frac{\varepsilon \bar{\mathbf{x}}}{2}, 0 \right) p_{\text{refl}} \left(t - \frac{\varepsilon \bar{t}}{2}, \mathbf{x} - \frac{\varepsilon \bar{\mathbf{x}}}{2}, 0 \right) \right\rangle.$$

The time and space offsets in the expectation are of order ε , which is the pulse width. For larger offsets the two factors in (2.38) are asymptotically uncorrelated in the limit. The definition of I anticipates these facts that come out of the asymptotic theory. Multiplying two expressions of the form (2.36) with integration variables $\omega_1, \omega_2, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ and changing variables to

$$(2.39) \quad \begin{aligned} \omega_1 &= \omega - \frac{\varepsilon h}{2}, \quad \omega_2 = \omega + \frac{\varepsilon h}{2}, \\ \boldsymbol{\kappa}_1 &= \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2}, \quad \boldsymbol{\kappa}_2 = \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2}, \end{aligned}$$

we get for I the asymptotic expression

$$(2.40) \quad \begin{aligned} I(t, \mathbf{x}, \bar{t}, \bar{\mathbf{x}}) &= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{4(2\pi)^3} \int \int \int |\hat{f}(\omega)|^2 \omega^2 e^{i\omega(\boldsymbol{\kappa} \cdot \bar{\mathbf{x}} - \bar{t})} \\ &\quad \left[, \begin{smallmatrix} N+M \\ I \end{smallmatrix} \tilde{W}^{N+1, M+1} - , \begin{smallmatrix} M+N+1 \\ I \end{smallmatrix} \tilde{W}^{N, M+1} \right. \\ &\quad \left. - , \begin{smallmatrix} M+N+1 \\ I \end{smallmatrix} \tilde{W}^{N+1, M} + , \begin{smallmatrix} M+N+2 \\ I \end{smallmatrix} \tilde{W}^{NM} \right] d\boldsymbol{\kappa} d\omega \end{aligned}$$

where $,_I = ,_I(0)$ and $\tilde{W}^{NM}$ is defined by

$$(2.41) \quad \begin{aligned} \tilde{W}^{NM}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) &\doteq \lim_{\varepsilon \downarrow 0} \frac{\omega^2}{(2\pi)^3} \int \int \int dh d\boldsymbol{\lambda} e^{iht - i\omega \boldsymbol{\lambda} \cdot \mathbf{x}} \\ &\quad \left\langle , \begin{smallmatrix} N \\ I \end{smallmatrix} (z, \omega - \frac{\varepsilon h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2}), \begin{smallmatrix} -M \\ I \end{smallmatrix} (z, \omega + \frac{\varepsilon h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2}) \right\rangle. \end{aligned}$$

The arguments of $\tilde{W}$ in (2.40) are evaluated at

$$(2.42) \quad \begin{aligned} z &= 0, \quad t \rightarrow t - \boldsymbol{\kappa} \cdot \mathbf{x} - c_0^{-1} (1 - c_0^2 \kappa^2)^{1/2} z_s, \\ \mathbf{x} &\rightarrow \mathbf{x} - \boldsymbol{\kappa} c_0 (1 - c_0^2 \kappa^2)^{1/2} z_s, \quad \boldsymbol{\kappa}, \text{ and } \omega \end{aligned}$$

and we will assume throughout that the interface reflection coefficients $,_I$ are real. They depend in general on the slowness vector $\boldsymbol{\kappa}$ and are complex for evanescent modes, but the evanescent spectrum contribution to the reflected signal is negligible in the asymptotic limit we consider here. We see, therefore, that we must compute the limit (2.41), to which we turn in the next section. The asymptotic behavior of the coherently reflected pressure (2.37) is obtained from knowledge of $\lim_{\varepsilon \downarrow 0} \left\langle , \begin{smallmatrix} N \\ I \end{smallmatrix} (0^-, \omega, \boldsymbol{\kappa}) \right\rangle$, which can also be obtained as a special case of $\tilde{W}^{NM}$, as we will see in §3.3.

3. Transport equations for the statistics of the reflected signal.

3.1. Equations for powers of the reflection coefficient. We will write the Riccati equation (2.32) as

$$(3.1) \quad \frac{d,}{dz} = -\frac{i\omega}{\varepsilon} n \left\{ e^{-2i\omega\tau/\varepsilon} - 2, + e^{2i\omega\tau/\varepsilon}, 2 \right\} + \text{negligible terms},$$

with the $(2\zeta_1)^{-1}$ term omitted since it is a rapidly oscillating term of order 1, with average value zero, and will vanish in the asymptotic analysis which entails averaging; details and references can be found in Appendix B.

The quantity of interest for the asymptotic analysis is $\tilde{W}^{NM}$ defined by (2.41) and we see that we need to derive from the nonlinear Riccati equation (3.1) equations for the moments of $\tilde{W}$ at two ω 's and two $\boldsymbol{\kappa}$'s. These moment equations are coupled, so we expect to obtain a hierarchy of equations. We therefore let

$$(3.2) \quad \tilde{W}^{NM} = \tilde{W}^N \left(z, \omega - \frac{\varepsilon h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2} \right), \tilde{W}^M \left(z, \omega + \frac{\varepsilon h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2} \right),$$

where N and M are integers as in the previous section, $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, and the offsets are proportional to ε as required in (2.41). From the Riccati equation (3.1), we find that the $\tilde{W}^{NM}$ satisfy the following infinite dimensional system of linear equations:

$$(3.3a) \quad \frac{d \tilde{W}^{NM}}{dz} = \frac{i\omega}{\varepsilon} n N \left\{ \tilde{W}^{NM} - e^{-2i\omega\tau/\varepsilon + ih\tau + i\omega \boldsymbol{\lambda} \cdot \nabla_{\boldsymbol{\kappa}} \tau} \tilde{W}^{N-1, M} \right. \\ \left. - e^{2i\omega\tau/\varepsilon - ih\tau - i\omega \boldsymbol{\lambda} \cdot \nabla_{\boldsymbol{\kappa}} \tau} \tilde{W}^{N+1, M} \right\} \\ - \frac{i\omega}{\varepsilon} n M \left\{ \tilde{W}^{NM} - e^{2i\omega\tau/\varepsilon + ih\tau + i\omega \boldsymbol{\lambda} \cdot \nabla_{\boldsymbol{\kappa}} \tau} \tilde{W}^{N, M-1} \right. \\ \left. - e^{-2i\omega\tau/\varepsilon - ih\tau - i\omega \boldsymbol{\lambda} \cdot \nabla_{\boldsymbol{\kappa}} \tau} \tilde{W}^{N, M+1} \right\} \\ + \text{negligible terms,}$$

$$(3.3b) \quad \tilde{W}^{NM} |_{z=-L} = (\tilde{W}^N(-L))^N (\tilde{W}^M(-L))^M e^{-2i\omega\tau(-L)(N-M)/\varepsilon} \\ \cdot e^{i\omega(N+M)\boldsymbol{\lambda} \cdot \nabla_{\boldsymbol{\kappa}} \tau} e^{i\omega(N+M)h\tau},$$

where $\nabla_{\boldsymbol{\kappa}} = (\partial/\partial\kappa_1, \partial/\partial\kappa_2)$ and the exponents are evaluated at offset slowness $\boldsymbol{\lambda} = \mathbf{0}$. Let $\mathbf{x} = (x, y)$ and define

$$(3.4) \quad \tilde{W}^{NM} (z, t + (N+M)\tau, \mathbf{x} - (N+M)\nabla_{\boldsymbol{\kappa}} \tau, \boldsymbol{\kappa}, \omega) \\ = \frac{\omega^2}{(2\pi)^3} \iiint e^{iht - i\omega \boldsymbol{\lambda} \cdot \mathbf{x}} \tilde{W}^{NM}(z, \omega, \boldsymbol{\kappa}, h, \boldsymbol{\lambda}) dh d\boldsymbol{\lambda}.$$

Then we can rewrite (3.3) as

$$(3.5) \quad \frac{\partial \tilde{W}^{NM}}{\partial z} + (N+M)\tau_z \left\{ \frac{\partial \tilde{W}^{NM}}{\partial t} - \frac{\nabla_{\boldsymbol{\kappa}} \tau_z}{\tau_z} \cdot \nabla \tilde{W}^{NM} \right\} \\ = \frac{i\omega}{\varepsilon} 2n(N-M) \tilde{W}^{NM} - \frac{i\omega}{\varepsilon} n \left\{ N e^{-2i\omega\tau/\varepsilon} \tilde{W}^{N-1, M} \right. \\ \left. + N e^{2i\omega\tau/\varepsilon} \tilde{W}^{N+1, M} - M e^{2i\omega\tau/\varepsilon} \tilde{W}^{N, M-1} - M e^{-2i\omega\tau/\varepsilon} \tilde{W}^{N, M+1} \right\} \\ + \text{negligible terms}$$

with initial conditions at $z = -L$ obtained from (3.3) and (3.4):

$$(3.6) \quad \tilde{W}^{NM} |_{z=-L} = (\tilde{W}^N(-L))^N (\tilde{W}^M(-L))^M e^{-2i\omega\tau(-L)(N-M)/\varepsilon} \delta(t) \delta(\mathbf{x}).$$

Note that the shift of the arguments (3.4) is zero at $z = 0$ because $\tau(0) = 0$ and hence the behavior of $E \left\{ \tilde{W}^{NM}(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) \right\}$ as $\varepsilon \rightarrow 0$ gives us $\tilde{W}^{NM}(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$.

3.2. The asymptotic limit of the moment equations. The moment equations (3.5) for the reflection coefficient are a system of equations for the $\tilde{\cdot}^{NM}$ of the form

$$(3.7) \quad \frac{dU^\varepsilon}{dz} = \frac{1}{\varepsilon} A \left(z, \frac{\tau(z)}{\varepsilon}, \frac{z}{\varepsilon^2} \right) U^\varepsilon + B \left(z, \frac{\tau(z)}{\varepsilon}, \frac{z}{\varepsilon^2} \right) U^\varepsilon,$$

where U^ε is a vector, A is a matrix (having zero mean with respect to its third, rapidly varying argument), and B is an operator that includes the drift terms in (3.5). In Appendix B we indicate how in the limit $\varepsilon \downarrow 0$ the mean of U^ε , denoted by $\bar{U}$, is given by

$$(3.8) \quad \frac{d\bar{U}(z)}{dz} = \overline{AA}(z) \bar{U}(z) + \bar{B}(z) \bar{U}(z),$$

where

$$(3.9) \quad \overline{AA}(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \int_0^\infty ds E \{ A(z, \tau, s) A(z, \tau, 0) \}$$

and

$$\bar{B}(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau E \{ B(z, \tau, s) \}.$$

In the expressions for the limit operators we have used the stationarity in s of $A(z, \tau, s)$ and $B(z, \tau, s)$. Equation (3.8) becomes the transport equation for

$$(3.10) \quad \tilde{W}^{NM} = \lim_{\varepsilon \rightarrow 0} E \{ \tilde{\cdot}^{NM} \}$$

when the formulas (3.9) are applied to (3.5).

The computation of the form that (3.8) takes when applied to (3.5) for the $\tilde{\cdot}^{NM}$ is straightforward. We record the results not for it, which corresponds to $\tilde{W}^{NM}$, (2.41), but for this quantity evaluated at the arguments (2.42), which is what we need for the reflected pressure intensity (2.40). We also remove the rapidly oscillating factor from the initial conditions and for simplicity we assume that $z_s = 0$, which means that the source is located at the origin (cf. (2.3)). We denote the $\tilde{W}$'s with shifted arguments and without the oscillatory factor by W^{NM} :

$$(3.11) \quad W^{NM}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) \doteq e^{2i\omega\tau(-L)(N-M)/\varepsilon} \tilde{W}^{NM}(z, t - \boldsymbol{\kappa} \cdot \mathbf{x}, \mathbf{x}, \boldsymbol{\kappa}, \omega).$$

We then find that W^{NM} satisfies the system

$$(3.12) \quad \begin{aligned} & \frac{\partial W^{NM}}{\partial z} + (N + M) \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^{NM}}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^{NM} \right\} \\ & - \frac{2\omega^2 \alpha N M}{c_1^2 (1 - c_1^2 \kappa^2)} \{ W^{N+1 M+1} - 2W^{NM} + W^{N-1 M-1} \} \\ & + \frac{6\omega^2 \alpha}{c_1^2 (1 - c_1^2 \kappa^2)} (N - M)^2 W^{NM} = 0, \end{aligned}$$

$$W^{NM} |_{z=-L} = (, {}_I(-L)){}^N (, {}_I(-L)){}^M \delta(t) \delta(\mathbf{x}),$$

where $N, M \geq 0$, $z < 0$, $z, t, \omega \in R^1$, $\mathbf{x} \in R^2$,

$$(3.13) \quad \alpha(z) \doteq \frac{1}{4} \int_0^\infty E \{ \nu(z, s) \nu(z, 0) \} ds,$$

and ν is the stochastic process that was defined⁷ by (2.16). Thus the noise intensity parameter α depends on z also. As an integral of a correlation function, α is nonnegative and we assume that it is positive in what follows.

In many situations we only need the diagonal part of W^{NM} , that is, W^{NN} for $N \geq 0$, which decouples from the off-diagonal part. Denoting W^{NN} by W^N , the transport equations (3.12) for the diagonal part are

$$(3.14) \quad \begin{aligned} \frac{\partial W^N}{\partial z} + 2N \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^N}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^N \right\} \\ - 2 \frac{\omega^2 \alpha N^2}{c_1^2 (1 - c_1^2 \kappa^2)} \{ W^{N+1} - 2W^N + W^{N-1} \} = 0, \end{aligned}$$

$$W^N|_{z=-L} = |_{, I(-L)}|^2 \delta(t) \delta(\mathbf{x}),$$

where $N \geq 0$, $z < 0$, $z, t, \omega \in R^1$, $\mathbf{x} \in R^2$.

3.3. The coherent reflected pressure. The reflected pressure is given by (2.36) and (2.37). Assume for simplicity that $|_{, I(0)} = 0$ which means that the medium properties are matched at the interface $z = 0$ and that the source is at the origin so that $z_s = 0$. Then

$$(3.15) \quad \langle p_{\text{refl}}(t, \mathbf{x}, 0) \rangle = \frac{1}{2(2\pi)^3} \frac{1}{\varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon} \langle |_{, (0^-, \omega, \boldsymbol{\kappa})} \rangle d\omega d\boldsymbol{\kappa}.$$

So we need the asymptotic behavior of $\langle |_{, (0^-, \omega, \boldsymbol{\kappa})} \rangle$ as $\varepsilon \rightarrow 0$. From (2.41) we see that

$$(3.16) \quad \langle |_{, (0^-, \omega, \boldsymbol{\kappa})} \rangle = \int \int \int dt d\mathbf{x} \tilde{W}^{10}(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$$

We now set $N = 1$, $M = 0$ in (3.12) and integrate over t and $\mathbf{x}$. The equations decouple and $\langle |_{, (0^-, \omega, \boldsymbol{\kappa})} \rangle$ behaves asymptotically like V which satisfies

$$(3.17) \quad \begin{aligned} \frac{\partial V}{\partial z} &= \frac{-6\alpha\omega^2}{c_1^2(1 - c_1^2\kappa^2)} V, \\ V|_{z=-L} &= |_{, I(-L)} e^{-2i\omega\tau(-L)/\varepsilon}. \end{aligned}$$

Here we have restored into the initial conditions the oscillatory factor in (3.11) that was removed from the initial conditions of (3.12). Let

$$(3.18) \quad \beta_1(L) = \int_{-L}^0 \frac{6\alpha}{c_1^2(1 - c_1^2\kappa^2)} dz.$$

Then integrating (3.17) we get to leading order in ε ,

$$(3.19) \quad \langle |_{, (0^-, \omega, \boldsymbol{\kappa})} \rangle \approx |_{, I(-L)} e^{-2i\omega\tau(-L)/\varepsilon} e^{-\omega^2 \beta_1(L)},$$

⁷ With macroscopic z dependence, in general.

and hence from (3.15)

$$(3.20) \quad \langle p_{\text{refl}}(t, \mathbf{x}, 0) \rangle \approx -\frac{1}{2(2\pi)^3 \varepsilon^{3/2}} \iint \int, {}_I(-L) \hat{f}(\omega) \omega^2 \cdot e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t - 2\tau(-L))/\varepsilon} e^{-\omega^2 \beta_1(L)} d\omega d\boldsymbol{\kappa}$$

The $\boldsymbol{\kappa}$ integral can be approximated by the stationary phase method [68]. To simplify the formulas and to see more clearly the physical meaning of the results let us assume that c_1 is a constant, which means that there are no large scale variations in the sound speed in the randomly layered region. Then from (2.24)

$$(3.21) \quad \tau(-L) = \frac{-L(1 - c_1^2 \kappa^2)^{1/2}}{c_1}$$

and the stationary phase approximation yields

$$(3.22) \quad \langle p_{\text{refl}}(t, \mathbf{x}, 0) \rangle \approx \frac{\sqrt{\varepsilon} {}_I(-L) \sin \theta}{8\pi L c_1} \cdot \frac{\partial}{\partial t} \left\{ \frac{1}{2\pi} \int d\omega \hat{f}(\omega) e^{-i\omega(t - 2L(c_1 \sin \theta)^{-1})/\varepsilon} e^{-\omega^2 \beta_1(L)} \right\},$$

where θ is defined by (cf. Fig. 3.1)

$$(3.23) \quad c_1 \kappa = \cos \theta = \frac{r/2}{\sqrt{L^2 + (r/2)^2}}$$

with $r = |\mathbf{x}|$. In the deterministic case $\alpha = 0$ we have $\beta_1(L) = 0$, in (3.18), and hence (3.22) gives

$$(3.24) \quad p_{\text{refl}}(t, \mathbf{x}, 0) \approx \frac{\sqrt{\varepsilon} {}_I(-L) \sin \theta}{8\pi L c_1} \frac{\partial}{\partial t} f\left(\frac{t - 2L(c_1 \sin \theta)^{-1}}{\varepsilon}\right),$$

which is the correct result for the reflection of a pulse from a plane interface in the geometrical acoustics approximation. Note that the interface reflection coefficient, ${}_I(-L)$ depends on θ and from (2.30), (2.25), and (3.23) we have

$$(3.25) \quad {}_I(-L) = \frac{\rho_1 c_1 \sin \theta_2 - \rho_2 c_2 \sin \theta_1}{\rho_1 c_1 \sin \theta_2 + \rho_2 c_2 \sin \theta_1},$$

where $\theta_1 = \theta$ and θ_2 is related to θ_1 by Snell's law

$$(3.26) \quad \frac{c_1}{c_2} = \frac{\cos \theta_1}{\cos \theta_2}.$$

When $\alpha > 0$ and so $\beta_1(L) > 0$, the result (3.22) tells us that the coherent reflected pressure pulse spreads with spreading factor proportional to $(\beta_1(L))^{-1/2}$. This is analogous to the results in [70] and [18] and is explored further in [69] and in §3.7.

It is also interesting to see what the W -equations (3.12) tell us about the behavior of the square of $p_{\text{refl}}(t, \mathbf{x}, 0)$ in the deterministic case. We consider this in the next section.

FIG. 3.1. Ray geometry.

3.4. The reflected pressure intensity. The two-point intensity function I for the reflected pressure is defined by (2.38) and is given in terms of $\tilde{W}^{NM}$ by (2.40). We rewrite it here in terms of W^{NM} , the solution of (3.12), taking into account the oscillatory phases in (3.11). We will assume again that $z_s = 0$ for simplicity. It is clear that when (3.11) is used in (2.40), the oscillatory factors eliminate all terms with $N \neq M$ in the limit $\varepsilon \rightarrow 0$. Therefore we can express I in terms of the solution W^N of (3.14) as follows:

$$(3.27) \quad I(t, \mathbf{x}; \bar{t}, \bar{\mathbf{x}}) = \frac{1}{4(2\pi)^3} \iint \int |\hat{f}(\omega)|^2 \omega^2 e^{i\omega[\boldsymbol{\kappa} \cdot \bar{\mathbf{x}} - \bar{t}]} \cdot \left[(, I(0))^2 W^0 + \left(, I(0) - \frac{1}{, I(0)} \right)^2 \sum_{N=1}^{\infty} (, I(0))^{2N} W^N \right] d\boldsymbol{\kappa} d\omega,$$

where W^N is evaluated at $z = 0$, t , $\mathbf{x}$, $\boldsymbol{\kappa}$, and ω . The $\boldsymbol{\kappa}$ integration is only over propagating modes for which $c_1 \boldsymbol{\kappa}$ is in the interval $[0, 1)$ and $, I(0)$ is real.

It is convenient to introduce the local power spectral density $\Lambda(t, \mathbf{x}, \omega)$ at $\mathbf{x}$ which is defined by

$$(3.28) \quad I(t, \mathbf{x}; \bar{t}, 0) = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 e^{-i\omega \bar{t}} \Lambda(t, \mathbf{x}, \omega) d\omega$$

so that

$$(3.29) \quad \Lambda(t, \mathbf{x}, \omega) = \frac{1}{4(2\pi)^2} \iint d\boldsymbol{\kappa} \cdot \left[, I(0)^2 W^0 + \left(, I(0) - \frac{1}{, I(0)} \right)^2 \sum_{N=1}^{\infty} , I(0)^{2N} W^N \right]$$

with the W^N evaluated at $z = 0$, t , $\mathbf{x}$, $\boldsymbol{\kappa}$, and ω . This quantity is most informative about the statistics of reflected signals because it characterizes their frequency content at a point $\mathbf{x}$ on the surface $z = 0$, within a time window centered at time t . The width of the time window is of order ε , which is also the width of the incident pulse.

A simple special case of the above general expression for the local power spectral density arises when the large-scale properties of the medium are matched at the interface $z = 0$ so that $,_I(0) = 0$. We then have

$$(3.30) \quad \Lambda(t, \mathbf{x}, \omega) = \frac{1}{4(2\pi)^2} \iint d\boldsymbol{\kappa} W^1(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) .$$

The function W^1 satisfies (3.14), which simplifies considerably when the medium is matched at $z = -L$ as well, so that $,_I(-L) = 0$. The initial conditions at $z = -L$ in (3.14) become in the matched case

$$(3.31) \quad W^N|_{z=-L} = \delta(t)\delta(\mathbf{x})\delta_{N0}.$$

We return to this problem in §3.7.

We will now calculate the asymptotic behavior of $(p_{\text{refl}}(t, \mathbf{x}, 0))^2$ for a pulse reflected from an interface at $z = -L$, in a deterministic medium so that $\alpha = 0$. We will use the W equations (3.14) to compute $I(t, \mathbf{x}; 0, 0)$ via (3.30) and (3.28). We will then compare the result with the asymptotic behavior of the square of $p_{\text{refl}}(t, \mathbf{x}, 0)$ given by (3.24). This is a simple but illuminating example that explains how the W equations capture the geometrical acoustics behavior at the level of intensities directly.

In the deterministic case the W equations (3.14) decouple and for W^1 we have

$$(3.32) \quad \frac{\partial W^1}{\partial z} + 2 \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^1}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^1 \right\} = 0,$$

$$W^1|_{z=-L} = |,_I(-L)|^2 \delta(t) \delta(\mathbf{x}).$$

The solution of this equation is

$$(3.33) \quad W^1(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) = (, _I(-L))^2 \delta \left(t - \frac{2L}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \right) \delta \left(\mathbf{x} - \frac{2L c_1}{(1 - c_1^2 \kappa^2)^{1/2}} \boldsymbol{\kappa} \right),$$

from which we obtain from (3.30)

$$(3.34) \quad \begin{aligned} \Lambda(t, \mathbf{x}, \omega) &= \frac{1}{4(2\pi)^2} \iint d\boldsymbol{\kappa} (, _I(-L))^2 \\ &\quad \cdot \delta \left(t - \frac{2L}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \right) \delta \left(\mathbf{x} - \frac{2L c_1}{(1 - c_1^2 \kappa^2)^{1/2}} \boldsymbol{\kappa} \right) \\ &= \left(\frac{, _I(-L)}{4\pi} \frac{\sin \theta}{2L c_1} \right)^2 \delta \left(t - \frac{2L}{c_1 \sin \theta} \right) \\ &= \left(\frac{, _I(-L)}{4\pi} \frac{1}{c_1^2 t} \right)^2 \delta \left(t - \frac{2L}{c_1 \sin \theta} \right) \end{aligned}$$

Here

$$(3.35) \quad c_1 \kappa = \cos \theta, \quad \sin \theta = \sqrt{1 - c_1^2 \kappa^2}$$

are determined as in Fig. 3.1, which is the ray diagram for the geometrical acoustics approximation

$$(3.36) \quad \cos \theta = \frac{r}{c_1 t}, \quad \sin \theta = \frac{2L}{c_1 t},$$

and the interface reflection coefficient $r_I(-L)$ is given by (3.25).

Note that the local power spectral density Λ is independent of ω in the deterministic case and, moreover, it is a delta function in the time variable. From (3.28) we see that the distribution limit of $(p_{\text{ref}}(t, \mathbf{x}, 0))^2$ as $\varepsilon \rightarrow 0$ is

$$(3.37) \quad I(t, \mathbf{x}; 0, 0) = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 \left(\frac{r_I(-L)}{4\pi} \frac{1}{2c_1^2 t} \right)^2 \delta \left(t - \frac{2L}{c_1 \sin \theta} \right) d\omega.$$

This is precisely what we get from (3.24) when we square and calculate the limit as $\varepsilon \rightarrow 0$ in the distribution sense. We have only to use Parseval's relation

$$\int (f'(t))^2 dt = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 d\omega$$

to complete the identification.

This example is important because it shows that the W -equations (3.14) contain all the geometrical acoustics information about reflected intensities in an efficient way. We get the result (3.37) directly from the W -equations whereas we had to use the stationary phase approximation to get (3.24) and then calculate the distribution limit of its square. When we deal with stochastic problems we want the geometrical acoustics to come into the picture in the most direct, geometrical way, and this is what the W -equations do.

3.5. Basic properties of the transport equations. We will now consider the W -equations (3.12) as an infinite-dimensional hyperbolic system and we will examine the existence of solutions and their properties. We will also give a probabilistic representation of the solution, which is very useful both analytically and in gaining a better physical understanding of what the W -equations mean. We will assume throughout this section that the medium is uniform in $-L \leq z \leq 0$ (i.e., $c_1(z)$ and $\alpha(z)$ are constants) and that there are no turning points (i.e., $1 - \kappa^2 c_1^2 > 0$). Turning points are considered in §3.8.

Let

$$(3.38) \quad \beta = \frac{2\alpha\omega^2}{c_1^2(1 - c_1^2\kappa^2)}, \quad v = c_1(1 - c_1^2\kappa^2)^{1/2}, \quad \mathbf{g} = \frac{c_1\kappa}{(1 - c_1^2\kappa^2)^{1/2}}$$

and define the operator Δ_D by

$$(3.39) \quad \Delta_D W^{NM} = \beta N M \{W^{N+1 \ M+1} - 2W^{N \ M} + W^{N-1 \ M-1}\}$$

and the diagonal operator (or potential) V by

$$(3.40) \quad V W^{NM} = -3\beta(N - M)^2 W^{NM}.$$

Then the W -equation (3.12) can be written in the form

$$(3.41) \quad \frac{\partial W^{NM}}{\partial z} = -(N + M) \left\{ \frac{1}{v} \frac{\partial W^{NM}}{\partial t} + \mathbf{g} \cdot \nabla W^{NM} \right\} + \Delta_D W^{NM} + V W^{NM}$$

in $-L \leq z \leq 0$ with

$$(3.42) \quad W^{NM}(-L, t, \mathbf{x}) = W_0^{NM}(t, \mathbf{x}).$$

The initial data W_0^{NM} are delta functions in (3.12). We will assume that the W_0^{NM} are smooth and bounded functions of $t \in \mathbf{R}^1$ and $\mathbf{x} \in \mathbf{R}^2$. It is easy to get distribution solutions of (3.41) from smooth ones since the coefficients in (3.41) do not depend⁸ on t and $\mathbf{x}$.

The operator Δ_D is the generator of a Markov chain $(N(z), M(z))$ on the nonnegative integers $N, M \geq 0$. The lines $N = 0$ and $M = 0$ are absorbing sets and the chain is degenerate because $N(z) - M(z)$ is constant and equal to its initial value $N - M$ at $z = -L$. Since the transition frequency of the Markov chain depends quadratically on the state variables, the chain is defined up to an “explosion” time z_∞ . We refer to the variable z as time because it plays that role in the Markov chain. It is, of course, the vertical coordinate. It can shown, as we indicate below, that with probability 1 $(N(z), M(z))$ is uniquely determined by its generator (3.39). We will call $(N(z), M(z))$ the canonical Markov chain of multiple scattering.

We can give a probabilistic representation of the solution $W^{NM}(z, t, \mathbf{x})$ of (3.41) in terms of the canonical Markov chain.

$$(3.43) \quad W^{NM}(z, t, \mathbf{x}) = E_{NM} \left\{ e^{\int_{-L}^z V(N(s), M(s)) ds} \cdot W_0^{N(z), M(z)} \left(t - \frac{1}{v} \int_{-L}^z (N(s) + M(s)) ds, \mathbf{x} - \mathbf{g} \int_{-L}^z (N(s) + M(s)) ds \right) \right\}.$$

Here E_{NM} stands for expectation over the trajectories of the Markov chain that start from N, M at $z = -L$. The right-hand side of (3.43) is well defined provided $|W_0^{NM}|$ is uniformly bounded, since the Markov chain is well defined. If $W^{NM}(z, t, \mathbf{x})$ is defined by (3.43), we must show that it satisfies (3.41) in a suitable sense. The uniqueness of bounded solutions of (3.41) follows from the maximum principle, which is immediately clear from (3.43) (since $V \leq 0$) and can be established directly. To show that (3.43) satisfies (3.41) we use the familiar renewal argument [71], [11] which converts (3.41) into an equivalent integral equation. It is in this sense that (3.43) solves (3.41). Solutions with distribution-valued initial data are obtained as distribution derivatives in t and $\mathbf{x}$ of smooth solutions since the coefficients are independent of t and $\mathbf{x}$.

The solution of the W -equations enters in the determination of the reflected pressure intensity through equations (3.27) and it is the asymptotic limit of moments of integrals of the reflection coefficient (z, ω, κ) defined by (3.1), when suitably interpreted as we explain in appendix B. It is somewhat surprising, therefore, that the hierarchy of moments equations, the W -equations, admits a probabilistic representation of their solutions. It is also difficult to guess the physical mechanism that emerges from our asymptotic limit. But once (3.43) is established its physical interpretation is important and we discuss it in more detail in §3.6, 3.7 and 3.8.

⁸ They depend on z in general but not in this section for simplicity. For nonplane layered media they depend on $\mathbf{x}$ as well.

Along with the W -equations (3.12), or in the form (3.41), we have their adjoint system relative to the inner product

$$(3.44) \quad \langle W(z), U(z) \rangle = \int dt \int d\mathbf{x} \sum_{NM} W^{NM}(z, t, \mathbf{x}) U^{MN}(z, t, \mathbf{x}).$$

Suppose $U^{NM}(z, t, \mathbf{x})$ satisfies the system

$$(3.45) \quad \frac{\partial U^{NM}}{\partial z} + (N + M) \left\{ \frac{1}{v} \frac{\partial U^{NM}}{\partial t} + \mathbf{g} \cdot \nabla U^{NM} \right\} + (\Delta_D^* + V) U^{NM} = 0$$

in $-L \leq z \leq 0$ with

$$(3.46) \quad U^{NM}(0, t, \mathbf{x}) = U_0^{NM}(t, \mathbf{x})$$

and with Δ_D^* the operator adjoint to Δ_D

$$(3.47) \quad \Delta_D^* W_{NM} = \beta \left\{ (N + 1)(M + 1) W^{N+1, M+1} - 2NM W^{NM} + (N - 1)(M - 1) W^{N-1, M-1} \right\}$$

Then

$$\frac{d}{dz} \langle W(z), U(z) \rangle = 0$$

and so

$$(3.48) \quad \begin{aligned} \sum_{NM} \int \int dt d\mathbf{x} U_0^{NM}(t, \mathbf{x}) W^{NM}(0, t, \mathbf{x}) \\ = \sum_{NM} \int \int dt d\mathbf{x} U^{NM}(-L, t, \mathbf{x}) W_0^{NM}(t, \mathbf{x}). \end{aligned}$$

Thus, if

$$U_0^{NM}(t, \mathbf{x}) = \delta_{NN_0} \delta_{MM_0} \delta(t - t_0, \mathbf{x} - \mathbf{x}_0)$$

then

$$(3.49) \quad W^{N_0 M_0}(0, t_0, \mathbf{x}_0) = \sum_{NM} ({}_{-L}(-L))^N ({}_{-L}(-L))^M U^{NM}(-L, 0, 0),$$

which means that the solution of the W -equations (3.12) can be obtained also by solving the adjoint W -equations (3.45). The adjoint equations are more convenient to use in the solution of inverse problems (§6).

A simple property of the solution of the W -equations that follows immediately from the probabilistic representation (3.43) is that

$$(3.50) \quad \lim_{L \rightarrow \infty} W^{NM} = 0, \quad N \neq M.$$

This is clear because $V(N, M) = -\beta(N - M)^2$ by (3.40) and $N(z) - M(z)$ is constant. Hence the exponential factor in (3.43) is a decaying deterministic exponential if $N \neq M$ and it goes to zero as $L \rightarrow \infty$. In view of this and also because of the oscillatory

factors in (3.11) it suffices for most applications to consider only the diagonal of the W -equations (3.14).

Let W^N stand for W^{NN} . The next property that we want to prove is that $W^N(0, t, \mathbf{x})$ is independent of the reflection coefficient $|, _I(-L)|$ if $2L > c_1 t$. This means that the reflected pressure intensity at some time t and position $\mathbf{x}$ on the surface of the randomly layered medium does not depend on the reflectivity properties of the medium at depth $z = -L$ if t is less than the time it takes for the wave to reach this depth and return to the surface. To prove this property of $W^N(0, t, \mathbf{x})$ recall that W^N satisfies the equation

$$(3.51) \quad \frac{\partial W^N}{\partial z} + 2N \left\{ \frac{1}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^N}{\partial t} + \frac{c_1 \kappa}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^N \right\} - \beta N^2 \{W^{N+1} - 2W^N + W^{N-1}\} = 0$$

$$(3.52) \quad W^N(-L, t, \mathbf{x}) = |, _I(-L)|^{2N} \delta(t) \delta(\mathbf{x}).$$

The probabilistic representation of the solution is

$$(3.53) \quad W^N(z, t, \mathbf{x}) = E_N \left\{ |, _I(-L)|^{2N(z)} \delta \left(t - \frac{2}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \int_{-L}^z N(s) ds \right) \cdot \delta \left(\mathbf{x} - \frac{2c_1 \kappa}{(1 - c_1^2 \kappa^2)^{1/2}} \int_{-L}^z N(s) ds \right) \right\},$$

which follows from (3.43), with the delta functions interpreted as distribution derivatives of Heaviside functions. Normally incident waves corresponding to $\kappa = \mathbf{0}$ travel fastest so we need only consider them. Then $W^N(0, t, \mathbf{x})$ simplifies to

$$E_N \left\{ |, _I(-L)|^{2N(0)} \delta \left(t - \frac{2}{c_1} \int_{-L}^0 N(s) ds \right) \right\} \delta(\mathbf{x}).$$

There are two kinds of paths that contribute to this expectation: those which start from $N(-L) = N \neq 0$ and stay positive up to $z = 0$ and those that become zero at some z in $-L < z \leq 0$ and therefore $N(0) = 0$. The first set of paths does not contribute to the expectation because $N(z) \geq 1$ for $-L \leq z \leq 0$ and $2L > c_1 t$, so the delta function never acts. Thus only the paths for which $N(0) = 0$ contribute and then W^N does not depend on $|, _I(-L)|$ as we wanted to show.

Now we want to return to the basic property of the canonical Markov chain which is that it is defined for all z : there is no finite z blowup. This is shown by a duality identity that is very useful in analyzing the Markov chain. We first note that the reflection coefficients $, _I$ are complex numbers of modulus less than or equal to 1 so they can be represented in the form

$$(3.54) \quad , _I = \tanh \frac{\bar{\theta}}{2} e^{i\phi}$$

with $\bar{\theta} \geq 0$ and $0 \leq \phi \leq 2\pi$. Let

$$(3.55) \quad V^{NM}(\bar{\theta}, \phi) = \left(\tanh \frac{\bar{\theta}}{2} e^{i\phi} \right)^N \left(\tanh \frac{\bar{\theta}}{2} e^{-i\phi} \right)^M = \left(\tanh \frac{\bar{\theta}}{2} \right)^{N+M} e^{i(N-M)\phi}$$

and introduce the diffusion generator

$$(3.56) \quad \Delta_C = \alpha \left[\frac{1}{\sinh \bar{\theta}} \frac{\partial}{\partial \bar{\theta}} (\sinh \bar{\theta} \frac{\partial}{\partial \bar{\theta}}) + \frac{1}{\sinh^2 \bar{\theta}} \frac{\partial^2}{\partial \phi^2} \right] + \alpha \frac{\partial^2}{\partial \phi^2},$$

acting on smooth functions of $\bar{\theta} \geq 0$ and $\phi \in [0, 2\pi]$, which are periodic in ϕ . If in the unit disc $|z| \leq 1$ we introduce polar coordinates (3.54) with the metric

$$(3.57) \quad |dz|^2 = \sinh^2 \bar{\theta} (d\bar{\theta})^2 + (d\phi)^2$$

and volume element

$$(3.58) \quad dv = \sinh \bar{\theta} d\bar{\theta} d\phi,$$

then we can write Δ_C in the form

$$(3.59) \quad \Delta_C = \alpha \Delta + \alpha \frac{\partial^2}{\partial \phi^2},$$

where Δ is the Laplace-Beltrami operator on the unit disc with metric (3.57), the hyperbolic disc.

The key observation in duality is the identity

$$(3.60) \quad \Delta_C V^{NM}(\bar{\theta}, \phi) = (\Delta_D + V) V^{NM}(\bar{\theta}, \phi),$$

where Δ_C acts on V^{NM} as a function of $\bar{\theta}, \phi$ and $\Delta_D + V$ as a function of the discrete indices N, M . This identity is easily verified by direct calculation and with the help of the identity

$$\frac{4}{\sinh^2 \bar{\theta}} = \tanh^2 \frac{\bar{\theta}}{2} - 2 + \frac{1}{\tanh^2 \frac{\bar{\theta}}{2}}.$$

The subscripts C and D on Δ stand for continuous and discrete, respectively. Let $(\bar{\theta}(z), \phi(z))$ be the diffusion process generated by the partial differential operator (3.59). We say that the random walk $(N(z), M(z))$ and the diffusion $(\bar{\theta}(z), \phi(z))$ are dual processes relative to the functions $V^{NM}(\bar{\theta}, \phi)$ if

$$(3.61) \quad E_{NM} \left\{ V^{N(z), M(z)}(\bar{\theta}, \phi) e^{\int_{-L}^z V(N(s), M(s)) ds} \right\} = E_{\bar{\theta}, \phi} \left\{ V^{NM}(\bar{\theta}(z), \phi(z)) \right\}$$

where the expectation on the left is relative to the law of the random walk starting at N, M when $z = -L$ while the expectation on the right is relative to the diffusion process starting at $\bar{\theta}, \phi$ at $z = -L$. We can verify directly that (3.60) is simply the infinitesimal version of (3.61). We see from (3.61) that properties of the random walk can be inferred from corresponding ones of the diffusion process and conversely. Duality is therefore like a transform, and desired properties may sometimes be easier to get, or more transparent, in the transformed problem. In §3.11 we discuss further the use of duality in explicit computations.

The process $\bar{\theta}(z)$ is transient, which means that as $z \rightarrow \infty$, $\bar{\theta}(z) \rightarrow \infty$ with probability 1. This can be seen from the Ito stochastic equation for $\bar{\theta}(z)$,

$$d\bar{\theta}(z) = \alpha \coth \bar{\theta}(z) dz + \sqrt{2\alpha} db(z),$$

that comes from (3.56). Here $b(z)$ is the standard one dimensional Brownian motion. The positive drift in this equation tends asymptotically to a constant and this gives the result $z^{-1}\bar{\theta}(z) \rightarrow \alpha$ with probability 1 as $z \rightarrow \infty$. A general class of asymptotic results like this is given in [72].

Assuming that $N = M$ initially so that $N(z) = M(z)$ for all z , we restrict attention to $N(z)$. Because of (3.61) we see that

$$(3.62) \quad E_N \left\{ \left(\tanh \frac{\bar{\theta}}{2} \right)^{N(z)} \right\} = E_{\bar{\theta}} \left\{ \left(\tanh \frac{\bar{\theta}(z)}{2} \right)^N \right\},$$

which is a simpler form of duality between the two processes $N(z)$ and $\bar{\theta}(z)$. Now the fact that $N(z)$ is defined for all z is an easy consequence of (3.62). To see the type of argument that is involved we will use (3.62) to prove that with probability 1 $N(z) \rightarrow 0$ as $z \rightarrow \infty$, which we use in §3.8. Note first that $\bar{\theta}(z) \rightarrow \infty$ as $z \rightarrow \infty$ with probability 1. Hence the right side of (3.62) goes to 1 for any $N < \infty$ and $\bar{\theta} < \infty$. The only way the left side can be identically equal to 1 for all $\bar{\theta} < \infty$ is if $\lim_{z \rightarrow \infty} N(z) = 0$ with probability 1. Duality makes the proof of absorption at $N = 0$ very simple.

3.6. Reflected pressure intensity for a uniform background. We will calculate the power spectral density $\Lambda(t, \mathbf{x}, \omega)$ and intensity function $I(t, \mathbf{x}; \bar{t}, 0)$ given by (3.29) and (3.28), respectively, for a randomly layered slab with c_1 constant, which means that the large scale sound speed profile is uniform. We assume again that $z_s = 0$ so the source is at the origin for simplicity, and that it is vertical $\mathbf{e} = (0, 0, 1)$ (cf. (2.3)). We must solve the W equations (3.14) and then calculate the κ integral (3.29) for Λ . First we note that the equations and the initial conditions in (3.14) are axially symmetric so we can eliminate angular dependence. Let $|\mathbf{x}| = r$ and recall that $|\kappa| = \kappa$. Then it is easily seen that the angular integral of W^N , which we denote again by W^N , depends on r and κ only and for $t \geq 0$ it satisfies

$$(3.63) \quad \frac{\partial W^N}{\partial z} + 2N \left\{ \frac{1}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^N}{\partial t} + \frac{c_1 \kappa}{(1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^N}{\partial r} \right\} - \beta N^2 \{ W^{N+1} - 2W^N + W^{N+1} \} = 0,$$

$$(3.64) \quad W^N|_{z=-L} = |_{, I(-L)}|^{2N} \delta(t) \frac{\delta(r)}{r}.$$

The parameter β is defined by (3.38), depends on κ , and is zero for deterministic problems. The local power spectral density (3.29) becomes

$$(3.65) \quad \Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2} \int_0^{1/c_1} \kappa d\kappa \left[|_{, I(0)} - \frac{1}{|_{, I(0)}} \right]^2 \sum_{N=1}^{\infty} (|_{, I(0)})^{2N} W^N \Big],$$

where we explicitly ignore the evanescent spectrum and omit the term W^0 since it is proportional to a delta function at zero and does not appear for $t > 0$. We also write the probabilistic representation (3.53) of W^N in polar coordinates:

$$(3.66) \quad W^N(0, t, r, \kappa, \omega) = \frac{1}{r} E_N \left\{ |_{, I(-L)}|^{2N(0)} \delta \left(t - \frac{2}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \int_{-L}^0 N(s) ds \right) \cdot \delta \left(r - \frac{2c_1 \kappa}{(1 - c_1^2 \kappa^2)^{1/2}} \int_{-L}^0 N(s) ds \right) \right\}.$$

Let us consider the case $L > c_1 t/2$, which means that the waves have not reached the bottom of the stack $z = -L$ by time t . Then we may as well set $z = -\infty$ since reflection statistics in $[0, t]$ are unaffected. Let

$$(3.67) \quad \mu = \int_{-\infty}^0 N(s) ds,$$

which is a random variable that is finite with probability 1 because $N(z)$ goes to zero (and stays at zero thereafter) with probability 1. This follows from the duality (3.62). In the case $L > c_1 t/2$, only those realizations for which $N(0) = 0$ contribute, and W^N has the form

$$(3.68) \quad W^N(0, t, r, \kappa, \omega) = \frac{1}{r} E_N \left\{ \delta \left(t - \frac{2\mu}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \right) \cdot \delta \left(r - \frac{2c_1 \kappa \mu}{(1 - c_1^2 \kappa^2)^{1/2}} \right) \right\}.$$

We must therefore compute the probability density of the random variable μ when $N(-L) = N$, which we denote by $P_N(\mu; \beta)$. Then for $0 < t < 2L/c_1$, $\Lambda(t, r, \omega)$ is given by

$$(3.69) \quad \Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2} \int_0^{1/c_1} \kappa d\kappa \left(, I(0) - \frac{1}{, I(0)} \right)^2 \sum_{N=1}^{\infty} (, I(0))^{2N} \cdot \int_0^{\infty} d\mu P_N(\mu; \beta) \frac{1}{r} \delta \left(t - \frac{2\mu}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \right) \delta \left(r - \frac{2c_1 \kappa \mu}{(1 - c_1^2 \kappa^2)^{1/2}} \right).$$

Recall that β defined by (3.38) depends on κ and that $, I(0)$ also depends on κ in general (cf. (2.25) and (2.30)). If we change variables in (3.69) by letting

$$c_1 \kappa = \cos \theta, \quad (1 - c_1^2 \kappa^2)^{1/2} = \sin \theta,$$

change the argument of delta function of t , and evaluate the integral in μ at the argument of the t delta function we get

$$(3.70) \quad \begin{aligned} \Lambda(t, r, \omega) &= \frac{1}{8(2\pi)^2} \frac{1}{c_1 r} \int_0^{\pi/2} d\theta \sin^2 \theta \cos \theta \delta(r - c_1 t \cos \theta) \\ &\quad \cdot \left(, I(0) - \frac{1}{, I(0)} \right)^2 \sum_{N=1}^{\infty} (, I(0))^{2N} P_N \left(\frac{c_1 t \sin \theta}{2}; \frac{2\alpha\omega^2}{c_1^2 \sin^2 \theta} \right) \\ &= \frac{1}{8(2\pi)^2} \frac{\sqrt{1 - (\frac{r}{c_1 t})^2}}{c_1 (c_1 t)^2} \left(, I(0) - \frac{1}{, I(0)} \right)^2 \sum_{N=1}^{\infty} (, I(0))^{2N} \\ &\quad \cdot P_N \left(\frac{c_1 t \sqrt{1 - (\frac{r}{c_1 t})^2}}{2}; \frac{2\alpha\omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)} \right), \end{aligned}$$

where the θ at which the integral is evaluated is as in the ray diagram in Fig. 3.1 and $, I(0)$ is a function of θ , (3.80).

When the medium is matched at $z = 0$ so that $, I(0) = 0$, (3.70) reduces to

$$(3.71) \quad \Lambda(t, r, \omega) = \frac{1}{8(2\pi)^2} \frac{\sqrt{1 - (\frac{r}{c_1 t})^2}}{c_1 (c_1 t)^2} P_1 \left(\frac{c_1 t \sqrt{1 - (\frac{r}{c_1 t})^2}}{2}; \frac{2\alpha\omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)} \right)$$

and this should be contrasted with the deterministic reflection from an interface at depth $-L$ given by (3.34). The expression (3.71) is an average of reflections from an “interface” at depth μ (defined by (3.67)).

The probability density $P_N(\mu; \beta)$ can be computed explicitly as follows. First note that because of the homogeneity in z of the Markov chain $N(z)$ we may replace the range of integration in (3.67) by $[0, \infty)$ without altering the law of μ . Let

$$(3.72) \quad \mu(z) = \bar{\mu} - \int_0^z N(s) ds$$

where $\bar{\mu}$ is a constant. With $0 \leq \mu \leq 1$ and f in $C_0^\infty(\mathbf{R})$ let

$$V^N(z, \bar{\mu}) = E_N \left\{ f(\mu(z)), \begin{matrix} N(z) \\ 0 \end{matrix} \right\}$$

Then V^N satisfies

$$(3.73) \quad \begin{aligned} \frac{\partial V^N}{\partial z} + N \frac{\partial V^N}{\partial \bar{\mu}} &= \beta N^2 \left\{ V^{N+1} - 2V^N + V^{N-1} \right\}, \\ V^N|_{z=0} &= f(\bar{\mu}), \begin{matrix} N \\ 0 \end{matrix}. \end{aligned}$$

Since $N(z) \rightarrow 0$ with probability 1 as $z \rightarrow \infty$,

$$V^N(z, \bar{\mu}) \rightarrow E_N \{ f(\bar{\mu} - \mu) \} = \int_0^\infty f(\bar{\mu} - \mu) P_N(\mu; \beta) d\mu,$$

where μ is defined by (3.67). P_N therefore satisfies

$$(3.74) \quad \begin{aligned} \frac{\partial P_N}{\partial \bar{\mu}} &= \beta N \{ P_{N+1} - 2P_N + P_{N-1} \}, \quad N \geq 1, \bar{\mu} \geq 0, \\ P_0 &= \delta(\bar{\mu}). \end{aligned}$$

The solution of (3.74) is

$$(3.75) \quad P_N(\mu, \beta) = \frac{\partial}{\partial \mu} \left[\left(\frac{\beta \mu}{1 + \beta \mu} \right)^N H(\mu) \right],$$

as can be verified by direct calculation. Here $H(\mu)$ is the Heaviside step function.

Returning to (3.70), summing the geometrical series, and using (3.38) we get for $t > 0$

$$(3.76) \quad \Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2} \frac{1}{c_1(c_1 t)^2} \frac{(1 - (I(0))^2)^2 \frac{\alpha \omega^2}{c_1^2(1 - (\frac{r}{c_1 t})^2)^{1/2}}}{\left[1 + (1 - (I(0))^2) \frac{\alpha \omega^2}{c_1^2(1 - (\frac{r}{c_1 t})^2)^{1/2}} c_1 t \right]^2},$$

which in the matched case, $I(0) = 0$ reduces to

$$(3.77) \quad \Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2} \frac{1}{c_1(c_1 t)^2} \frac{\frac{\alpha \omega^2}{c_1^2(1 - (\frac{r}{c_1 t})^2)^{1/2}}}{\left[1 + \frac{\alpha \omega^2}{c_1^2(1 - (\frac{r}{c_1 t})^2)^{1/2}} c_1 t \right]^2}.$$

This formula is particularly important in this work because it is used as a benchmark to validate the numerical calculations of §4 and 5. To understand it better we now

express it in terms of a localization length as follows. For fixed t and r let θ be determined by the ray diagram (Fig. 3.1) and define the reduced localization length

$$(3.78) \quad l(\omega, \theta) = \frac{c_1^2 \sin^2 \theta}{2\alpha\omega^2} = \tilde{l}(\omega) \sin^2 \theta.$$

In terms of l (3.76) becomes

$$(3.79) \quad \Lambda(t, r, \omega) = \frac{1}{8(2\pi)^2} \frac{\sin \theta}{c_1(c_1 t)^2} \frac{(1 - (\gamma_I(0))^2)^2 l(\omega, \theta)}{\left[l(\omega, \theta) + (1 - (\gamma_I(0))^2) \frac{c_1 t \sin \theta}{2} \right]^2},$$

where the interface reflection coefficient $\gamma_I(0)$ depends on θ . From (2.30), (2.25), and (3.35) it has the form

$$(3.80) \quad \gamma_I(0) = \frac{\rho_1 c_1 \sin \theta_0 - \rho_0 c_0 \sin \theta_1}{\rho_1 c_1 \sin \theta_0 + \rho_0 c_0 \sin \theta_1},$$

where $\theta_1 = \theta$ and θ_0 is related to θ_1 by Snell's law

$$(3.81) \quad \frac{c_1}{c_0} = \frac{\cos \theta_1}{\cos \theta_0}.$$

For time harmonic plane waves of frequency ω and normally incident on a randomly layered medium, the localization length $\tilde{l}(\omega) = l(\omega, \pi/2)$ characterizes the exponential decay of fields inside the random medium and away from the interface. The natural appearance of the reduced localization length $l(\omega, \theta)$ in the power spectral density Λ in (3.79) allows us to give an interesting physical interpretation [15] of localization phenomena in the time domain. Consider the case $\gamma_I(0) = 0$ for simplicity. Then the maximum of $\Lambda(t, r, \omega)$ as a function of ω occurs at that frequency $\omega_{\max}(t, r)$ (or $\omega_{\max}(t, \theta)$) for which

$$(3.82) \quad l(\omega_{\max}, \theta) = \frac{c_1 t \sin \theta}{2}.$$

This is a resonance condition that among other things allows us to estimate efficiently localization lengths from the statistics of reflected signals by simply locating maxima of estimated local power spectral densities. From (3.82), (3.78), and Fig. 3.1 we see that

$$(3.83) \quad \omega_{\max} = \left(\frac{c_1}{\alpha t} \right)^{1/2} \left(1 - \left(\frac{r}{c_1 t} \right)^2 \right)^{1/4},$$

which means that noise power in the reflected signal shifts eventually to lower frequencies as t increases, with r fixed. It also shifts to lower frequencies as r increases, with t fixed and with $r < c_1 t$.

3.7. Analysis of the reflected signal near a coherent arrival. Let us consider again a uniform background and a source at the origin, as in §3.6, but now allow the interface at $z = -L$ to affect the reflected signal. We will therefore consider times $t > 2L/c_1$, longer than it takes the waves to reach depth L and return. For simplicity we will assume that $\gamma_I = 0$ so that the medium is matched at the interface $z = 0$.

The local power spectral density Λ of the reflected signal on the surface $z = 0$ is given by (3.65) with $,_I(0) = 0$,

$$(3.84) \quad \Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2} \int_0^{1/c_1} \kappa \, d\kappa W^1(0, t, r, \kappa, \omega).$$

Here $W^N(z, t, r, \kappa, \omega)$ is the solution of (3.63), (3.64). If we let

$$(3.85) \quad \mu_L = \int_{-L}^0 N(s) \, ds$$

then the probabilistic representation of W^1 in terms of the random walk $N(z)$ generated by the operator on the second line of (3.63) is

$$(3.86) \quad W^1(0, t, r, \kappa, \omega) = \frac{1}{r} E_1 \left\{ (, _I(-L))^{2N(0)} \delta \left(t - \frac{2\mu_L}{c_1(1 - c_1^2 \kappa^2)^{1/2}} \right) \cdot \delta \left(r - \frac{2c_1 \kappa \mu_L}{(1 - c_1^2 \kappa^2)^{1/2}} \right) \right\}.$$

The average on the right is over all paths $N(z)$, $-L \leq z \leq 0$, of the random walk starting from $N = 1$ at $z = -L$. The calculation (3.70) can be repeated, without the condition $t < 2L/c_1$, and we get

$$(3.87) \quad \Lambda(t, r, \omega) = \frac{\sin \theta}{8(2\pi)^2 c_1 (c_1 t)^2} E_1 \left\{ (, _I(-L))^{2N(0)} \delta \left(\frac{c_1 t}{2} \sin \theta - \mu_L \right) \right\},$$

where

$$(3.88) \quad c_1 \kappa = \cos \theta = \frac{r}{c_1 t}$$

as in the ray diagram, Fig. 3.1, and $,_I(-L)$ depends on θ as in (3.25), (3.26).

When there are no random fluctuations, the noise intensity parameter⁹ α in (3.13) is zero, and hence β in (3.38) is zero. Then, $N(z) = 1$ in $-L \leq z \leq 0$, $\mu_L = L$ and (3.87) reduces to (3.34):

$$(3.89) \quad \begin{aligned} \Lambda(t, r, \omega) &= \frac{\sin \theta (, _I(-L))^2}{8(2\pi)^2 c_1 (c_1 t)^2} \delta \left(\frac{c_1 t}{2} \sin \theta - L \right) \\ &= \left(\frac{, _I(-L)}{4\pi c_1^2 t} \right)^2 \delta \left(t - \frac{2L}{c_1 \sin \theta} \right). \end{aligned}$$

In the random case there are two possibilities: With r and t fixed and θ determined by (3.88), either $c_1 t \sin \theta < 2L$ or $c_1 t \sin \theta \geq 2L$. In the first case the interface at $z = -L$ has not been reached by the waves and we have the situation of §3.6 so that (3.87) reduces to (3.77). In the second case the reflected signal contains contributions from multiply scattered waves as well as the coherent reflection from the interface at $z = -L$.

Let us assume that $c_1 t \sin \theta$ is slightly larger than $2L$ so that we can examine the form of the local power spectral density near a coherent arrival. From (3.87) we see

⁹ It does not depend on z here because we have uniform background properties.

that there are two kinds of paths of the random walk $N(z)$ that contribute: those that start from 1 and switch to zero, $N(-L) = 1$, $N(0) = 0$, and those that start from 1 and do not switch. If we look sufficiently near the arrival time, that is, $c_1 t \sin \theta \approx 2L$, then only the path $N(z) \equiv 1$ contributes in the second case. This is because paths that switch to $N > 1$ must return to $N = 1$ quickly or else the delta function will be zero and such paths occur with small probability. Given a trajectory of the random walk, let z_L^* be the smallest $z \geq -L$ for which $N(z) = 0$ for $z \geq z_L^*$. We can then write the local power spectral density (3.87) in the form

$$(3.90) \quad \Lambda(t, r, \omega) \approx \frac{\sin \theta}{8(2\pi)^2 c_1 (c_1 t)^2} E_1 \left\{ \delta \left(\frac{c_1 t \sin \theta}{2} - \mu_L \right), z_L^* \leq 0 \right\} \\ + \frac{\sin \theta (I(-L))^2}{8(2\pi)^2 c_1 (c_1 t)^2} \delta \left(\frac{c_1 t \sin \theta}{2} - L \right) e^{-2\beta L}.$$

The approximation here is that $c_1 t \sin \theta \approx 2L$ with $c_1 t \sin \theta \geq 2L$. The expectation in (3.90) is over paths of the first kind and for them μ_L can be replaced by $\mu = \mu_\infty$ as in §3.6. The decomposition (3.90) gives the multiply scattered and the coherent contributions to the local power spectral density. The exponential factor in the coherent component is the probability that $N(z) = 1$ in $-L \leq z \leq 0$ given that $N(-L) = 1$ and from (3.38) and (3.88)

$$(3.91) \quad \beta = \frac{2\alpha\omega^2}{c_1^2 \sin^2 \theta}.$$

The incoherent component in (3.90) is similar to (3.71) but the conditional law of μ when $z_L^* \leq 0$ is not known explicitly. It has, however, a continuous density, as seen at the end of this section.

The reflected signal intensity function is given by (3.28) with $\bar{t} = 0$,

$$(3.92) \quad I(t, \mathbf{x}; 0, 0) = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 \Lambda(t, \mathbf{x}, \omega) d\omega.$$

Using (3.90) in (3.92) we see that near the arrival of the coherent reflection, the signal intensity is a delta function multiplied by the factor

$$(3.93) \quad \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 e^{-2\beta L} d\omega \frac{\sin \theta (I(-L))^2}{8(2\pi)^2 c_1 (c_1 t)^2}.$$

Let us compare this with the square of the coherently reflected field (3.22). We see, as in §3.4, that the square of $\langle p_{\text{refl}}(t, r, 0) \rangle$ behaves like a delta function times a factor that differs from (3.93) because the exponentials are different in the ω integral. In view of (3.18), the exponential in the factor that comes from the square of $\langle p_{\text{refl}}(t, r, 0) \rangle$ is

$$(3.94) \quad \exp \left\{ -\frac{12\alpha\omega^2}{c_1^2 \sin^2 \theta} L \right\},$$

which is different from

$$(3.95) \quad \exp \left\{ -\frac{4\alpha\omega^2}{c_1^2 \sin^2 \theta} L \right\}$$

in (3.93). This observation is interesting for two reasons. First, it allows us to interpret the exponential factor in (3.93) as pulse spreading, seen at the level of intensities, by analogy with what happens when (3.22) is squared (§3.4). Second, it tells us that the coherent component of the reflected intensity has a larger spreading factor than the coherent field itself, because of the difference in (3.94) and (3.95). We might have expected the two coherent components to have the same spreading, but this is not so. The coherent field spreads less than the coherent part of the reflected intensity.

Let us now explain how to compute the expectation of the incoherent part of Λ in (3.90). Let

$$(3.96) \quad \mu(z) = \bar{\mu} - \int_{-L}^z N(s) ds$$

where $\bar{\mu}$ is a constant and let $V^N(z, \bar{\mu})$ be the solution of

$$(3.97) \quad \begin{aligned} \frac{\partial V^N}{\partial z} + N \frac{\partial V^N}{\partial \bar{\mu}} &= \beta N^2 \{V^{N+1} - 2V^N + V^{N-1}\}, \\ V^N|_{z=-L} &= \delta(\bar{\mu}), \quad \frac{N}{0} \end{aligned}$$

with $0 < \frac{N}{0} < 1$ and $N \geq 0$. Then

$$(3.98) \quad V^N(0, \bar{\mu}) = E_N \left\{ \delta(\bar{\mu} - \mu_L), \frac{N(0)}{0} \right\}.$$

where μ_L is given by (3.85). Similarly, let $V_D^N(z, \bar{\mu})$ for $N \geq 1$ satisfy (3.97) with the additional boundary condition

$$(3.99) \quad V_D^0(z, \bar{\mu}) = 0, \quad -L \leq z \leq 0.$$

Then

$$(3.100) \quad V_D^N(0, \bar{\mu}) = E_N \left\{ \delta(\bar{\mu} - \mu_L), \frac{N(0)}{0}, N(s) \geq 1, -L \leq s \leq 0 \right\},$$

that is¹⁰, $V_D^N(0, \bar{\mu})$ is the expectation over paths starting with $N(-L) = N$ and subject to the restriction $N(s) \geq 1$ in $-L \leq s \leq 0$. The subscript D stands for “Dirichlet” because of the boundary condition (3.99). Now

$$(3.101) \quad \begin{aligned} V^N(0, \bar{\mu}) &= E_N \left\{ \delta(\bar{\mu} - \mu_L), \frac{N(0)}{0}, N(s) \geq 1, -L \leq s \leq 0 \right\} \\ &\quad + E_N \left\{ \delta(\bar{\mu} - \mu_L), N(s) = 0, \text{ for some } s \text{ in } -L \leq s \leq 0 \right\}. \end{aligned}$$

Therefore, since the events $\{N(s) = 0, \text{ for some } s \text{ in } -L \leq s \leq 0\}$ and $\{z_L^* \leq 0\}$ are the same, we have

$$(3.102) \quad E_1 \{ \delta(\bar{\mu} - \mu_L), z_L^* \leq 0 \} = V^1(0, \bar{\mu}) - V_D^1(0, \bar{\mu}).$$

Note that both V^1 and V_D^1 have singular components but their difference is continuous. This is seen by writing the integral equation form of (3.97) for $V^N(z, \bar{\mu})$ and the one for $V_D^N(z, \bar{\mu})$, expanding them in a Neumann series and noting that the singular terms cancel.

¹⁰ See for example [72].

3.8. Structure of the local power spectral density for a general background. In this section we will explain the general structure of the solution of the W equations (3.14), in the axially symmetric form (3.63), (3.64), when the sound speed profile $c_1(z)$ is not constant. We will then use this structure in (3.65) in order to deduce the general form of the local power spectral density Λ . We again assume $z_s = 0$.

First we explain the geometrical meaning of the transport equation (3.63). Let S be the phase in (2.29)

$$(3.103) \quad S(x, y, z) = \boldsymbol{\kappa} \cdot \mathbf{x} \pm \tau, \quad \mathbf{x} = (x, y)$$

with the travel time τ defined by (2.24). To fix ideas, we take the $+$ sign in (3.103). It is easily seen that S satisfies the eiconal equation

$$(3.104) \quad S_x^2 + S_y^2 + S_z^2 = \frac{1}{c_1^2(z)}$$

in $z < 0$ with $S(x, y, 0) = \boldsymbol{\kappa} \cdot \mathbf{x}$. Let

$$(3.105) \quad \cos \theta(z) = \kappa c_1(z), \quad \sin \theta(z) = (1 - \kappa^2 c_1^2(z))^{1/2}$$

and assume that κ and $c_1(z)$ are such that $\theta(z)$ is in $(0, \frac{\pi}{2}]$ for all z in $[-L, 0]$, so that there are no turning points. The ray equations for (3.104) are

$$(3.106) \quad \dot{x} = S_x = \kappa_1, \quad \dot{y} = S_y = \kappa_2, \quad \dot{z} = \tau_z = \frac{\sin \theta(z)}{c_1(z)}.$$

If we parametrize the rays by the depth variable z then they become

$$(3.107) \quad \frac{dx}{dz} = \frac{c_1 \kappa_1}{\sin \theta}, \quad \frac{dy}{dz} = \frac{c_1 \kappa_2}{\sin \theta}$$

and if $r = (x^2 + y^2)^{1/2}$ then

$$(3.108) \quad \frac{dr}{dz} = \frac{c_1 \kappa}{\sin \theta} = \cot \theta.$$

The arclength ρ of the ray trajectory satisfies

$$d\rho = \sqrt{dr^2 + dz^2} = \frac{1}{\sin \theta} dz,$$

and therefore the travel time t along the trajectory satisfies

$$(3.109) \quad \frac{dt}{dz} = \frac{1}{c_1 \sin \theta}.$$

We note that (3.107) and (3.109) are exactly the characteristic equations for (3.14) (and (3.108), (3.109) for (3.63)). In the case of variable $c_1(z)$, the coefficient β defined by (3.38) is not constant so the effects of multiple scattering cannot be seen clearly.

We introduce a new depth variable l , letting

$$(3.110) \quad l = l(z) = \int_z^0 \frac{2\alpha(s)\omega^2}{c_1^2(s) \sin^2 \theta(s)} ds$$

in $-L \leq z \leq 0$ so that l ranges over $0 \leq l \leq l_L$, where $l_L = l(-L)$. This variable l is dimensionless and it is depth in units of the localization length (cf. (3.78)) at frequency ω and slowness κ , since $l = l(z; \kappa, \omega)$ and of course $r = c_1^{-1}(0) \sin \theta(0)$, for example. This is the intrinsic scale on which multiple scattering is universal and the W -equations in the axisymmetric form (3.63) become

$$(3.111) \quad \frac{\partial W^N}{\partial l} - N \left\{ \frac{\tilde{c}_1 \sin \tilde{\theta}}{\tilde{\alpha} \omega^2} \frac{\partial W^N}{\partial t} + \frac{\tilde{c}_1^2 \cos \tilde{\theta} \sin \tilde{\theta}}{\tilde{\alpha} \omega^2} \frac{\partial W^N}{\partial r} \right\} + N^2 \{W^{N+1} - 2W^N + W^{N-1}\} = 0$$

in $0 \leq l \leq l_L$ with terminal condition

$$(3.112) \quad W^N|_{l=l_L} = |_{I(-L)}|^{2N} \frac{1}{r} \delta(t) \delta(r).$$

In (3.111) $\tilde{\alpha}$, $\tilde{\theta}$, and $\tilde{c}_1$ are now functions of l through the inverse of the map (3.110) that gives $z = z(l)$, so that $\tilde{\theta} = \theta(z(l))$, $\tilde{\alpha} = \alpha(z(l))$, and $\tilde{c}_1 = c_1(z(l))$. With localization length scaling the ray equations (3.108), (3.109) become

$$(3.113) \quad \frac{dr}{dl} = -\frac{\tilde{c}_1^2 \sin \tilde{\theta} \cos \tilde{\theta}}{2\tilde{\alpha} \omega^2}, \quad \frac{dt}{dl} = -\frac{\tilde{c}_1 \sin \tilde{\theta}}{2\tilde{\alpha} \omega^2}.$$

The probabilistic representation of the solution of (3.111) is most informative. First let $N(z)$ be the canonical Markov chain of §3.5 with parameters $\beta = 1$ (localization scaling). Then define

$$(3.114) \quad \begin{aligned} \bar{t}(l) &= -\int_0^l \frac{\tilde{c}_1(s) \sin \tilde{\theta}(s)}{2\tilde{\alpha}(s) \omega^2} N(s) ds \\ \bar{r}(l) &= -\int_0^l \frac{\tilde{c}_1^2(s) \cos \tilde{\theta}(s) \sin \tilde{\theta}(s)}{2\tilde{\alpha}(s) \omega^2} N(s) ds \end{aligned}$$

The solution W^N at $z = 0$ is now given by

$$(3.115) \quad W^N(0, t, r; \kappa, \omega) = \frac{1}{r} E_N \left\{ |_{I(-L)}|^{2N(l_L)} \delta(t + \bar{t}(l_L)) \delta(r + \bar{r}(l_L)) \right\}$$

and the local power spectral density of (3.65) by

$$(3.116) \quad \begin{aligned} \Lambda(t, r, \omega) &= \frac{1}{4(2\pi)^2 c_1^2(0)} \int_0^{\pi/2} d\theta_0 \cos \theta_0 \sin \theta_0 \\ &\quad \cdot \left(|_{I(0)} - \frac{1}{|_{I(0)}} \right)^2 \sum_{N=1}^{\infty} |_{I(0)}|^{2N} W^N(0, t, r; \theta_0, \omega) \end{aligned}$$

Here we have omitted the W^0 term in (3.65) because it is always zero if t is positive (it is proportional to $\delta(t)$) and we have changed variables of integration from κ to $\theta_0 = \cos^{-1}(\kappa c_1(0))$. Recall also that $|_{I(0)}$ depends on θ_0 , (3.80).

We may now give a precise interpretation of formula (3.116) for the local power spectral density. Fix an observation point r on the surface $z = 0$ (the source is at the origin so $z_s = 0$), a time of observation t , and a frequency ω . Assume for simplicity that $|_{I(0)} = 0$ so that only W^1 enters in (3.116) and that

$$(3.117) \quad \int_{-L}^0 \frac{1}{c_1(s)} ds > \frac{t}{2}$$

so that the influence of the interface at $z = -L$ is not felt and we can set ${}_I(-L) = 0$. Let $N(l)$ be a path of the canonical Markov chain with generator Δ_{DD} defined by

$$(3.118) \quad \Delta_{DD} U^N = N^2 \left\{ U^{N+1} - 2U^N + U^{N-1} \right\},$$

which is the diagonal part of (3.39) with $\beta = 1$. Assume that $N(0) = 1$ and that l^* is the smallest point l for which $N(l) = 0$, so that $N(l) = 0$ thereafter. We know that $l^* < \infty$ with probability 1. Now N is piecewise constant on a finite number of intervals of random length, so

$$\begin{aligned} \bar{t}(l^*) &= - \int_0^{l^*} \frac{\tilde{c}_1(s) \sin \tilde{\theta}(s)}{2\tilde{\alpha}(s)\omega^2} N(s) ds, \\ \bar{r}(l^*) &= - \int_0^{l^*} \frac{\tilde{c}_1^2(s) \cos \tilde{\theta}(s) \sin \tilde{\theta}(s)}{2\tilde{\alpha}(s)\omega^2} N(s) ds \end{aligned}$$

are piecewise arcs of the geometrical acoustics rays (3.113) in which the localization length parameter is multiplied by N . We then average over paths

$$W^1(0, t, r; \theta_0, \omega) = \frac{1}{r} E_1 \left\{ \delta(t + \bar{t}(l_L)) \delta(r + \bar{r}(l_L)) \right\}$$

and over initial angles θ_0 ,

$$\Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2 c_1^2(0)} \int_0^{\pi/2} d\theta_0 \cos \theta_0 \sin \theta_0 W^1(0, t, r; \theta_0, \omega).$$

The main difference between the general results of this section and the special case $c_1(z) \equiv \text{constant}$ of §3.6 is that when the geometrical acoustics rays are straight lines only the random variable

$$\mu = \int_0^{l^*} N(s) ds$$

enters and the paths $\bar{t}(l_L)$ and $\bar{r}(l_L)$ simplify to

$$\bar{t}(l^*) = -\frac{\tilde{c}_1 \sin \tilde{\theta}}{2\tilde{\alpha}\omega^2} \mu, \quad \bar{r}(l^*) = -\frac{\tilde{c}_1^2 \cos \tilde{\theta} \sin \tilde{\theta}}{2\tilde{\alpha}\omega^2} \mu.$$

This allows us to determine immediately the starting angle for the random ray that contributes to W^1 since the ray direction is constant.

3.9. The transport equation with turning points. Up to now we have avoided the issue of what happens at turning points that are points L_T in $[-L, 0]$ for which $1 - c_1^2(-L_T)\kappa^2 = 0$. Since we have to integrate over κ in (3.29) to get the power spectral density Λ , there will inevitably be κ 's for which the W -equations (3.14) have singular coefficients. For general velocity profiles $c_1(z)$ there will be κ 's for which $1 - c_1^2\kappa^2 = 0$ and the turning points L_T will depend on κ . In this section we will prove the following theorem.

Let $c_1(z)$ be a smooth velocity profile in $[-L, 0]$ and suppose that for a fixed $\kappa > 0$, $1 - c_1^2(z)\kappa^2 > 0$ in $(-L_T(\kappa), 0]$ and

$$(3.119) \quad \lim_{z \rightarrow -L_T} [1 - c_1^2(z)\kappa^2] = 0.$$

Let γ_T be a fixed reflection coefficient with $0 \leq |\gamma_T| \leq 1$ and let L_n be any sequence of points with $-L_T < -L_n \leq 0$ that decreases to $-L_T$,

$$\lim_{n \rightarrow \infty} L_n = L_T.$$

Let $W_n^N(z, t, r; \kappa, \omega)$ be the solution of (3.14) in $[-L_n, 0]$ with

$$(3.120) \quad W_n^N|_{z=-L_n} = |\gamma_T|^{2N} g(t, r),$$

where $g(t, r)$ is a bounded continuous function¹¹ on $\mathbf{R}^1 \times \mathbf{R}^1$. Then for any z in $(-L_T, 0]$, W_n^N converges, pointwise in the other variables, to a limit W^N that is independent of γ_T .

Before proving this theorem we note that it says roughly that the W equations (3.14), with coefficients singular at $-L_T$, are well posed in $(-L_T, 0]$ and have a unique distribution solution when $g = \delta(t)\delta(r)/r$.

The proof of the theorem is based on the property that $N(z) \rightarrow 0$ as $z \rightarrow \infty$ with probability 1 for the canonical Markov chain. This was proven with the help of the duality (3.62). We change variables to localization length scaling (3.110) so that the W-equations have the form (3.111). We write

$$(3.121) \quad l_n = l(-L_n)$$

and note that $l_n \rightarrow +\infty$ as $n \rightarrow \infty$. The probabilistic representation of W_n^N is given by (3.115), which here is

$$(3.122) \quad W_n^N(0, t, r; \kappa, \omega) = E_N \left\{ |\gamma_T|^{2N(l_n)} g(t + \bar{t}(l_n), r + \bar{r}(l_n)) \right\}$$

But since $l_n \rightarrow \infty$ we have $N(l_n) \rightarrow 0$ with probability 1 and hence

$$(3.123) \quad \lim_{n \rightarrow \infty} W_n^N(0, t, r; \kappa, \omega) = E_N \{ g(t + \bar{t}^*, r + \bar{r}^*) \},$$

where

$$\begin{aligned} \bar{t}^* &= - \int_0^\infty \frac{\tilde{c}_1(s) \sin \tilde{\theta}(s)}{2\tilde{\alpha}(s)\omega^2} N(s) ds \\ \bar{r}^* &= - \int_0^\infty \frac{\tilde{c}_1^2(s) \cos \tilde{\theta}(s) \sin \tilde{\theta}(s)}{2\tilde{\alpha}(s)\omega^2} N(s) ds. \end{aligned}$$

These are finite random variables with probability 1 because they extend only up to l^* , which is the first point l for which $N(l) = 0$.

It is clear from (3.123) that the W_n^N have a limit and that the limit is independent of γ_T . So we may take $\gamma_T = 0$ and then solve (3.63) in $-L_T < z \leq 0$ as a boundary value problem with

$$W^0(z, t, r; \kappa, \omega) = \delta(t) \frac{\delta(r)}{r}.$$

This is what gives the limit (3.123) with the appropriate choice of the generalized function g .

¹¹ For example g is product of functions of the form $h(s) = \min \{ \max \{ s, 0 \}, 1 \}$

3.10. Dissipative media. When there is dissipation the scaled acoustic equations (2.17) are

$$(3.124a) \quad \rho \frac{\partial \mathbf{u}}{\partial t} + \sigma \mathbf{u} + \nabla p = \varepsilon^{1/2} f\left(\frac{t}{\varepsilon}\right) \delta(x) \delta(y) \delta(z - z_s) \mathbf{e},$$

$$(3.124b) \quad K^{-1} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0,$$

where

$$(3.125) \quad \rho(z) = \begin{cases} \rho_0, & z > 0, \\ \rho_1, & -L < z < 0, \\ \rho_2, & z < -L, \end{cases}$$

$$(3.126) \quad \sigma(z) = \begin{cases} \sigma_0, & z > 0, \\ \sigma_1(z, \frac{z}{\varepsilon^2}), & -L < z < 0, \\ \sigma_2, & z < -L, \end{cases}$$

and

$$(3.127) \quad K^{-1}(z) = \begin{cases} K_0^{-1}, & z > 0, \\ K_1^{-1}(z) \left(1 + \nu(z, \frac{z}{\varepsilon^2})\right), & -L < z < 0, \\ K_2^{-1}, & z < -L. \end{cases}$$

All the calculations in §§2.3, 2.4, 3.1, and 3.2 extend to the case where dissipation is present and the W -equations (3.14) become¹²

$$(3.128) \quad \begin{aligned} & \frac{\partial W^N}{\partial z} + 2N \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \left(\frac{\partial W^N}{\partial t} + \frac{\bar{\sigma}_1}{\rho_1} W^N \right) + \frac{c_1 \kappa}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^N \right\} \\ & - 2 \frac{\omega^2 \alpha N^2}{c_1^2 (1 - c_1^2 \kappa^2)} \{W^{N+1} - 2W^N + W^{N-1}\} = 0, \end{aligned}$$

$$W^N|_{z=-L} = |, I(-L)|^{2N} \delta(t) \delta(\mathbf{x}),$$

where $\bar{\sigma}_1 = \bar{\sigma}_1(z)$ is the mean of the dissipation coefficient σ_1 .

Actually, σ_1 , and hence $\bar{\sigma}_1$, can depend on ω , in which case $\sigma \mathbf{u}$ in (3.124) is interpreted as the time convolution of σ and $\mathbf{u}$. In exploration geophysics [70], the dependence of $\bar{\sigma}_1$ is modeled well by a linear function over the range 10-50Hz.

In the case of a randomly layered half space with a uniform background so that c_1 , α , and $\bar{\sigma}_1$ are constants, the local power spectral density (3.76) becomes

$$(3.129) \quad \Lambda(t, r, \omega) = \frac{e^{-\frac{\bar{\sigma}_1(\omega)t}{\rho_1}} (1 - , I^2(0))^2 \frac{\alpha \omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)^{1/2}}}{4(2\pi)^2 c_1 (c_1 t)^2 \left[1 + (1 - , I^2(0))^2 \frac{\alpha \omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)^{1/2}} c_1 t \right]^2},$$

which in the matched case $, I(0) = 0$ reduces to

$$(3.130) \quad \Lambda(t, r, \omega) = \frac{e^{-\frac{\bar{\sigma}_1(\omega)t}{\rho_1}} \frac{\alpha \omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)^{1/2}}}{4(2\pi)^2 c_1 (c_1 t)^2 \left[1 + \frac{\alpha \omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)^{1/2}} c_1 t \right]^2}.$$

¹² Here $z_s = 0$ and $\mathbf{e} = (0, 0, 1)$, pointing in the vertical direction.

3.11. Duality and generating functions. The duality relation between the Markov chain $N(z)$ and the diffusion process $\bar{\theta}(z)$ that we introduced in §3.5 (equation (3.62)) allows us to calculate the local power spectral density Λ , (3.29), directly without having to sum a geometric series as we did in §3.6. In our numerical calculations we prefer to work with (3.14) and its adjoint (3.45) or (5.1) but it is useful to see the generating function version of the W -equations. We need for this the adjoint W -equations (3.14) which are as follows. Let $V^N(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$ satisfy

$$(3.131) \quad \frac{\partial V^N}{\partial z} + 2N \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial V^N}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla V^N \right\} \\ + 2 \frac{\omega^2 \alpha}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ (N+1)^2 V^{N+1} - 2N^2 V^N + (N-1)^2 V^{N-1} \right\} = 0$$

in $-L \leq z < 0$ with $V^N \equiv 0$ for $N < 0$ and

$$V^N|_{z=0} = {}_I^{2N}(0) \delta(t) \delta(\mathbf{x}),$$

where $N \geq 0, t, \omega \in R^1, \mathbf{x} \in R^2$. Then it is easy to see that for all t and $\mathbf{x}$,

$$(3.132) \quad \sum_{N=0}^{\infty} {}_I^{2N}(-L) V^N(-L, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) = \sum_{N=0}^{\infty} {}_I^{2N}(0) W^N(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega),$$

where the W^N satisfy (3.14). Let

$$(3.133) \quad {}_I(0) = \tanh \frac{\bar{\theta}_0}{2}, \quad {}_I(-L) = \tanh \frac{\bar{\theta}_L}{2},$$

which depend on κ and are assumed to be real since only propagating modes are considered. Define the generating function of the V^N by

$$(3.134) \quad V(z, t, \mathbf{x}, \bar{\theta}, \boldsymbol{\kappa}, \omega) = \sum_{N=0}^{\infty} \left(\tanh \frac{\bar{\theta}}{2} \right)^{2N} V^N(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega).$$

Then, using the basic duality identity (3.60) (with $\phi = 0$ and $N = M$), we see that V satisfies the degenerate parabolic partial differential equation

$$(3.135) \quad \frac{\partial V}{\partial z} + \sinh \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial V}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla V \right\} \\ + \frac{2\alpha\omega^2}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ \frac{\partial^2 V}{\partial \bar{\theta}^2} + \coth \bar{\theta} \frac{\partial V}{\partial \bar{\theta}} \right\} = 0$$

in $-L \leq z < 0, \bar{\theta} > 0, t \in R$ and $\mathbf{x} \in R^2$ with

$$V|_{z=0} = \frac{1}{1 - \left(\tanh \frac{\bar{\theta}}{2} \tanh \frac{\bar{\theta}_L}{2} \right)^2} \delta(t) \delta(\mathbf{x}).$$

The local power spectral density Λ in (3.29) can now be written in terms of the generating function V , using the identity (3.132). Thus, for $t > 0$,

$$(3.136) \quad \Lambda(t, \mathbf{x}, \omega) = \frac{1}{4(2\pi)^2} \iint d\boldsymbol{\kappa} \frac{4}{\sinh^2 \bar{\theta}_0} V(0, t, \mathbf{x}, \bar{\theta}_L; \boldsymbol{\kappa}, \omega).$$

Note that $\bar{\theta}_0$ and $\bar{\theta}_L$ depend on the angle of incidence θ (cf. (3.25) and (3.80)) and the κ integration is restricted to $c_1(0)\kappa = \sin \theta$ with $0 \leq \theta \leq \pi/2$. Note also that if we restrict attention to $t > 0$, we see from (3.132) that

$$(3.137) \quad V(-L, t, \mathbf{x}, \bar{\theta}_L, \kappa, \omega) = \sum_{N=1}^{\infty} W^N(0, t, \mathbf{x}, \kappa, \omega) \left(\tanh \frac{\bar{\theta}_0}{2} \right)^{2N},$$

which we have used to get (3.136). We may also replace the condition for V at $z = 0$ in (3.135) by

$$V|_{z=0} = \frac{\left(\tanh \frac{\bar{\theta}}{2} \tanh \frac{\bar{\theta}_L}{2} \right)^2}{1 - \left(\tanh \frac{\bar{\theta}}{2} \tanh \frac{\bar{\theta}_L}{2} \right)^2} \delta(t) \delta(\mathbf{x})$$

because the sum in (3.137) starts from $N = 1$ and we can replace the solution V in (3.135) by $V - \delta(t)\delta(\mathbf{x})$ because we have assumed that t is positive.

As we show in more detail in the next section, if we integrate the solution V of (3.135) over all t and $\mathbf{x}$ then we get the mean intensity of the reflection coefficient for a plane, time harmonic wave, normally incident on the random medium occupying the region $-L \leq z \leq 0$. Let us denote it by $\bar{I}(L, \omega)$. If $\bar{\theta}(z)$ denotes the diffusion process generated by the operator on the second line of (3.135), starting from $\bar{\theta}_L$ at $z = -L$, then we have

$$(3.138) \quad \bar{I}(L, \omega) = \frac{4}{\sinh^2 \bar{\theta}_0} E_{\bar{\theta}_L} \left\{ \frac{\left(\tanh \frac{\bar{\theta}_0}{2} \tanh \frac{\bar{\theta}(0)}{2} \right)^2}{1 - \left(\tanh \frac{\bar{\theta}_0}{2} \tanh \frac{\bar{\theta}(0)}{2} \right)^2} \right\}.$$

Note that we have multiplied the right side by the factor $4/\sinh^2 \bar{\theta}_0$ which comes from (3.136). When the medium is matched at $z = 0$, so that $\bar{\theta}_0 = 0$, then

$$(3.139) \quad \begin{aligned} \bar{I}(L, \omega) &= E_{\bar{\theta}_L} \left\{ \left(\tanh \frac{\bar{\theta}(0)}{2} \right)^2 \right\} \\ &= E_1 \left\{ \left(\tanh \frac{\bar{\theta}_L}{2} \right)^{2N(0)} \right\}, \end{aligned}$$

where in the last equality we have used the duality with the Markov chain $N(z)$ starting from $N = 1$ at $z = -L$, (3.62). We see, therefore, that the statistics of the Markov chain are closely related to the reflected intensity of time harmonic plane waves. Such problems are considered in detail in [35] and in [8], [57], although the scaling is different, as we discuss in Appendix C. When $\bar{\theta}_L = 0$ and the medium is matched at $z = -L$ as well, then

$$(3.140) \quad \begin{aligned} \bar{I}(L, \omega) &= E_0 \left\{ \left(\tanh \frac{\bar{\theta}(0)}{2} \right)^2 \right\} \\ &= E_1 \left\{ \chi_{\{N(0)=0\}} \right\} \\ &= \frac{4e^{-L/4\bar{l}}}{\sqrt{\pi}} \int_0^\infty \frac{\eta^2 e^{-\eta^2} d\eta}{\cosh \left(\eta \sqrt{L/\bar{l}} \right)}, \end{aligned}$$

where χ_A is the indicator function of the set A and $\tilde{l} = c_1^2/2\alpha\omega^2$ is the localization length (3.78) [33].

3.12. Time harmonic point source. Time harmonic, point source problems are important in many applications in wave propagation. They are also important in understanding localization phenomena because they have a simpler form than problems in the time domain. In §3.6 we saw how localization phenomena appear in the time domain through results like (3.79).

With the time factor $e^{-i\omega t/\varepsilon}$ omitted, the scaled acoustic equations are

$$(3.141a) \quad -\frac{i\omega\rho}{\varepsilon}\mathbf{u} + \nabla p = \varepsilon f\delta(x)\delta(y)\delta(z - z_s)\mathbf{e},$$

$$(3.141b) \quad -\frac{i\omega}{\varepsilon K}p + \nabla \cdot \mathbf{u} = 0$$

with ρ and K given by (2.18), (2.19). Note that the source strength is now proportional to ε and not $\sqrt{\varepsilon}$ as in (2.17) and f is a constant. This scaling makes the energy released per unit time independent of ε . We will again assume for simplicity that the source is at the origin, $z_s = 0$, and that it is directed in the z direction so that $\mathbf{e} = (0, 0, 1)$. As in §2.4, the integral representation of the reflected pressure is now

$$(3.142) \quad p_{\text{refl}}(\mathbf{x}, 0) = -\frac{f\omega^2}{2(2\pi)^2\varepsilon} \iint e^{i\omega\boldsymbol{\kappa}\cdot\mathbf{x}/\varepsilon} \frac{, (0^-, \omega, \boldsymbol{\kappa}) - , I(0, \boldsymbol{\kappa})}{1 - , (0^-, \omega, \boldsymbol{\kappa}), , I(0, \boldsymbol{\kappa})} d\boldsymbol{\kappa}.$$

When $|, I, (0^-)| < 1$, then the expanded form of (3.142) is

$$\begin{aligned} p_{\text{refl}}(\mathbf{x}, 0) &= \frac{-f\omega^2}{2(2\pi)^2\varepsilon} \iint e^{i\omega\boldsymbol{\kappa}\cdot\mathbf{x}/\varepsilon} \\ &\quad \cdot \sum_{N=0}^{\infty} \left[(, (0^-, \omega, \boldsymbol{\kappa}))^N (, I(0))^N - (, (0^-, \omega, \boldsymbol{\kappa}))^{N+1} (, I(0))^N \right] d\boldsymbol{\kappa}. \end{aligned}$$

The two-point, surface intensity function is

$$I(\mathbf{x}; \bar{\mathbf{x}}, \omega) = \lim_{\varepsilon \downarrow 0} \left\langle p_{\text{refl}}\left(\mathbf{x} + \frac{\varepsilon\bar{\mathbf{x}}}{2}, 0\right) p_{\text{refl}}\left(\mathbf{x} - \frac{\varepsilon\bar{\mathbf{x}}}{2}, 0\right) \right\rangle,$$

whose expression analogous to (2.40) and (3.27) is now

$$(3.143) \quad \begin{aligned} I(\mathbf{x}; \bar{\mathbf{x}}, \omega) &= \frac{f^2\omega^2}{4(2\pi)^2} \iint d\boldsymbol{\kappa} e^{i\omega\boldsymbol{\kappa}\cdot\bar{\mathbf{x}}} \\ &\quad \cdot \left[(, I(0))^2 W^0 + \left(, I(0) - \frac{1}{, I(0)} \right)^2 \sum_{N=1}^{\infty} (, I(0))^{2N} W^N(0, \mathbf{x}, \boldsymbol{\kappa}, \omega) \right]. \end{aligned}$$

Here the $\boldsymbol{\kappa}$ integration is over only propagating modes for which $c_1(0^-)\kappa$ is in the interval $[0, 1)$ and $, I(0)$ is real and depends on κ (cf. (3.80), (3.81)). The functions W^N satisfy now the following system of transport equations which is the analogue of (3.14):

$$(3.144) \quad \begin{aligned} \frac{\partial W^N}{\partial z} + 2N \frac{c_1\boldsymbol{\kappa}}{(1 - c_1^2\kappa^2)^{1/2}} \cdot \nabla W^N \\ - 2 \frac{\omega^2\alpha N^2}{c_1^2(1 - c_1^2\kappa^2)} \left\{ W^{N+1} - 2W^N + W^{N-1} \right\} = 0, \end{aligned}$$

$$W^N|_{z=-L} = |{}_I(-L)|^{2N} \delta(\mathbf{x}),$$

where $N \geq 0$, $-L \leq z < 0$, $\omega \in R^1$, $\mathbf{x} \in R^2$.

When the randomly layered medium in $z < 0$ has a constant mean sound speed profile, $c_1 = \text{constant}$, and $L = \infty$ so that there are no deterministic interfaces; then the angular integral of the solution W^N of (3.144), which we also denote by W^N , does not depend on z and in polar coordinates satisfies the equation

$$(3.145) \quad c_1 \kappa \frac{\partial W^N}{\partial r} - \frac{\omega^2 \alpha N}{c_1^2 (1 - c_1^2 \kappa^2)^{1/2}} \{W^{N+1} - 2W^N + W^{N-1}\} = 0$$

for $N \geq 1$ with

$$W^0 = \frac{1}{r} \delta(r).$$

The solution of (3.145) can be gotten the same way we solved (3.75) in §3.6. In fact, the surface intensity function $I(r; 0, \omega)$ can be calculated directly by integrating (3.76) or (3.79) over t . Thus

$$(3.146) \quad \begin{aligned} I(r; 0, \omega) &= f^2 \omega^2 \int_{r/c_1}^{\infty} dt \frac{1}{4(2\pi)^2} \frac{1}{c_1 (c_1 t)^2} \frac{(1 - ({}_I(0))^2)^2 \frac{\alpha \omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)^{1/2}}}{\left[1 + (1 - ({}_I(0))^2) \frac{\alpha \omega^2}{c_1^2 (1 - (\frac{r}{c_1 t})^2)^{1/2}} c_1 t\right]^2} \\ &= \frac{f^2 \omega^2}{c_1^2 \tilde{l}(\omega) r} \int_0^1 dy y^2 \sqrt{1 - y^2} \left[y \sqrt{1 - y^2} + (1 - ({}_I(0))^2) \frac{r}{2\tilde{l}(\omega)} \right]^{-2} \frac{(1 - ({}_I(0))^2)^2}{8(2\pi)^2}, \end{aligned}$$

where $\tilde{l}(\omega)$ is the localization length for normally incident plane waves, $\tilde{l}(\omega) = l(\omega, \frac{\pi}{2})$ in (3.78) and we have assumed that $c_1 \geq c_0$.

In the matched case, ${}_I = 0$ and when

$$(3.147) \quad \frac{r}{2\tilde{l}(\omega)} \gg 1,$$

then the integral in (3.146) can be evaluated approximately and we get

$$(3.148) \quad I(r; 0, \omega) \approx \frac{\tilde{l}(\omega) f^2 \omega^2}{128\pi c_1^2 r^3}.$$

Condition (3.147) says that the observation point is at a distance from the source that is much larger than the localization length $\tilde{l}(\omega)$. The surface intensity (3.148) is then proportional to r^{-3} which means that there is a lot of reflected energy returning to the surface $z = 0$, even far from the source. This is produced entirely by the random layering. In the next section we consider the pressure release boundary conditions which correspond roughly to ${}_I(0) = 1$. We will see then that the analogue of (3.148) is (3.164) which displays clearly the waveguide, or ducting, properties that localization produces.

In the matched case, ${}_I = 0$ but when

$$(3.149) \quad \frac{r}{2\tilde{l}(\omega)} \ll 1,$$

that is, when the observation point on the surface is near the source at the origin (in localization length units), then the integral in (3.146) is approximately

$$(3.150) \quad I(r; 0, \omega) \approx \frac{f^2 \omega^2}{64\pi c_1^2 \tilde{l}(\omega) r}.$$

The surface intensity has therefore cylindrical spreading behavior near the source, which is a point source in three dimensions. Since $\gamma_I = 0$, it is the random layering that causes this behavior.

When the randomly layered medium in $z < 0$ has constant mean sound speed profile it is possible to get in considerable generality an explicit expression for the two-point reflected pressure correlation function on the interface $z = 0$, denoted by $I(\mathbf{x}; \bar{\mathbf{x}}, \omega)$. The source can be located at a distance $z_s > 0$ above the layered medium, the directivity of the source $\mathbf{e} = (e_1, e_2, e_3)$ need not be vertical and dissipation effects can be included as in §3.10. We will not give the detailed derivation of this general result because it is rather long. However, the calculations are similar to the ones we have presented. The result is:

$$(3.151) \quad I(\mathbf{x}; \bar{\mathbf{x}}, \omega) = \frac{f^2 \omega^2 e^{i\omega c_0^{-1} \hat{\mathbf{r}} \cdot \bar{\mathbf{x}} \cos \Theta_0}}{(4\pi c_0 R)^2} (\mathbf{e} \cdot \hat{\mathbf{r}} \cos \Theta_0 - e_3 \sin \Theta_0)^2 |\gamma_I(0; \Theta)|^2 \\ + \frac{f^2 \omega^2}{8(2\pi)^2 c_1 c_0 r} \int_{\max(\Theta_0, \Theta_c)}^{\pi/2} \frac{\sin \theta_1 l(\omega, \theta_1) (\mathbf{e} \cdot \hat{\mathbf{r}} \cos \theta_0 - e_3 \sin \theta_0)^2}{\sin \theta_0 \left[l(\omega, \theta_1) + (1 - \gamma_I^2(0; \theta)) \frac{d(\theta_0) \sin \theta_1}{2} \right]^2} \\ \cdot (1 - \gamma_I^2(0; \theta))^2 e^{i\omega c_0^{-1} \hat{\mathbf{r}} \cdot \bar{\mathbf{x}} \cos \theta_0} e^{-\frac{\gamma_1 d(\theta_0)}{\rho_1 c_1}} d\theta_0.$$

Here, $|\mathbf{x}| = r$, z_s is the source elevation above the origin, $R = (r^2 + z_s^2)^{1/2}$, $\hat{\mathbf{r}}$ is a unit vector in the radial direction on the interface $z = 0$, and

$$(3.152) \quad \cos \Theta_0 = \frac{r}{R}.$$

The angles Θ_0 and Θ_1 are related by Snell's law (3.81):

$$(3.153) \quad \frac{c_0}{c_1} = \frac{\cos \Theta_0}{\cos \Theta_1},$$

while Θ_c is the critical angle defined by

$$(3.154) \quad \cos \Theta_c = \frac{c_0}{c_1}$$

when $c_1 > c_0$ and zero otherwise. The interface reflection coefficient $\gamma_I(0) = \gamma_I(0; \Theta)$ depends on the angles Θ_0 and Θ_1 and is given by (3.80)

$$(3.155) \quad \gamma_I(0; \Theta) = \frac{\rho_1 c_1 \sin \Theta_0 - \rho_0 c_0 \sin \Theta_1}{\rho_1 c_1 \sin \Theta_0 + \rho_0 c_0 \sin \Theta_1}.$$

In the integral over θ_0 , the angle θ_1 is related to θ_0 by Snell's law (3.153) and $\gamma_I(0; \theta)$ depends on θ_0 and θ_1 as in (3.155) with the Θ 's replaced by the θ 's. The reduced localization length $l(\omega, \theta)$ is given by (3.78) and equals $(2\alpha\omega^2)^{-1} c_1^2 \sin^2 \theta$, the distance d is given by

$$(3.156) \quad d(\theta_0) = \frac{c_0}{c_1} \left[\frac{r}{\cos \theta_0} - \frac{z_s}{\sin \theta_0} \right],$$

and $\bar{\sigma}_1$ is the mean dissipation coefficient as defined in §3.10.

The first term in (3.151) is the coherent reflection from the interface $z = 0$ and does not involve dissipation. Energy that penetrates the half space undergoes dissipative attenuation and ultimately is scattered back to the surface as incoherent energy. The second term arises due to the random multiple scattering and vanishes when the noise intensity $\alpha = 0$. In the case $z_s = 0$, $\mathbf{e} \cdot \hat{\mathbf{r}} = 0$, $e_3 = 1$, $c_1 \geq c_0$, and $\bar{\mathbf{x}} = \mathbf{0}$, the first term in (3.151) is zero and the second term can be written in the form (3.146), after a change of variables. Note that $d(\theta_0)$ is the distance travelled by the refracted ray in the effective medium so that the dissipative factor has a form analogous to the one in §3.10 for pulse problems.

3.13. Pressure release boundary conditions. In geophysical applications the density ρ_0 in the homogeneous half space $z > 0$ is much smaller than the density ρ_1 in the randomly layered medium in $z < 0$. It is interesting then to calculate the fields on the interface $z = 0$ when the source is also on the interface ($z_s = 0$) and $\rho_0 \rho_1^{-1} \rightarrow 0$. In this limit the pressure in $z > 0$ goes to zero and hence, by continuity, the pressure at $z = 0^-$, just below the surface, is also zero. The quantity of interest is now the normal velocity on the surface, $u_3 = \mathbf{u} \cdot (0, 0, 1)$. We assume that the directivity of the source $\mathbf{e}$ is in the z direction so $\mathbf{e} = (0, 0, 1)$.

To get an integral representation for u_3 we follow the same steps as in §2.4 except that we now put the source at a depth $z_s < 0$, at first. We then let $z_s \uparrow 0$. After introducing Fourier transforms and the upgoing and downgoing representation (2.26) for $\hat{p}$ and $\hat{u}_3 = \hat{u}$, we find that equations (2.27) now have a source term

$$(3.157a) \quad \frac{dA}{dz} = \frac{i\omega}{\varepsilon} n \left[A - e^{-2i\omega\tau/\varepsilon} B \right] + \frac{1}{2\zeta_1} \frac{d\zeta_1}{dz} e^{-2i\omega\tau/\varepsilon} B + J_- \delta(z - z_s),$$

$$(3.157b) \quad \frac{dB}{dz} = -\frac{i\omega}{\varepsilon} n \left[-e^{2i\omega\tau/\varepsilon} A + B \right] + \frac{1}{2\zeta_1} \frac{d\zeta_1}{dz} e^{2i\omega\tau/\varepsilon} A - J_+ \delta(z - z_s),$$

where

$$(3.158) \quad J_{\pm} = \varepsilon^{3/2} \frac{1}{2} \zeta_1^{-1/2} \hat{f}(\omega) e^{\pm i\omega\tau(z_s)/\varepsilon}.$$

Since $\hat{p}$ is continuous at $z = 0$ and $\hat{p} = 0$ in $z > 0$, we see from (2.26) that

$$A(0^-) = B(0^-)$$

and

$$(3.159) \quad \hat{u}|_{z=0} = \zeta^{-1/2} (A + B) = 2\zeta^{-1/2} A(0^-).$$

The jump conditions in (3.157) tell us that

$$A(z_s^-) = A(z_s^+) - J_-, \quad B(z_s^-) = B(z_s^+) + J_+.$$

Hence, with $\gamma = A/B$ as in (2.31), we have

$$\gamma(z_s^-) = \frac{A(z_s^+) - J_-}{A(z_s^+) + J_+}.$$

Letting $z_s \uparrow 0$ and noting that $A(z_s^+) \rightarrow A(0^-)$, we get

$$, (0^-) = \frac{A(0^-) - J_-}{A(0^-) + J_+}$$

which, using (3.158), gives

$$A(0^-) = \varepsilon^{3/2} \frac{1}{2} \zeta_1^{-1/2} \hat{f}(\omega) \frac{1 + , (0^-)}{1 - , (0^-)}$$

and by (3.159),

$$\hat{u} \mid_{z=0-} = \varepsilon^{3/2} \zeta_1^{-1} \hat{f}(\omega) \frac{1 + , (0^-)}{1 - , (0^-)}.$$

Inverting the Fourier transformation we get the analog of (2.35) for u_3

$$(3.160) \quad u_3(t, \mathbf{x}, 0) = \frac{1}{(2\pi)^3 \varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon} \zeta_1^{-1} \frac{1 + , (0^-, \boldsymbol{\kappa}, \omega)}{1 - , (0^-, \boldsymbol{\kappa}, \omega)} d\omega d\boldsymbol{\kappa}.$$

Note that this is different from (2.35), not only because of the reciprocal impedance factor ζ_1^{-1} that converts pressure to velocity, but also because the ratio $(1 + , (0^-))(1 - , (0^-))^{-1}$ is not a special case of (2.33).

To expand the ratio of reflection coefficients in (3.160) we introduce a convergence¹³ parameter γ in $(0, 1)$ and write

$$(3.161) \quad u_3(t, \mathbf{x}, 0) = \lim_{\gamma \uparrow 1} \frac{1}{(2\pi)^3 \varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon} \zeta_1^{-1} \cdot \left[1 + 2 \sum_{N=1}^{\infty} (\gamma, (0^-, \boldsymbol{\kappa}, \omega))^N \right] d\omega d\boldsymbol{\kappa}.$$

The term 1 in the bracket in the second line produces a coherent field component, so it is better to subtract it in computing two-point intensities. Let u_3 stand for (3.161) with this term omitted. Then, as in (2.38), (2.40), (3.27), and (3.29) we have that

$$\begin{aligned} I(t, \mathbf{x}, \bar{t}, 0) &= \lim_{\gamma \uparrow 1} \lim_{\varepsilon \downarrow 0} \left\langle u_3 \left(t + \frac{\varepsilon \bar{t}}{2}, \mathbf{x} + \frac{\varepsilon \bar{\mathbf{x}}}{2}, 0 \right) u_3 \left(t - \frac{\varepsilon \bar{t}}{2}, \mathbf{x} - \frac{\varepsilon \bar{\mathbf{x}}}{2}, 0 \right) \right\rangle \\ &= \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 e^{-i\omega \bar{t}} \Lambda(t, \mathbf{x}, \omega) d\omega, \end{aligned}$$

where the local power spectral density Λ of the incoherent, normal velocity on the surface $z = 0$ is now

$$(3.162) \quad \Lambda(t, \mathbf{x}, \omega) = \lim_{\gamma \uparrow 1} \frac{4}{(2\pi)^2} \iint d\boldsymbol{\kappa} \zeta_1^{-2} \sum_{N=1}^{\infty} \gamma^N W^N(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$$

with W^N the solution of (3.14), as before, and with $\zeta_1 = \rho_1 c_1 (1 - c_1^2 \kappa^2)^{-1/2}$.

In the case of a mean sound speed profile that is uniform, $c_1 = \text{const}$, and $L > c_1 t/2$ so that the interface $z = -L$ is not reached by the waves by time t , we can calculate

¹³ A more physical way to do this regularization is by dissipation. The result is the same.

the local power spectral density $\Lambda(t, \mathbf{x}, \omega)$ the same way we did to get (3.76). We now get, with $|\mathbf{x}| = r$,

$$(3.163) \quad \begin{aligned} \Lambda_{PR}(t, r, \omega) &= \frac{1}{2(2\pi)^2} \frac{\sin \theta}{c_1 (c_1 t)^2 \zeta_1^2 l(\omega, \theta)} \\ &= \frac{\sin \theta}{2(2\pi)^2 c_1 (c_1 t)^2 (\rho_1 c_1)^2 \tilde{l}(\omega)}, \end{aligned}$$

where we have used the subscript PR for pressure release boundary conditions,

$$\sin \theta = \sqrt{1 - \left(\frac{r}{c_1 t} \right)^2}$$

as in the ray diagram (Fig. 3.1) and $l(\omega, \theta)$ and $\tilde{l}(\omega)$ are the reduced and usual localization length, respectively, defined by (3.78).

We can also consider the time-harmonic point source problem with pressure release boundary conditions. We simply adapt the calculations of §3.12 to the setup of this section. In the pressure release case the surface intensity function, analogous to (3.143) for the normal velocity, is given by

$$I_{PR}(r; 0, \omega) = f^2 \omega^2 \int_{r/c_1}^{\infty} \Lambda_{PR}(t, r, \omega) dt.$$

Using (3.163) and performing the elementary integrations, we get

$$(3.164) \quad I_{PR}(r; 0, \omega) = \frac{f^2 \omega^2}{16\pi c_1^2 (\rho_1 c_1)^2 \tilde{l}(\omega) r}.$$

This result is exact, in our asymptotic limit, and valid for all $r > 0$, in contrast to (3.148) which is valid only for r large, satisfying the condition (3.147). The surface intensity function (3.164) shows clearly the waveguide or ducting behavior of the fields. The randomly layered half space behaves as if a reflecting surface is placed at a depth equal to the localization length $\tilde{l}(\omega)$. Since the waves are also reflected at $z = 0$ because of the pressure release boundary condition, we have an effective waveguide [58] and the intensity of the normal velocity behaves like $(\tilde{l}r)^{-1}$ which is characteristic of two-dimensional radiation in the radial direction.

3.14. Transmitted fields. So far we have considered only reflected fields, observed on the surface $z = 0$. In this section we will analyze the pressure transmitted through a slab of randomly layered medium occupying the region $-L \leq z \leq 0$. The pressure and velocity satisfy again the scaled equations (2.17) with the density ρ and the bulk modulus K given by (2.18) and (2.19), respectively. We are interested in the statistics of the pressure in the homogeneous half space $z < -L$ when a pulsed point source is located at the origin (we take $z_s = 0$ for simplicity). As in previous sections, the density is piecewise constant and only the bulk modulus varies randomly.

The analysis of transmitted fields is very similar to that of reflected fields, so we will only summarize it here, pointing out the additional steps that must be taken. The pressure p and normal velocity u_3 have again the representation (2.29) with the upgoing and downgoing wave amplitudes A and B satisfying (2.27); for simplicity we will not show dependence on ω and κ explicitly in some formulas. The local reflection

coefficient $\hat{p}$, is defined by (2.31) as before. In order to calculate the pressure at $z = -L$ we introduce the additional quantity

$$(3.165) \quad Z(z) = \hat{p}(-L)B^{-1}(z)$$

in $-L \leq z \leq 0$. Using (2.26) we see that at $z = -L$,

$$Z(-L) = (\zeta_1(-L))^{1/2} \left[(-L)e^{i\omega\tau(-L)/\varepsilon} - e^{-i\omega\tau(-L)/\varepsilon} \right]$$

so, from (2.32),

$$(3.166) \quad Z(-L) = (\zeta_1(-L))^{1/2} [{}_I(-L) - 1] e^{-i\omega\tau(-L)/\varepsilon}.$$

By direct computation we get the following ordinary differential equation for $Z(z)$

$$(3.167) \quad \frac{dZ(z)}{dz} = \left[\frac{i\omega n}{\varepsilon} (1 - e^{2i\omega\tau/\varepsilon}) - \frac{1}{2\zeta_1} \frac{d\zeta_1}{dz} e^{2i\omega\tau/\varepsilon} \right] Z$$

in $-L < z \leq 0$ with initial condition (3.166). Note that $Z(z)$ and $\hat{p}(z)$ are solutions of the coupled system (2.32) and (3.167) where Z depends on $\hat{p}$, but $\hat{p}$ is independent of Z .

The quantity Z is introduced because we can calculate the Fourier transform of the pressure at $z = -L$ by evaluating the solution of the system for Z , $\hat{p}$, at $z = 0$. Note that

$$(3.168) \quad \hat{p}(-L) = B(0^-)Z(0^-)$$

and that the downgoing amplitude $B(0^-)$ can be expressed in terms of $\hat{p}$, (0^-) . In fact, from the continuity of $\hat{p}$ and $\hat{u}$ at $z = 0$ we have that

$$(3.169) \quad \begin{aligned} \zeta_0^{1/2} [A(0^+) - B(0^+)] &= (\zeta_1(0))^{1/2} [A(0^-) - B(0^-)] , \\ \zeta_0^{-1/2} [A(0^+) + B(0^+)] &= (\zeta_1(0))^{-1/2} [A(0^-) + B(0^-)] . \end{aligned}$$

From the first of these conditions we have

$$(3.170) \quad \zeta_0^{1/2} [{}_I(0^+) - 1] B(0^+) = (\zeta_1(0))^{1/2} [{}_I(0^-) - 1] B(0^-).$$

Using (2.33) and the expression for $B(0^+)$ given by (A.1) (with $z_s = 0$) we can solve for $B(0^-)$ in terms of $\hat{p}$, (0^-) . Thus, from (3.168),

$$(3.171) \quad \hat{p}(-L) = (\zeta_1(0))^{-1/2} \frac{\varepsilon^{3/2}}{2} \hat{f}(\omega) \left(\frac{1 + {}_I(0)}{1 - {}_I(0^-)} \right) Z(0^-)$$

and the transmitted pressure is

$$(3.172) \quad p(t, \mathbf{x}, -L) = \frac{1}{2(2\pi)^3} \frac{1}{\varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 (\zeta_1(0))^{-1/2} e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon} \cdot \left(\frac{1 + {}_I(0)}{1 - {}_I(0^-)} \right) Z(0^-) d\omega d\boldsymbol{\kappa},$$

which is the analogue of (2.35) with $z_s = 0$. Expanding the fraction in (3.172) in a geometric series as we did in (2.36), we get

$$(3.173) \quad p(t, \mathbf{x}, -L) = \frac{1}{2(2\pi)^3} \frac{1}{\varepsilon^{3/2}} \iiint \hat{f}(\omega) \omega^2 (\zeta_1(0))^{-1/2} e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon} \cdot (1 + \cdot, I(0)) Z(0^-, \omega, \boldsymbol{\kappa}) \sum_{N=0}^{\infty} (\cdot, I(0))^N (\cdot, (0^-, \omega, \boldsymbol{\kappa}))^N d\omega d\boldsymbol{\kappa}.$$

We see from (3.173) that to calculate the coherent transmitted pressure we must calculate the expectation of $Z(0^-)$, $N(0^-)$ in the limit $\varepsilon \rightarrow 0$. For the transmitted pressure intensity

$$(3.174) \quad I_{TR}(t, \mathbf{x}, \bar{t}, \bar{\mathbf{x}}) = \lim_{\varepsilon \downarrow 0} \left\langle p \left(t + \frac{\varepsilon \bar{t}}{2}, \mathbf{x} + \frac{\varepsilon \bar{\mathbf{x}}}{2}, -L \right) p \left(t - \frac{\varepsilon \bar{t}}{2}, \mathbf{x} - \frac{\varepsilon \bar{\mathbf{x}}}{2}, -L \right) \right\rangle$$

we get an expression similar to (2.40), after the change of variables (2.39),

$$(3.175) \quad I_{TR}(t, \mathbf{x}, \bar{t}, \bar{\mathbf{x}}) = \frac{1}{4(2\pi)^3} \iiint d\omega d\boldsymbol{\kappa} |\hat{f}(\omega)|^2 \omega^2 (\zeta_1(0))^{-1} e^{i\omega(\boldsymbol{\kappa} \cdot \bar{\mathbf{x}} - \bar{t})} \cdot (1 + \cdot, I(0))^2 \sum_{N_1=0}^{\infty} \sum_{M_1=0}^{\infty} e^{-2i\omega(N_1 - M_1)\tau(-L)/\varepsilon} \cdot, {}^{N_1+M_1}_I(0) W^{N_1, M_1, 1, 1}(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$$

with the range of integration restricted so that $c_1 \kappa < 1$. Here $W^{N_1, M_1, 1, 1}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$ is defined as follows. Analogously to (3.4) and (3.10) let

$$(3.176) \quad \tilde{W}^{N_1, M_1, N_2, M_2} \left(z, t + \left[N_1 + M_1 + \frac{N_2 + M_2}{2} \right] \tau, \mathbf{x} - \left[N_1 + M_1 + \frac{N_2 + M_2}{2} \right] \nabla_{\boldsymbol{\kappa}} \tau, \boldsymbol{\kappa}, \omega \right) = \lim_{\varepsilon \downarrow 0} \frac{\omega^2}{(2\pi)^3} \iiint dh d\boldsymbol{\lambda} e^{iht} e^{-i\omega \boldsymbol{\lambda} \cdot \mathbf{x}} \cdot \left\langle \cdot, {}^{N_1}_I \left(z, \omega - \frac{\varepsilon h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2} \right), {}^{-M_1}_I \left(z, \omega + \frac{\varepsilon h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2} \right) \cdot Z^{N_2} \left(z, \omega - \frac{\varepsilon h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2} \right) \bar{Z}^{M_2} \left(z, \omega + \frac{\varepsilon h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2} \right) \right\rangle$$

and like (3.11) let

$$(3.177) \quad W^{N_1, M_1, N_2, M_2}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) = e^{2i\omega \tau(-L)[(N_1 - M_1) + \frac{1}{2}(N_2 - M_2)]/\varepsilon} \cdot \tilde{W}^{N_1, M_1, N_2, M_2}(z, t - \boldsymbol{\kappa} \cdot \mathbf{x}, \mathbf{x}, \boldsymbol{\kappa}, \omega).$$

Then W^{N_1, M_1, N_2, M_2} satisfies the analogue of (3.12), where we now omit the superscripts unless they change:

$$(3.178) \quad \begin{aligned} & \frac{\partial W}{\partial z} + (N_1 + M_1 + \frac{1}{2}(N_2 + M_2)) \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W \right\} \\ & - \frac{2\omega^2 \alpha}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ (N_1 + N_2)(M_1 + M_2) W^{N_1+1, M_1+1} \right. \\ & - (N_1^2 + M_1^2 + N_1 N_2 + M_1 M_2 + \frac{1}{2}(N_2 + M_2)) W \\ & \left. + N_1 M_1 W^{N_1-1, M_1-1} \right\} \\ & + \frac{\omega^2 \alpha}{c_1^2 (1 - c_1^2 \kappa^2)} [2(N_1 - M_1) + (N_2 - M_2)]^2 W = 0, \end{aligned}$$

$$W|_{z=-L} = (\zeta_1(-L))^{(N_2+M_2)/2} ({}_{,I}(-L))^{N_1} ({}_{,I}(-L))^{M_1} \cdot ({}_{,I}(-L)-1)^{N_2} ({}_{,I}(-L)-1)^{M_2} \delta(t) \delta(\mathbf{x}).$$

The off-diagonal terms in the sum in (3.175) are negligible in the limit $\varepsilon \rightarrow 0$. If we denote by W^N the diagonal part of W with $N_2 = M_2 = 1$, $W^{N,N,1,1}$, then, as in (3.28), (3.29),

$$(3.179) \quad I_{TR}(t, \mathbf{x}; \bar{t}, 0) = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 e^{-i\omega \bar{t}} \Lambda_{TR}(t, \mathbf{x}, \omega) d\omega$$

with

$$(3.180) \quad \Lambda_{TR}(t, \mathbf{x}, \omega) = \frac{1}{4(2\pi)^2} \iint d\boldsymbol{\kappa} (\zeta_1(0))^{-1} (1 + {}_{,I}(0))^2 \cdot \sum_{N=0}^{\infty} {}_{,I}^{2N}(0) W^N(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega),$$

where $W^N(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$ satisfies the analogue of (3.14)

$$(3.181) \quad \frac{\partial W^N}{\partial z} + (2N+1) \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^N}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^N \right\} - 2 \frac{\omega^2 \alpha}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ (N+1)^2 W^{N+1} - [N^2 + (N+1)^2] W^N + N^2 W^{N-1} \right\} = 0,$$

$$W^N|_{z=-L} = \zeta_1(-L) {}_{,I}^{2N}(-L) ({}_{,I}(-L)-1)^2 \delta(t) \delta(\mathbf{x}),$$

where $N \geq 0$, $-L < z < 0$, $t, \omega \in R^1$, $\mathbf{x} \in R^2$.

The solution W^N of (3.181) has again a probabilistic representation in terms of a canonical Markov chain $N(z)$ which is generated by the operator on the second line of (3.181). Note that this is different from the canonical Markov chain of the reflected signal (generated by the second line in (3.14)) discussed in §3.5; in particular, $N = 0$ is not an absorbing state now. We will not go into the details of the structure of the transmitted signal. We will only examine the form of the front of the pulse when it arrives at $z = -L$, as we did in §3.7 for the reflected signal.

As is §3.3, the coherent transmitted pressure $\langle p(t, \mathbf{x}, -L) \rangle$ can be obtained by averaging (3.173). Assuming for simplicity that ${}_{,I}(0) = 0$, which is the matched interface case at $z = 0$ that we used also in §3.3, we must calculate $\langle Z(0^-, \boldsymbol{\kappa}, \omega) \rangle$ in the limit $\varepsilon \rightarrow 0$. As in (3.16) this is given by

$$(3.182) \quad \langle Z(0^-, \boldsymbol{\kappa}, \omega) \rangle = \iiint dt d\mathbf{x} \tilde{W}^{0,0,1,0}(0, t, \mathbf{x}, \boldsymbol{\kappa}, \omega),$$

where $\tilde{W}^{N_1, M_1, N_2, M_2}$ is defined by (3.176) and is computed using (3.177) and the equations (3.178). The equations decouple again and we find that, as in (3.19), to leading order in ε ,

$$\langle Z(0^-, \boldsymbol{\kappa}, \omega) \rangle \approx (\zeta_1(-L))^{1/2} ({}_{,I}(-L)-1) e^{-i\omega \tau(-L)/\varepsilon} e^{-\omega^2 \beta_2(L)},$$

where

$$(3.183) \quad \beta_2(L) = \int_{-L}^0 \frac{2\alpha}{c_1^2 (1 - c_1^2 \kappa^2)} dz.$$

Thus

$$\langle p(t, \mathbf{x}, -L) \rangle \approx \frac{1}{2(2\pi)^3 \varepsilon^{3/2}} \iint \int d\omega d\boldsymbol{\kappa} \hat{f}(\omega) \omega^2 \left(\frac{\zeta_1(-L)}{\zeta_1(0)} \right)^{1/2} (, I(-L) - 1) \\ \cdot e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t - \tau(-L))/\varepsilon} e^{-\omega^2 \beta_2(L)},$$

which is the analogue of (3.20). The $\boldsymbol{\kappa}$ integral can again be evaluated approximately by the stationary phase method and to get explicit formulas we will assume from now on that the mean speed is uniform, $c_1 = \text{constant}$. Then $\tau(-L)$ is given by (3.21) and, as in (3.22), the stationary phase approximation yields

$$(3.184)$$

$$\langle p(t, \mathbf{x}, -L) \rangle \approx \frac{\sqrt{\varepsilon} \sin \theta}{4\pi L c_1} \frac{\partial}{\partial t} \left\{ \frac{1 - , I(-L)}{2\pi} \int d\omega \hat{f}(\omega) e^{-i\omega(t - L(c_1 \sin \theta)^{-1})/\varepsilon} e^{-\omega^2 \beta_2(L)} \right\}$$

Here θ is defined by

$$(3.185) \quad c_1 \kappa = \cos \theta = \sqrt{1 - \left(\frac{L}{c_1 t} \right)^2}$$

with $|\mathbf{x}| = r$, c_1 , t , L and θ related as in Fig. 3.1 except that for the case of transmission, $r/2$ is replaced by r and $c_1 t/2$ by $c_1 t$.

As in §3.7, we can calculate the local power spectral density Λ_{TR} in (3.180) (with $, I(0) = 0$) approximately for times just after the mean arrival time of the pulse at $z = -L$. Thus, we consider times t and horizontal distances r such that $(c_1 t)^2 \approx L^2 + r^2$ with $(c_1 t)^2 \geq L^2 + r^2$. A calculation very similar to the one that gave (3.90) now gives

$$(3.186) \quad \Lambda_{TR}(t, r, \omega) \approx \left(\frac{1 - , I(-L)}{4\pi c_1^2 t} \right)^2 \delta \left(t - \frac{L}{c_1 \sin \theta} \right) e^{-\beta L},$$

where β is given again by (3.91). The exponential term that involves β is now determined by the middle term in the braces in the second line of (3.181) with $N = 0$, the same way as in (3.90). From (3.179) the transmitted intensity (with $\bar{t} = 0$) near the arrival time at $z = -L$ is thus

$$(3.187) I_{TR}(t, r; 0, 0) \approx \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \omega^2 e^{-\beta L} d\omega \left(\frac{1 - , I(-L)}{4\pi c_1^2 t} \right)^2 \delta \left(t - \frac{L}{c_1 \sin \theta} \right).$$

When this is compared with the behavior of $\langle p(t, r, -L) \rangle^2$ from (3.184), we see that they differ only in the exponential factors in the ω integrals: $-2\omega^2 \beta_2(L)$ for the square of the coherent field and $-\beta L$ for the intensity. This is qualitatively similar to what we found for the reflected signal in §3.7 but the factors are different there (cf. (3.94), (3.95)). We conclude that when the statistics of the pulse are calculated a little after the deterministic (mean) arrival time at $z = -L$, the pulse has spread twice as much for the intensity as for the square of the mean field.

4. Direct simulation of the randomly reflected signals.

4.1. Introduction. In this section we will describe the numerical method used to solve the acoustic equations (2.17) for a given realization of the random medium. With a constant macroscopic material profile, we assume that the random fluctuations of the acoustical properties described by ν in (2.16) are piecewise constant in each microscopic layer. In each fine layer, ν will be taken as an independent random variable with a uniform distribution. More generally, we are interested in cases where the macroscopic medium properties are also variable and change smoothly with the depth. They can then be approximated quite accurately in the discrete problem by taking them to be piecewise constant on the z/ϵ^2 microscale. We place the point source at an elevation z_s in the homogeneous half space above the stack of layers (see Fig. 1.2) and compute the reflection of a pulse.

Many authors have written computer codes for simulating waves in layered media [73], [75], [55] and such codes are widely used in seismic exploration, for example. Our code was written to address the difficulties inherent in our multiple scales problem: we must compute waves scattered through hundreds, or even thousands, of subwavelength microlayers, and their effects must be computed accurately. That is, computation of spatially averaged properties of the wave field is not sufficient for our purposes. We must be able to construct statistical fluctuations accurately. Furthermore, the code must be efficient enough to be rerun many times for different members of a statistical ensemble.

Following the usual methods for layer codes, we first reduce the wave problem to an ordinary differential equation in depth z by Fourier transformation in time and then in the two lateral space dimensions. These transformations are analogous to our analytical treatment above. Several subtle numerical problems, however, need to be addressed and our approaches to them are explained more fully below. We basically follow the formalism of Kennett [55] for the elastic case, which we have adapted for the acoustic equations. This method avoids the computation of large exponentials that would result from straightforward solution of an ordinary differential equation in z . We give a brief outline of the method in §4.2. We now discuss some computational aspects of Fourier transforms.

A first numerical difficulty with Fourier transforms is that they will produce poles on the real axis for wave problems. This effect can be most easily seen in the homogeneous case where the solution is the well-known Weyl integral (cf. Appendix A). Most numerical schemes (see, for example, [55]) circumvent this difficulty by introducing a small attenuation coefficient which complexifies the velocity $c = \sqrt{K/\rho}$ and therefore moves the poles away from the axis of integration ω . We prefer to use another well-known numerical trick whose advantage is that it is entirely transparent in the physical domain $(\mathbf{x}, t)$. We multiply the original unknowns p and u by a small decaying exponential and solve for the new unknowns¹⁴

$$(4.1) \quad \tilde{\mathbf{u}}(t, \mathbf{x}) = e^{-\xi t} \mathbf{u}(t, \mathbf{x}) \text{ and } \tilde{p}(t, \mathbf{x}) = e^{-\xi t} p(t, \mathbf{x}).$$

This also moves the poles away from the axis of integration in the frequency domain.

The vertical direction of the point source makes the solution radially symmetric. Therefore we work in polar coordinates and equivalently use the Hankel transform of

¹⁴ Afterwards, it is a simple matter to revert back to the original unknowns by multiplication by $e^{+\xi t}$.

order zero instead of the spatial Fourier transform:

$$(4.2) \quad \tilde{g}(k) = \frac{1}{2\pi} \int_0^\infty J_0(kr) g(r) r dr,$$

where J_0 is the zeroth order Bessel function. The original unknowns p and u (the vertical component of $\mathbf{u}$) are recovered by the inverse transform

$$(4.3) \quad p(t, r, z) = \frac{1}{2\pi} e^{\xi t} \int \int_0^\infty e^{-i\omega t} J_0(kr) \tilde{p}(k, z, \omega) k dk d\omega$$

and a similar formula for u . Here $\tilde{p}$ and $\tilde{u}$ are solutions of a system of equations similar to (2.22), which, omitting tildes, we write in matrix form as

$$(4.4) \quad \frac{d}{dz} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 & \frac{\omega^c}{K^c} \\ \Omega^c \rho & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} + \delta(z - z_s) \hat{f}(\omega^c) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where we use the superscript c to denote complexified frequency and bulk modulus

$$(4.5) \quad \omega^c = \omega + i\xi,$$

$$(4.6) \quad \Omega^c = \omega + i \left(\xi + \frac{\sigma}{\rho} \right),$$

$$(4.7) \quad \frac{1}{K^c} = \frac{1}{K} - \frac{k^2}{\rho \omega^c \Omega^c}.$$

We have also allowed for the simulation of dissipative media in which σ is the rate of dissipation (cf. (2.1)).

The frequency domain Fourier integral in (4.3) is easy to perform because the pulse is bandlimited. On the other hand, the wave number integral is the delicate step of the numerical scheme. The range of k depends on several other parameters besides the frequency range, namely, the position of the receivers and the properties of the layers, and requires some tuning. Several algorithms are available to perform the Hankel transform (4.2) (which is its own inverse). Candel proposed [76] a scheme which is competitive when several orders of the Hankel transform are needed -such as in elastic medium computations. DiNapoli and Davenport [73] and Schmidt [75] proposed equivalent schemes based on the asymptotic approximation of the Bessel function for large argument and are therefore valid only for receivers very distant from the source. We use a method proposed originally by Tsang et.al. [74], valid for kernels containing a term decreasing at least exponentially, which is the case here because the trick (4.1) produces exponential decay in k as a result. In this method we use Parseval's relation to do the k integral in (4.3). Repeated evaluation of the Bessel function is avoided because the Fourier transform of $e^{-\gamma k} k J_0(kr)$ with respect to k is an elementary function. Here γ is a small positive number smaller than the decay rate in k of $\tilde{p}$. This method is computationally expensive compared with the other two but is valid for all ranges r . Furthermore, in our case since the number of layers is very large -of the order of ϵ^{-2} - most of the time is spent in computing the kernel of (4.3) and the time spent in the Hankel transform is not critical.

For the numerical solution of (4.4) we need to restrict the stack of layers to a finite depth. We set the "bottom" deep enough not to be detected by the pulse within the recording time span and assume that the remainder of the half space $z < 0$ is homogeneous. This enables us to impose the same radiation condition below as above, i.e., there is no upcoming wave at $z = -\infty$ and no downgoing wave at $z = +\infty$.

4.2. Layer code for computation of reflected signals. As in §2 (equation (2.27) for unknowns A and B), we change the unknowns into up- and downgoing waves but now we do not distinguish between the smooth and the rapid variation of the solution. We obtain a diagonal system of ordinary differential equations

$$(4.8) \quad \frac{d}{dz} \begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} i\beta & 0 \\ 0 & -i\beta \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix} + \delta(z - z_s) \begin{bmatrix} F^U(\omega, k) \\ F^D(\omega, k) \end{bmatrix}$$

with boundary conditions at infinity given by

$$(4.9) \quad \lim_{z \rightarrow \infty} D = \lim_{z \rightarrow -\infty} U = 0.$$

U and D stand for “up” and “down” and will be related from one layer to the other by reflection and transmission coefficients. If we denote by $v = (\tilde{u}, \tilde{p})$ the vector of physical unknowns, then $V = (U, D)$ is related to v by $V = M^{-1}v$ and we have a similar relation for the point force. The matrix M is obtained from diagonalization and normalization of (4.4):

$$(4.10) \quad \begin{bmatrix} 0 & \frac{\omega^c}{K^c} \\ \Omega^c \rho & 0 \end{bmatrix} = M \begin{bmatrix} i\beta & 0 \\ 0 & -i\beta \end{bmatrix} M^{-1}$$

with

$$(4.11) \quad M = \begin{bmatrix} (2\rho c_c)^{-\frac{1}{2}} & -(2\rho c_c)^{-\frac{1}{2}} \\ (\rho c_c/2)^{\frac{1}{2}} & (\rho c_c/2)^{\frac{1}{2}} \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} (\rho c_c/2)^{\frac{1}{2}} & (2\rho c_c)^{-\frac{1}{2}} \\ -(\rho c_c/2)^{\frac{1}{2}} & (2\rho c_c)^{-\frac{1}{2}} \end{bmatrix},$$

where

$$(4.12) \quad c_c^2 = \frac{\Omega^c K^c}{\omega^c \rho}, \quad \beta = \frac{\Omega^c}{c_c}.$$

Whenever the physical properties ρ and K are constant, as assumed in the previous section, (4.10) can be easily integrated. Consider two adjacent layers, $l-1$ above l . Denote by V_l^+ (respectively, V_l^-) the value of V just above (respectively, below) the interface l between the layers. Integration of (4.10) between interfaces l and $l+1$ (“in” the layer l) provides

$$(4.13) \quad V_l^- = \begin{bmatrix} e^{i\beta h_l} & 0 \\ 0 & e^{-i\beta h_l} \end{bmatrix} V_{l+1}^+.$$

On the other hand, continuity of v at the interfaces give the relation

$$(4.14) \quad V_l^+ = M_{l-1}^{-1} M_l V_l^-$$

where M_l is the matrix M , for layer l , defined by (4.11) with properties ρ_l and K_l and thickness h_l . Let us denote by Q^{ln} the matrix relating vectors V between two levels l and n separated either by an interface (4.14) or a layer (4.13). We have $V_l = Q^{ln} V_n$.

A possible method for the solution of the overall problem when the total depth of the stack of layers is not too large, is to multiply all the elementary matrices $Q^{l,l+1}$ to obtain the global reflection-transmission coefficient of the stack. This, however, introduces numerical instabilities because of the multiplication of occasional large exponentials, when many layers are to be computed.

An equivalent formulation that is more stable numerically can be given in terms of scattering from a large stack of microlayers, from level l to level n , as described above. The reflection and transmission coefficients for scattering from the stack relate the vector (U_n, D_l) of waves entering the stack, to the vector (U_l, D_n) of waves exiting. Thus

$$(4.15) \quad \begin{bmatrix} U_l \\ D_n \end{bmatrix} = \begin{bmatrix} T_u^{ln} & R_d^{ln} \\ R_u^{ln} & T_d^{ln} \end{bmatrix} \begin{bmatrix} U_n \\ D_l \end{bmatrix}.$$

The two formulations are compared schematically in Fig. 4.1.

FIG. 4.1. *Reflection-transmission diagram.*

Starting with

$$(4.16) \quad Q^{L\infty} = M_L^{-1} M_\infty$$

at the interface between the bottom layer L and an underlying homogeneous half space with unit density and bulk modulus, we obtain equating matrices

$$(4.17) \quad \begin{bmatrix} T_u^{L\infty} & R_d^{L\infty} \\ R_u^{L\infty} & T_d^{L\infty} \end{bmatrix} = \frac{1}{Q_{22}^{L\infty}} \begin{bmatrix} Q_{11}^{L\infty} Q_{22}^{L\infty} - Q_{21}^{L\infty} Q_{12}^{L\infty} & Q_{12}^{L\infty} \\ -Q_{21}^{L\infty} & 1 \end{bmatrix}.$$

The transfer matrix is then updated from one level to the next using the following recursion relations:

$$(4.18) \quad \begin{bmatrix} T_u^{jn} & R_d^{jn} \\ R_u^{jn} & T_d^{jn} \end{bmatrix} = \frac{1}{Q_{11}^{lj} - R_d^{ln} Q_{21}^{lj}} \cdot \begin{bmatrix} T_u^{ln} & R_d^{ln} Q_{22}^{lj} - Q_{12}^{lj} \\ R_u^{ln} Q_{11}^{lj} + (T_d^{ln} T_u^{ln} - R_u^{ln} R_d^{ln}) Q_{21}^{lj} & T_d^{ln} (Q_{22}^{lj} Q_{11}^{lj} - Q_{21}^{lj} Q_{12}^{lj}) \end{bmatrix},$$

where $j = l - 1$ and $l < n$. The source is introduced through a fictitious interface zero at the level z_s , where we have

$$(4.19) \quad V_0^+ = V_0^- + F$$

with

$$(4.20) \quad F = \begin{bmatrix} F^U \\ F^D \end{bmatrix} = M^{-1} \hat{f} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and we use the following discretized version of the boundary condition (4.9):

$$(4.21) \quad D_0^+ = U_L^- = 0.$$

Eventually (4.15)-(4.21) provide the global reflection coefficient for the complete stack of layers

$$(4.22) \quad \begin{bmatrix} U_1 \\ D_\infty \end{bmatrix} = \begin{bmatrix} T_u^{1\infty} & R_d^{1\infty} \\ R_u^{1\infty} & T_d^{1\infty} \end{bmatrix} \begin{bmatrix} U_\infty \\ D_1 \end{bmatrix}.$$

We note that in the case of a plane wave incident from above the stack of layers, with an angle of incidence θ , there is a single summation over ω and the wavenumber is taken such that $k/\omega = \sin \theta$.

The quantity of interest is the “downgoing” reflection coefficient $R_d^{1\infty}$. In the case of a point source we want to recover the physical unknowns, particularly the pressure at the interface 1 where the receivers are located. We get

$$(4.23) \quad v_1 = M_0 V_1^+ = e^{-i\beta_0 z_s} F^D M_0 \begin{bmatrix} R_d^{1\infty} \\ 1 \end{bmatrix}.$$

If all we want is the reflected pressure as defined in (2.35) we write

$$(4.24) \quad \tilde{p}_{\text{refl}}(k, 0^+, \omega) = \frac{\hat{f}}{2} e^{-i\beta_0 z_s} R_d^{1\infty}$$

and use (4.3) to obtain the reflected pressure as a function of space and time.

4.3. Flowchart of the computation.

- ⇒ Define the layered medium properties ρ_l , K_l for the L layers σ and ξ , choose a sampling rate in time and space, and deduce the correct sampling in frequency $d\omega$ and wavenumber dk .
- ⇒ Loop on ω .
 - Loop on k . Compute $Q^{L\infty}(k)$ (equation (4.16)).
 - Loop on k . Compute $T^{L\infty}(k)$, $R^{L\infty}(k)$ (equation (4.17)).
 - Loop on layer $l = L, 1$; loop on k . Update $T^{l-1\infty}(k)$, $R^{l-1\infty}(k)$ (equation (4.18)).
 - Loop on k . Compute $\tilde{p}_{\text{refl}}(\omega, k)$ (equation 4.24)
 - Compute $\tilde{p}_{\text{refl}}(\omega, r)$ by Hankel transform.
- ⇒ Loop on r . Compute $p_{\text{refl}}(t, r, 0^+)$ by Fourier transform.

4.4. Periodic versus random medium: A numerical experiment.

We want to use the layer code to study waves propagating in a large number of very thin layers. When the properties of the layers vary randomly, the quantities of interest are the statistics of the reflection coefficient. On the other hand, when the acoustic properties vary periodically from one layer to the other the effective medium theory (or homogenization theory) of §2.2 tells us that the stack of layers behaves as

a *single* homogeneous layer having *average* properties (when probed by a wave having wavelength much larger than one elementary layer.) In the nondissipative, one dimensional case the properties of the equivalent homogeneous layer can be obtained in the wavenumber-frequency domain by averaging equation (4.4) over the stack of layers

$$(4.25) \quad \frac{d}{dz} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} 0 & \frac{\omega^c}{\tilde{K}^c} \\ \omega^c \tilde{\rho} & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} + \delta(z - z_s) \begin{bmatrix} 0 \\ \hat{f}(\omega^c) \end{bmatrix},$$

where the averaged properties are defined by

$$(4.26) \quad \tilde{\rho} = \int_{\text{stack}} \rho(z) dz \text{ and } \frac{1}{\rho_h} = \int_{\text{stack}} \frac{1}{\rho(z)} dz$$

and

$$(4.27) \quad \frac{1}{\tilde{K}^c} = \frac{1}{\tilde{K}} - \frac{k^2}{\rho_h \omega^{c2}} \text{ and } \frac{1}{\tilde{K}} = \int_{\text{stack}} \frac{1}{K(z)} dz$$

with complex frequency ω^c defined by (4.5).

To illustrate the theory we present first a set of numerical simulations that serve also as validation of the numerical method. We compare the propagation of a pulse of the form¹⁵

$$f^{0.1}(t) = e^{-0.5(5\pi t)^2}$$

coming from a source at $z_s = 0.2$ above the surface in a homogeneous half space overlying three different media:

- A single layer of thickness 3 where the acoustic properties are $\rho = 1$, $K = 2$
- A stack of 300 layers of thickness $\varepsilon^2 = 10^{-2}$ and density $\rho = 1$, where the bulk modulus takes alternatively the values $K_1 = 1.66$ and $K_2 = 2.5$. This medium behaves asymptotically like a uniform layer with parameters computed according to the effective medium theory of §2.2, using (4.26), (4.27).
- A stack of 300 layers of thickness $\varepsilon^2 = 10^{-2}$ and density $\rho = 1$, where $1/K$ takes random values uniformly distributed between 0.35 and 0.65.

In all three cases the kernel (4.24) is computed for 512 frequencies ranging from -255 to 256 radians per second and 8192 wavenumbers ranging from -409.5 to 409.6 . The reflected pressure is computed at 5 receivers on the interface $z = 0$, at radial distances varying from $r = 0$ to $r = 2$, the source being 0.2 units above the origin. The two 300-layer computations required 3.5 minutes of cpu-time on the Cray Y-MP.

In Fig. 1.3a we show the reflected pressure recorded for both the homogeneous and periodic case since they are indistinguishable. The left axis is distance of the receiver from the origin. The bottom axis is time. The check marks are the theoretical arrival times of the reflected waves in the case of a homogeneous layer. Consider, for instance, the third curve starting from the bottom of the figure. It shows the reflected pressure measured at a receiver placed at $z = 0$ and $r = 1$. The first event is the arrival of the head wave at $t = 0.84$. The next arrival, at $t = 1$, comes from the reflection from the interface; it coincides with the direct arrival that has been subtracted. The last arrival at $t = 4.5$ is the reflection from the bottom of the layer at $z = -2.7$. As mentioned in

¹⁵ The superscript on f denotes the value of ε .

the Introduction, §1, it is surprising that even the head wave is well accounted for by the effective medium theory (homogenization) in the periodic medium.

In Fig. 1.3b we show the results for the wave propagating in a stack of random layers with randomly changing parameters. The main reflections are seen clearly, at the correct arrival times based on effective medium theory. In contrast with the periodic case, however, multiple scattering produces a long noisy tail.

4.5. Statistics of randomly scattered signals. In this section we examine how statistical properties, namely, the intensity and the power spectrum, of numerically simulated random signals agree with the theory. We have done simulations for a random medium with a constant background so that the comparison with the theory is simpler since in this case there is a formula for the power spectral density (3.77). We have chosen to simulate the matched case, i.e., the background velocity and density are constant and equal to 1 everywhere. Only the bulk modulus varies randomly and $\alpha = \bar{\nu}^2/24$, where $\bar{\nu} = 0.3$ is the maximum of the uniformly distributed random variable ν . This leads to $\alpha = 3.75 \times 10^{-3}$. The signals are generated by propagating a pulse of the form

$$f^{0.05}(t) = e^{-0.5(10\pi t)^2}$$

from a source placed at $z = 0.2$ above the interface between the homogeneous half space and a stack of 400 layers of width $\varepsilon^2 = 2.5 \times 10^{-3}$. Receivers are located on the interface, at radial distances 0.4 and 0.8 from the origin located below the source. The results presented here show the statistics of the pressure field computed over 22 independent realizations of the random medium. Each realization requires 8 minutes of cpu time (single processor) on the Cray Y-MP. For each realization the reflection coefficient (4.24) is computed for 4096 wavenumbers ranging from -409.4 to 409.6 and 512 frequencies ranging from -255. to 256. $rad.s^{-1}$. In Fig. 4.2 the intensity ((2.38), with $\bar{t} = \bar{x} = 0$) of the reflected pressure is shown for three receivers as a function of time. The theoretical curve is obtained by numerical integration of (3.28) with $\bar{t} = 0$. The power spectrum (3.77) is shown in Fig. 4.3 as a function of time for a fixed frequency $\omega = 3.5$. We have used the windowing scheme described in Appendix E with windows of 128 samples.

5. Numerical solution of the transport equations.

5.1. Introduction. The local power spectral density contains information about the large-scale structure of the probed medium because the coefficients of the transport equations (3.14) depend on the mean sound speed. We want to solve the transport equations numerically, evaluate the local power spectral density, and interpret the results. This will give us an understanding of multiple scattering in physically interesting cases where we have no analytical solutions. It will also validate the observations of §3.8 concerning turning points and explain what the results mean physically. Most important, as we show in §6, the transport equations are needed in the solution of inverse problems.

The adjoint equations (3.45) introduced in §3.5 are more convenient for numerical calculations. When $I(0) = 0$, so that the medium is matched at the interface $z = 0$, we must solve

FIG. 4.2. Comparison of reflected intensity with the theory (point source).

FIG. 4.3. Comparison of local power spectral densities with the theory (point source).

$$(5.1) \quad \frac{\partial W^N}{\partial z} + 2N \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^N}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^N \right\} \\ + \frac{2\omega^2 \alpha}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ (N+1)^2 W^{N+1} - 2N^2 W^N + (N-1)^2 W^{N-1} \right\} = 0$$

in $z < 0$ with

$$W^N|_{z=0} = \delta(t) \delta(\mathbf{x}) \delta_{N1}.$$

The local power spectral density (3.30) is now given by

$$(5.2) \quad \Lambda(t, \mathbf{x}, \omega) = \frac{1}{4(2\pi)^2} \iint d\boldsymbol{\kappa} W^0(-L^*, t, \mathbf{x}, \boldsymbol{\kappa}, \omega),$$

where L^* is larger than the depth the waves penetrate up to time $t/2$.

There are three basic difficulties associated with equations (5.1). The characteristics depend on z , the terminal conditions are singular, and the system is infinite. To straighten the characteristics we introduce the travel time

$$(5.3) \quad \tau = \tau(z, \kappa) = 2 \int_0^z \frac{ds}{c_1(s)(1 - c_1^2(s)\kappa^2)^{1/2}}.$$

Note that this is actually twice the travel time and that it is different from the oblique, plane wave travel time (2.24). It is twice the travel time along the geometrical acoustics rays as described in §3.7. This will make the coefficients of the time derivative in (5.1) constant. However, those of the space derivatives will now depend on τ and it is not possible to straighten both of them. This difficulty and part of the singularity in the terminal conditions is handled by Fourier transformation of the $\mathbf{x}$ variables. Of course, we need a lot of resolution in the Fourier transform to capture the singular form of the terminal conditions, but as the calculation advances we can reduce the resolution. Finally, we must truncate the infinite system to a finite one. After repeated convergence tests we have found that setting $W^N = 0$ for $N > 10$ does not affect the overall accuracy of Λ . The remaining singularity in t of the terminal data is handled by a change of dependent variables based on the progressing wave expansion explained below.

With these changes, in particular with τ as independent variable, the adjoint transport equations (5.1) become

$$(5.4) \quad \frac{\partial \hat{W}^N}{\partial \tau} + N \left\{ \frac{\partial \hat{W}^N}{\partial t} + ic_1^2 \boldsymbol{\kappa} \cdot \mathbf{I} \hat{W}^N \right\} \\ + \frac{\omega^2 \alpha}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \left\{ (N+1)^2 \hat{W}^{N+1} - 2N^2 \hat{W}^N + (N-1)^2 \hat{W}^{N-1} \right\} = 0$$

in $\tau < 0$ and with

$$\hat{W}^N|_{\tau=0} = \delta(t) \delta_{N1}$$

and

$$\hat{W}^N = \iint d\mathbf{x} e^{-i\mathbf{l}\cdot\mathbf{x}} W^N(\tau, t, \mathbf{x}, \boldsymbol{\kappa}, \omega).$$

The local power spectral density is given by

$$(5.5) \quad \Lambda(t, \mathbf{x}, \omega) = \frac{1}{4(2\pi)^4} \iiint d\mathbf{l} d\boldsymbol{\kappa} e^{i\mathbf{l}\cdot\mathbf{x}} \hat{W}^0(-t, t, \mathbf{l}, \boldsymbol{\kappa}, \omega).$$

Note that the argument τ is evaluated at $-t$ in (5.5), which is the depth (in twice travel time units) beyond which the waves do not penetrate by time t .

Except for the fact that for times t near the first arrival of the wave at $\mathbf{x}$ we need fine resolution in $\mathbf{l}$ to get an accurate determination of Λ , the main difficulty in solving (5.4) is due to the delta function in the terminal conditions. Since $\hat{W}^0$ is evaluated on the characteristic in (5.5) this means that we have to resolve the singularity in the solution. This difficulty is eased somewhat if we do the $\boldsymbol{\kappa}$ integration first in (5.5). Singularities in the coefficients of (5.4), now considered functions of τ , do not cause problems as we saw in §3.8. The numerical calculations show clearly that the solutions converge as we refine them near turning points. In the actual computations [78], the Fourier transform in $\mathbf{l}$ is not two dimensional as in (5.4), (5.5) but only one-dimensional, in the direction $\boldsymbol{\kappa}$. We omit the detailed description of this elementary simplification.

In what follows we solve (5.4) only for $N = 1, 2, \dots, 10$ without the $N = 0$ equation. It decouples from the rest and $\hat{W}^0$ is given by

$$(5.6) \quad \hat{W}^0 = -\omega^2 \int_0^\tau \frac{\alpha(s)}{c_1(s) \sqrt{1 - c_1^2(s) \kappa^2}} \hat{W}^1(s, t, \mathbf{l}, \boldsymbol{\kappa}, \omega) ds.$$

5.2. The progressing wave expansion. We denote the truncated¹⁶ vector of $\hat{W}^N$ by $\hat{\mathbf{W}}$ and decompose it into two parts, a singular and a regular part. First we construct a representation of the solution in terms of generalized functions¹⁷, a progressing wave expansion [77], which determines analytically the singular part. If we retain only a finite number of terms in this representation then we get an equation for the remainder that is progressively less singular as we keep more terms. In our case we need only two terms but it is instructive to see the form of the expansion, in particular, the form it has after inverting the Fourier transform in $\mathbf{l}$.

Briefly (details can be found in [12], [78]), we write

$$(5.7) \quad \hat{\mathbf{W}} = \hat{\mathbf{W}}_s + \hat{\mathbf{W}}_r,$$

where $\hat{\mathbf{W}}_s$ is the singular part and $\hat{\mathbf{W}}_r$ is the regular remainder. Let

$$(5.8) \quad \hat{\mathbf{W}}(\tau, t, \mathbf{l}, \boldsymbol{\kappa}, \omega) = \sum_{M=1}^{\infty} \sum_{j=-1}^{\infty} S_j(t - M\tau) \hat{\mathbf{g}}^{M,j}(\tau, t, \mathbf{l}, \boldsymbol{\kappa}, \omega),$$

where the S_j 's are a sequence of generalized functions of increasing smoothness. They are functions of the characteristic coordinates $t - M\tau$ of the hyperbolic system (5.4) and are given by

$$S_{-1}(x) = \delta(x), \quad S_0(x) = H(x), \quad \dots$$

¹⁶ This is a ten-dimensional vector in our numerical computations.

¹⁷ In our case delta functions, Heaviside functions, and their integrals.

$$S_{j+1}(x) = \int_{-\infty}^x S_j(s) ds = \frac{x^j}{j!} H(x),$$

where $\delta(x)$ is the Dirac delta function and $H(x)$ is the Heaviside unit function. The $\hat{\mathbf{g}}^{M,j}$ are *smooth* vector functions. In order to construct generalized solutions $\hat{\mathbf{W}}$ of the progressing wave type (5.8), we substitute the expansion into the differential equation (5.4) and differentiate the generalized functions. Then we demand that the resulting smooth factors of the distributions S_j vanish at each order. Thus we solve for the unknown coefficients $\hat{\mathbf{g}}^{M,j}$. By increasing the number j , we can ensure that the remaining terms will be as smooth as desired. When we carry out the inverse Fourier transform in $\mathbf{l}$ we see that delta functions enter at all orders in the expansion. So the desingularization is easy in the Fourier domain but there is no obvious way to collect the delta functions in the physical variables.

The equation for the remainder, which is nonsingular after the singular terms have been extracted, is again (5.4) except that the right-hand side is no longer zero but is equal to an explicitly known function $\mathbf{R}$, coming from the singular part. The terminal conditions for the remainder are identically zero. In vector form the problem for $\hat{\mathbf{W}}_r$ becomes

$$(5.9) \quad \frac{\partial \hat{\mathbf{W}}_r}{\partial \tau} + \mathcal{L}_0 \left\{ \frac{\partial \hat{\mathbf{W}}_r}{\partial t} + i c_1^2 \boldsymbol{\kappa} \cdot \mathbf{l} \hat{\mathbf{W}}_r \right\} + \frac{\alpha \omega^2}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \mathcal{L}_1 \hat{\mathbf{W}}_r = \mathbf{R}$$

in $\tau < 0$ and with

$$\hat{\mathbf{W}}_r |_{\tau=0} = \mathbf{0},$$

where

$$\hat{\mathbf{W}}_r = [\hat{W}^1, \dots, \hat{W}^{10}]^T$$

and

$$\mathcal{L}_0 = \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & \ddots & \end{pmatrix},$$

$$\mathcal{L}_1 = \begin{pmatrix} -2 & 4 & 0 & 0 & \dots \\ 1 & -8 & 9 & 0 & \dots \\ 0 & 4 & -18 & 16 & \dots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

5.3. Numerical integration of the regularized equations. Because of the size of the problem, we have chosen simple, low-order integration schemes to limit the number of floating point operations. We obtain a tridiagonal linear system by integrating the regularized equations (5.9) along characteristics, as in [78] and [12]. This system must be solved at each step of the computation, i.e., for each fixed¹⁸

¹⁸ We denote the vectors $\mathbf{l}$ and $\boldsymbol{\kappa}$ by l and κ since they are scalar in the actual computations.

$\tau, t, \kappa, l, \omega$. We use a second-order accurate trapezoidal integration scheme. The κ integration in the evaluation of the power spectral density (5.5) is done by Simpson's rule. We can use a higher-order scheme here since we want to integrate right up to turning points and this integration is not performed at each τ and t .

The natural way to solve the regularized equations (5.9) is to fix a triple of the independent parameters (l, κ, ω) and solve the discrete equations by marching in travel time τ and time t . This method does *not* vectorize because of the recursiveness of timestepping schemes, where the current step is computed from the previous one. Moreover, the outermost loops on the parameters (l, κ, ω) are the most numerous (see below). An inner loop is one that vectorizes or parallelizes with maximum efficiency since there is nothing else inside an inner loop which can hinder the simultaneous evaluation of multiple operations on a vector or matrix. Thus we want to have the loop with the highest repeat count as the innermost loop. We can achieve this goal by "turning the code inside-out." To this end we define a new parameter

$$p = (l, \kappa, \omega)$$

to be a triple of fixed parameter values. For optimal vectorization, the loop order should then be:

- loop on τ – travel time
 - loop on t – time
 - * loop on p – parameters
 - * next p
 - next t
- next τ

The actual solution of the linear system and the integration to obtain Λ are exactly as before, except that they are now performed entirely within the inner p -loop, i.e., for *fixed* τ and t we solve a large number -typically about 15000- of independent problems, parametrized by p . With this approach, on a large vector supercomputer we are able to attain a speed-up factor of 500 over a fast scalar machine. A typical simulation with a travel time-time grid of (100×100) , 128 values each of κ and l and a single frequency runs in 15 minutes on a Cray Y-MP. Here we have solved 16384 different plane wave problems with each one taking only 0.06 seconds.

5.4. Numerical results. Recall first the form of the exact solution for the homogeneous, matched case from §3.6, equation (3.77) with $c_1 = c_0 = 1$:

$$(5.10) \quad \Lambda(t, r, \omega) = \frac{1}{4(2\pi)^2} \frac{\alpha \omega^2 \left(1 - \frac{r^2}{t^2}\right)^{-1/2}}{t^2 \left[1 + \alpha \omega^2 t \left(1 - \frac{r^2}{t^2}\right)^{-1/2}\right]^2} .$$

The code reproduces this formula very accurately [78]. In particular, we observe the well-defined arrival of the signals along the line $r = t$, the overall $1/t^2$ decay, and the fact that while varying r/t we "scan" all the angles of the point source. In a one-dimensional simulation, we required multiple frequencies in order to capture the complete behavior of the solution (see [11]).

If we look at the transport equation (3.14), we notice that a turning point¹⁹ always exists when $c_1^2 \kappa^2$ approaches 1. Since we will not integrate through turning points,

¹⁹ A detailed analysis is presented in §3.9.

this restricts the range of the κ -integration. Moreover, if c_1 is an *increasing* function of depth, then the permissible range of integration will *decrease* with depth. Physically, this means that once a ray at a fixed slowness κ has been totally reflected, it will no longer contribute to the multiple scattering and hence to the power spectral density at later times.

FIG. 5.1. *Decreasing (on left) and increasing (on right) step profiles for the sound speed.*

We consider two profiles for the sound speed as a function of depth. Figure 5.1 shows a decreasing and an increasing profile. They both begin at the surface, $z = 0$, with speed 1, have a sharp step at depth $z = -2.5$, halfway down the profile, and then continue with a uniform speed of $1/2$ and 2 , respectively. We choose these two profiles since they represent two differing physical situations and the contrast between them will highlight some phenomena associated with turning points. Also, the first portions of both profiles are identical and are the same as a uniform background with mean sound speed 1. Thus we can compare our results, for early times, with the above formula (5.10).

The series of graphs in Fig. 5.2(a)-(i) are slices of the local power spectral density surface $\Lambda(r, t, \omega)$ for a fixed r and a fixed $\omega = 1$. On each graph we compare three curves: the solution for the increasing step with depth-dependent turning point, the solution for the decreasing step, and the solution (5.10) for the homogeneous case (which has a uniform sound speed, equal to 1, through the whole profile). There are three groups of graphs, each depicting a different zone in the profile and having a different vertical scale. The first group (a)-(c) are for ranges very close to the point source. We see that the curves for both steps lie on top of each other and agree

FIG. 5.2. Local power spectral density for the step profiles and the uniform background - "before" the step.

FIG. 5.2. Local power spectral density for the step profiles and the uniform background - "at" the step.

FIG. 5.3. Surface plot of the local power spectral density for the step profile with turning point.

FIG. 5.4. Surface plot of the local power spectral density for the decreasing step profile.

FIG. 5.5. Surface plot of the local power spectral density for the linear profile with turning point.

well with the homogeneous solution. Note that the first arrivals are at $t = r$, which is correct for a mean sound speed of 1. The second group, Figs. (d)-(f), is at ranges which “see” the start of the step around $z = -2$. In the vicinity of the first arrival, all three curves agree but starting at $t = 2.5$ the effect of the increasing or decreasing sound speed begins to be detected and the curves separate. The nature of this separation is clarified in the third group, Figs. (g)-(i) which is for ranges that are sensitive to the varying part of the sound speed profile. The first thing to notice is that the arrivals of the increasing step are earlier, while those of the decreasing step are delayed. Looking at the tails of the curves, we see that the decreasing step and the homogeneous solution approach asymptotically the same value which is *higher* than that for the increasing step. This is the lowering of the energy content of the reflected wave due to the total internal reflection that we expect. We also note that the amplitude of the first arrival for the increasing profile is markedly higher than that of the decreasing step. This is due to the enhanced reflection by the total internal reflection at the turning point.

We have thus a good physical understanding of what happens to a wave that approaches a turning point in a randomly layered medium. The surface plots in Fig. 5.3 (increasing step) and 5.4 (decreasing step) show the phenomena synoptically. Initially, for early times and small ranges, the arrivals are along the line $r = t$; around $t = 2.5$ the slope of the arrival line changes to reflect the speeding up or slowing down of the sound speed profile; finally, we see the increased amplitude due to the enhanced reflection at the turning point and then the rapid decay for large t and r in the case of the increasing step.

Clearly, the local power spectral density does indeed contain information about the large-scale structure of the medium. In fact, just by looking at the surfaces, we can get a good idea of what the mean sound speed profile actually is. An interesting case, which is close to what happens in the earth [79], is a sound speed profile that

increases linearly with depth from speed 1 at the surface to speed 2, say. Then we have turning points at all depths. The surface plots of Fig. 5.5 exhibits what we expect: a curved line of arrival and then a rapid decay due to the internal reflection by turning points.

6. Inverse problems.

6.1. Inversion from reflection of normally incident plane waves. Using the theory presented above we can now formulate and solve a statistical inverse problem. We will determine the macroscopic properties of a randomly stratified medium from waves which have been reflected back to the surface $z = 0$. Our method is intended to complement existing deterministic methods [80], [81] that attempt to recover all variations of the material properties without regard to the spatial scales on which these properties vary. While these methods can theoretically recover all microscopic details of the medium, they are subject to numerical instabilities which we hope to mitigate by attempting to recover only the large scale medium properties.

If we consider scattering from the the macroscopic variations as “signal” embedded in the “noise” of random multiple scattering from the microstructure then we have a theory in which signal and noise are modeled from first principles. Our goal is then to devise a technique by which the signal can be extracted from the noise in an optimal way, where optimal is suitably defined. In this view of the problem, Fig. 1.1b shows a typical signal extraction to be extremely difficult; scattering from a sound speed profile with a microstructure yields a reflection time series that is more than three orders of magnitude larger than the reflection from the profile when the microstructure has been removed. It could be naively assumed that the signal-to-noise ratio is a small fraction of a percent. Yet, as we will show, the problem is not as hopeless as it might appear because scattering from the small scale structure also contains information about the macroscale. That is, the noise statistics contain useful information about the signal.

In this section we consider the case of a normally incident plane wave, which reduces to a problem in one space dimension. Three dimensional effects are considered in §§6.2 and 6.3. We thus use the adjoint of the transport equations (3.14) (cf. (3.45)) with $\kappa = 0$:

$$(6.1) \quad \frac{\partial W^N}{\partial z} + \frac{2N}{c_1(z)} \frac{\partial W^N}{\partial t} + \frac{2\omega^2 \alpha(z)}{c_1^2(z)} \left\{ (N+1)^2 W^{N+1} - 2N^2 W^N + (N-1)^2 W^{N-1} \right\} = 0,$$

$$W^N(0, t, \omega) = \delta(t) \delta_{N,1},$$

where for simplicity we have considered the case of a matched medium, $I = 0$. The power spectral density $\Lambda(t, \omega)$ of the reflection process is then recovered by computing

$$(6.2) \quad \Lambda(t, \omega) = \lim_{z \rightarrow -\infty} W^0(z, t, \omega).$$

By setting $\kappa = 0$ in (2.24) we obtain the normal incidence travel time function for $z < 0$,

$$(6.3) \quad \tau(z) = \int_0^z \frac{ds}{c_1(s)}.$$

Letting $\tilde{z}(\tau)$ be the inverse function of $\tau(z)$, we may change variables in (6.1) to obtain:

$$(6.4) \quad \frac{\partial W^N}{\partial \tau} + 2N \frac{\partial W^N}{\partial t} + 2\omega^2 \tilde{\alpha}(\tau) \left\{ (N+1)^2 W^{N+1} - 2N^2 W^N + (N-1)^2 W^{N-1} \right\} = 0,$$

$$W^N(0, t, \omega) = \delta(t) \delta_{N,1},$$

where

$$(6.5) \quad \tilde{\alpha}(\tau) = \frac{\alpha(\tilde{z}(\tau))}{c_1(\tilde{z}(\tau))}$$

and

$$W^N = W^N(\tau, t, \omega).$$

The significance of the change of variables (6.3) is twofold. First, the characteristic curves of (6.4) are now the straight lines $t = -2N\tau$. A consequence of this is that, because of the hyperbolic nature of the transport equations (see §3.7), equation (6.2) can be replaced by the simple expression

$$(6.6) \quad \Lambda(t, \omega) = W^0(\tau = -t/2, t, \omega).$$

A second consequence of the change of variables is that the resulting equation (6.4) is driven only by the single input function $\tilde{\alpha}(\tau)$ given by (6.5). Thus it is seen that any two statistical ensembles that have the same $\tilde{\alpha}(\tau)$ will yield indistinguishable reflection statistics, to our order of approximation. Thus the most we can expect from a statistical inversion is the recovery of $\tilde{\alpha}(\tau)$.

We next show that $\tilde{\alpha}(\cdot)$ can indeed be recovered from knowledge of the power spectrum Λ if the pulse contains low frequency information. To see this, we compute a low frequency approximation to W^N , that is, we expand W^N as a power series in ω^2 :

$$(6.7) \quad W^N \approx W^{N,0} + \omega^2 W^{N,1} + \dots$$

Substitution of (6.7) into (6.4) yields an equation for $W^{N,0}$ that is just (6.4) with ω set to zero. The solution is then

$$(6.8) \quad W^{N,0}(\tau, t, \omega) = \delta(t + 2\tau) \delta_{N,1}.$$

Now for all ω the equation for W^0 uncouples from those for W^N , $N = 1, 2, \dots$, so that

$$(6.9) \quad W^0(\tau, t, \omega) = -2\omega^2 \int_0^\tau \tilde{\alpha}(\tau') W^1(\tau', t, \omega) d\tau'.$$

Putting (6.8) into (6.9) and then into (6.6) yields a leading order approximation for Λ as $\omega \rightarrow 0$:

$$(6.10) \quad \Lambda(t, \omega) \approx \omega^2 \tilde{\alpha}(-t/2) \text{ as } \omega \rightarrow 0.$$

Hence

$$(6.11) \quad \tilde{\alpha}(-t/2) = \lim_{\omega \rightarrow 0} \frac{\Lambda(t, \omega)}{\omega^2}.$$

Thus $\tilde{\alpha}(\tau)$ is uniquely determined by $\Lambda(t, \omega)$ for small ω .

While (6.11) is of theoretical importance, it does not lead to a practical inversion scheme, since errors in the estimation of the power spectral density are magnified in (6.11) by the vanishing small denominator.

A more realistic problem is formulated as follows: for a finite set of macroscopic times $t_j, j = 1, 2, \dots, N_t$ and for a finite set of frequencies $\omega_l, l = 1, 2, \dots, N_f$ we construct estimates $\hat{\Lambda}(t_j, \omega_l)$ of the local power spectral density $\Lambda(t_j, \omega_l)$. In the remainder of this section, the estimated local power spectral density $\hat{\Lambda}$ and the theoretical local power spectral density Λ will be the ones for signals generated by a pulse with shape function f (as in §2.3). They contain, therefore, the factor $|\hat{f}(\omega)|^2$, the power spectrum of the incident pulse, which was not needed in the discussion above. The estimates $\hat{\Lambda}$ are constructed from sampled reflection series data either by the conventional technique of applying a fast Fourier transform to a small window of data received near time t , or by the windowing process described in Appendix E. From these estimates we wish to recover $\tilde{\alpha}(\tau)$. We set $\alpha^j = \tilde{\alpha}(-t_j/2)$ and assume that $\tilde{\alpha}(\tau)$ is piecewise linear between gridpoints $t_j, j = 1, 2, \dots, N_t$ so that we need only to recover the finite set of variables $\alpha^j, j = 1, 2, \dots, N_t$. We will construct the maximum likelihood estimator for these variables.

Now if $\tilde{\alpha}(\cdot)$ is known, Λ may be determined from (6.4) and (6.6), i.e., by numerical solution of the transport equation which can be accomplished by the methods of §5. We emphasize this dependence of Λ on $\tilde{\alpha}$ by writing $\Lambda(t, \omega; \tilde{\alpha}(\cdot))$.

To construct the maximum likelihood estimator of $\tilde{\alpha}(\cdot)$ we need to know the probability density of $\hat{\Lambda}(t_j, \omega_l)$ when $\tilde{\alpha}(\cdot)$ is known. By a well-known result from signal processing theory [82], if the reflection coefficient is Gaussian then $\hat{\Lambda}(t_j, \omega_l)$ are independent exponentially distributed random variables with means $\Lambda(t_j, \omega_l)$. We have not succeeded in showing rigorously that the reflection process is Gaussian, although there is a heuristic derivation in [10] and Monte Carlo simulations [11] also support the Gaussian hypothesis. More directly, simulation results are also supportive of the exponential distribution for $\hat{\Lambda}$, as evidenced in Fig. E.5. We therefore write the likelihood function as

$$(6.12) \quad P\{\hat{\Lambda}(t_j, \omega_l), j = 1, 2, \dots, N_t; l = 1, 2, \dots, N_f | \tilde{\alpha}(\cdot)\} \\ = \prod_{j=1}^{N_t} \prod_{l=1}^{N_f} \frac{e^{-\hat{\Lambda}(t_j, \omega_l)/\Lambda(t_j, \omega_l; \tilde{\alpha}(\cdot))}}{\Lambda(t_j, \omega_l; \tilde{\alpha}(\cdot))}.$$

The maximum likelihood estimator for $\tilde{\alpha}(\cdot)$ may therefore be constructed by maximizing the expression in (6.12) over $\alpha^1, \alpha^2, \dots, \alpha^{N_t}$. This may be accomplished by using a numerical optimization code with an objective function given by (6.12) to be maximized in N_t dimensions. Note that each evaluation of (6.12) involves the numerical solution of the transport equation up to time t_{N_t} and for all frequencies $\omega_l, l = 1, 2, \dots, N_f$.

The amount of computation necessary for the inversion can be reduced considerably by a layer peeling algorithm which makes use of the hyperbolicity of (6.4), i.e., the fact that $\Lambda(t, \omega; \tilde{\alpha}(\cdot))$ depends on $\tilde{\alpha}(\tau)$ only for travel times $\tau \geq -t/2$. Thus, using our

piecewise linear representation for $\tilde{\alpha}(\cdot)$, $\Lambda(t_j, \omega_l; \tilde{\alpha}(\cdot))$ depends on $\alpha^1, \alpha^2, \dots, \alpha^j$ but not on $\alpha^{j+1}, \dots, \alpha^{N_t}$. We thus may determine α^j recursively. That is, if $\alpha^1, \alpha^2, \dots, \alpha^{j-1}$ have been determined, α^j can be determined by maximizing the likelihood

$$(6.13) \quad P\{\hat{\Lambda}(t_j, \omega_l), l = 1, 2, \dots, N_f | \tilde{\alpha}(\cdot)\} = \prod_{l=1}^{N_f} \frac{e^{-\hat{\Lambda}(t_j, \omega_l)/\Lambda(t_j, \omega_l; \tilde{\alpha}(\cdot))}}{\Lambda(t_j, \omega_l; \tilde{\alpha}(\cdot))},$$

where expression (6.13) is to be maximized only over the single variable α^j .

A practical problem in the implementation of the algorithm based on (6.13) is associated with frequencies ω_l for which the pulse spectrum $\hat{f}$ is small, since the local power spectral densities contain the factor $|\hat{f}(\omega)|^2$. At these frequencies the ratio $\hat{\Lambda}/\Lambda$ is a ratio of small numbers and considerable inaccuracy can be introduced. This difficulty can be eliminated by restricting ω_l to be well within the passband of $\hat{f}$. We have found, however, that the inversion is largely independent of the choice of frequencies if allowance is made for a small white noise term in the local power spectrum in addition to the spectrum caused by random scattering. To accomplish this, the expression for Λ in (6.13) is replaced by $\Lambda + (\sigma^j)^2$ and the resulting likelihood is maximized over the two variables α^j and $(\sigma^j)^2$.

Another modification of (6.13) results if we have prior information on $\tilde{\alpha}(\cdot)$ and wish to incorporate it into our inversion scheme using a Bayesian approach. As an example, suppose we limit the variation in $\tilde{\alpha}(\cdot)$ by postulating the prior distribution

$$(6.14) \quad P\{\alpha^j | \alpha^1, \alpha^2, \dots, \alpha^{j-1}\} = \frac{1}{\sqrt{2\pi\beta^2(\alpha^{j-1})^2}} e^{-\frac{(\alpha^j - \alpha^{j-1})^2}{2\beta^2(\alpha^{j-1})^2}}.$$

That is, we assume that the prior distribution of α^j is Gaussian with mean α^{j-1} and standard deviation proportional to α^{j-1} . Such a prior distribution would penalize overly large increments in $\tilde{\alpha}$ between gridpoints t_{j-1} and t_j . The maximum a posteriori (MAP) estimator is then obtained, again recursively, by maximizing over α^j the product of expressions (6.13) and (6.14).

6.2. Inversion from reflection of obliquely incident plane waves. For a plane wave at oblique incidence that is arriving on the interface at an angle θ with the normal to the interface, the corresponding initial slowness is $\kappa = \sin \theta / c_1(0)$. The power spectrum of the reflected pressure is determined from the equation

$$(6.15) \quad \begin{aligned} \frac{\partial W^N}{\partial z} + \frac{2N(1 - c_1^2(z)\kappa^2)^{1/2}}{c_1(z)} \frac{\partial W^N}{\partial t} \\ + \frac{2\omega^2\alpha(z)}{c_1^2(z)(1 - c_1^2\kappa^2)} \left\{ (N+1)^2 W^{N+1} - 2N^2 W^N + (N-1)^2 W^{N-1} \right\} = 0, \end{aligned}$$

where we have retained the matched medium assumption. The power spectrum at time t is determined by

$$(6.16) \quad \Lambda(t, \omega, \kappa) = \lim_{z \rightarrow -\infty} W^0(z, t, \omega, \kappa).$$

We transform depth z to travel time $\tau(z, \kappa)$ using (2.24) in (6.15) to obtain:

$$(6.17) \quad \frac{\partial W^N}{\partial \tau} + 2N \frac{\partial W^N}{\partial t} + 2\omega^2 \tilde{\alpha}(\tau, \kappa) \left\{ (N+1)^2 W^{N+1} - 2N^2 W^N + (N-1)^2 W^{N-1} \right\} = 0,$$

$$(6.18) \quad W^N(0, t, \omega, \kappa) = \delta(t) \delta_{N,1},$$

where, denoting by $\tilde{z}(\tau, \kappa)$ the inverse function, for fixed κ , of $\tau(z, \kappa)$, we compute $\tilde{\alpha}(\tau, \kappa)$ by

$$(6.19) \quad \tilde{\alpha}(\tau, \kappa) = \frac{\alpha(\tilde{z}(\tau, \kappa))}{c_1(\tilde{z}(\tau, \kappa))(1 - c_1^2(\tilde{z}(\tau, \kappa))\kappa^2)^{3/2}}.$$

Note that (6.18) is now formally identical to (6.4), the corresponding normal incidence transport equation. Therefore the limit as $z \rightarrow -\infty$ in (6.16) can be replaced, in τ coordinates, by evaluation at $\tau = -t/2$, as in (6.6). Furthermore, the inverse problem may be solved as in the normal incidence case, where an estimate of the power spectrum $\Lambda(t, \omega, \kappa)$ determines $\tilde{\alpha}(-t/2, \kappa)$ recursively in t . Note that *only* $\tilde{\alpha}(\tau, \kappa)$ can be recovered from the reflection data, *not* the two functions $c_1(z)$ and $\alpha(z)$ that define the macroscopic properties of the medium.

We next show that scattering data from two distinct incidence angles is theoretically sufficient for determination of $c_1(z)$ and $\alpha(z)$, and as functions of depth z rather than travel time τ . Thus, let $\tilde{\alpha}(\tau, \kappa_1)$, $\tilde{\alpha}(\tau, \kappa_2)$ be given for $\kappa_1 \neq \kappa_2$. We assume these functions are known exactly and do not consider how errors in their estimation will propagate through our algorithm.

Let

$$(6.20) \quad \tilde{\beta}(\tau, \kappa) = \int_0^\tau [\tilde{\alpha}(\tau', \kappa)]^{1/3} d\tau'$$

so that $\tilde{\beta}(\tau, \kappa_1)$, $\tilde{\beta}(\tau, \kappa_2)$ are known. Since for each κ , $\tilde{\beta}$ is monotone in τ and takes $[0, \infty) \rightarrow [0, \infty)$ (assuming $\tilde{\alpha}$ is bounded below) there exists a function $h(\tau) = h(\tau, \kappa_1, \kappa_2)$ such that

$$(6.21) \quad \tilde{\beta}(\tau, \kappa_1) = \tilde{\beta}(h(\tau), \kappa_2)$$

$h(\tau)$ is a “time warp” function that, as we shall show, matches up travel times to the same depth z for the two different angles (see (6.25) below). First note that the derivative of h may be constructed by differentiating (6.21) and using (6.20)

$$(6.22) \quad h'(\tau) = (\tilde{\alpha}(\tau, \kappa_1) / \tilde{\alpha}(h(\tau), \kappa_2))^{1/3}.$$

We next use the travel time transformation (2.24) to change variables in (6.20). Putting (6.19) in this expression produces

$$(6.23) \quad \tilde{\beta}(\tau, \kappa) = \int_0^{\tilde{z}(\tau, \kappa)} (\alpha(z) / c_1^4(z))^{1/3} dz \equiv \beta(\tilde{z}(\tau, \kappa)),$$

where $\beta(z)$ is monotone in z . From (6.21) and (6.23) we obtain that

$$(6.24) \quad \beta(\tilde{z}(\tau, \kappa_1)) = \beta(\tilde{z}(h(\tau), \kappa_2))$$

which, by the monotonicity of β , implies that

$$(6.25) \quad \tilde{z}(\tau, \kappa_1) = \tilde{z}(h(\tau), \kappa_2).$$

Since $\tilde{z}$ is inverse to τ , the derivative $\tilde{z}_\tau$ can be computed from (2.24) as the reciprocal of τ_z . Thus

$$(6.26) \quad \tilde{z}_\tau(\tau, \kappa) = \frac{c_1(\tilde{z}(\tau, \kappa))}{\sqrt{1 - c_1^2(\tilde{z}(\tau, \kappa))\kappa^2}}.$$

Solving (6.26) for c_1 yields

$$(6.27) \quad c_1(\tilde{z}(\tau, \kappa)) = \frac{\tilde{z}_\tau(\tau, \kappa)}{\sqrt{1 + \kappa^2 \tilde{z}_\tau^2(\tau, \kappa)}}.$$

Now differentiation of (6.25) and use of (6.25) and (6.26) produces

$$(6.28) \quad \tilde{z}_\tau(\tau, \kappa_1) = h'(\tau) \frac{c_1(\tilde{z}(\tau, \kappa_1))}{\sqrt{1 - c_1^2(\tilde{z}(\tau, \kappa_1))\kappa_2^2}}.$$

Substitution of (6.27) into (6.28) gives an expression for $\tilde{z}_\tau$

$$(6.29) \quad \tilde{z}_\tau(\tau, \kappa_1) = \sqrt{\frac{1 - h'(\tau)^2}{\kappa_2^2 - \kappa_1^2}}.$$

We can now construct $\tilde{z}(\tau, \kappa_1)$ by integration of (6.29).

To summarize our algorithm: From the given $\tilde{\alpha}(\tau, \kappa_1)$, $\tilde{\alpha}(\tau, \kappa_2)$ we first construct $\tilde{\beta}(\tau, \kappa_1)$, $\tilde{\beta}(\tau, \kappa_2)$ from (6.20). We then construct the time warp $h(\tau)$ from (6.21), i.e., by determining the times $h(\tau)$ that map $\tilde{\beta}(h, \kappa_2)$ into $\tilde{\beta}(\tau, \kappa_1)$. The derivative $h'(\tau)$ is computed from equation (6.22). Now for each τ the value of z corresponding to it (with reference to κ_1) is obtained by integration of (6.29). The value of c_1 corresponding to this depth z is determined from (6.29) substituted into (6.27) with $\kappa = \kappa_1$. The value of α at this depth z can now be determined from (6.19) with $\kappa = \kappa_1$ from knowledge of c_1 and the original data $\tilde{\alpha}(\tau, \kappa_1)$.

Figure 6.2a is a simulated sound speed profile, displaying sound speed (km/s) as a function of depth (km). On the macroscopic scale we see a low velocity region, descending to 1.5 km/s in the trough, embedded in a constant velocity background of 3 km/s. Superimposed on the macroscopic profile, which covers about 12 km, are microscopic variations consisting of 4500 small layers with constant width $12/4500=2.7$ m. The sound speed fluctuates randomly and independently from microlayer to microlayer with the reciprocal bulk modulus having a uniform distribution fluctuating $\pm 30\%$ about the mean. We take $\varepsilon = .03$ so that a simulated pulse consistent with the theory should have frequencies of order ε^{-1} in nondimensional macroscopic units.

We implemented the layer-peeling inversion algorithm, as described in §6.1, for three angles, 0° (normal incidence), 10° , and 20° . We used the windowing algorithm of Appendix E, with $\varepsilon = .03$ and for $N_f = 30$ pulse frequencies, starting with $\omega_1 = 1$ and with $\omega_{l+1} - \omega_l = .36$, to effect the spectral estimation. Macroscopic times were advanced by an increment $t_{j+1} - t_j = .326$. To improve inversion accuracy, we used 90 independent realizations of this random medium, and maximized the combined

FIG. 6.2A. *Noisy velocity profile.*

FIG. 6.2B. *Inversion results for plane waves incident at different angles.*

FIG. 6.2c. *Inversion results for plane waves incident at different angles without knowledge of α .*

likelihood function, the product of 90 likelihoods, each given by equation (6.13). The results are plotted in Fig. 6.2b as functions of depth. As discussed above, each of these three inversions yields only $\tilde{\alpha}(\tau, \kappa)$ where $\kappa = \sin(\theta)/c_1(0)$. To plot $c_1(z)$ we assumed that (the constant) $\alpha = .00375$ was known, allowing us to solve for $c_1(\tilde{z}(\tau, \kappa))$ from (6.19). The depth to travel time transformation was then effected by integration of (6.26).

It is apparent from Fig. 6.2b that the inversion errors, for inversions using different angles but scattering from the same media, are highly correlated. It is also apparent that inversion errors tend to increase with increasing angle of incidence. This effect can be understood from (6.18) and (6.19). The only appearance of ω in (6.18) is in the term containing $\omega^2 \tilde{\alpha}(\tau, \kappa)$. From (6.19), $\tilde{\alpha}(\tau, \kappa)$ depends on κ only through the factor $(1 - \kappa^2 c_1^2)^{-3/2}$. At, for example, $z = 0$, this last factor is just $(\cos(\theta))^{-3}$ where θ is the angle of incidence. Thus larger incidence angles effectively increase ω through factors analogous to $\omega^2 (\cos(\theta))^{-3}$. However, the most valuable information for inversion is at lower frequencies, as is apparent from (6.11).

In Fig. 6.2c we recovered c_1 as a function of z without assuming that α is known. We used two of the plane wave inversions, for normal incidence and for $\theta = 20^\circ$, and applied the algorithm of this section for the recovery of $c_1(z)$. Although, as shown in Fig. 6.2c, reasonable results can be obtained, we encountered some numerical difficulties. If $\kappa_1 = 0$, we would like κ_2 substantially different from zero so that $\tilde{\alpha}(\tau, \kappa_1)$ and $\tilde{\alpha}(\tau, \kappa_2)$ are substantially different. This fact is reflected in (6.22) for $h'(\tau)$, which should be less than 1 for use in equation (6.29). In particular, if κ is small then $h'(\tau)$, from (6.22), differs from unity only by $O(\kappa^2)$. In this case, errors in the inversion can easily cause the square root in (6.29) to become imaginary, and the inversion fails. However, use of a larger κ_2 gives inaccurate estimates of $\tilde{\alpha}(\tau, \kappa_2)$ for reasons discussed above. Thus successful two plane wave inversions require an

intermediate value of κ_2 .

Ideally, to make the best use of data from two or more incidence angles, we should construct the likelihood function and maximize it, as was done for a single angle. However, we do not at present understand how errors in spectral estimates from data scattered at different angles are related, and their correlations may be considerable. Furthermore, a layer-peeling algorithm would present special difficulties since the travel-time to depth transformation is dependent on incidence angle. We defer these problems and turn instead to point source problems.

6.3. Inversion from reflections excited by a point source. Recall that if $c_1(z)$, $\alpha(z)$ are known, then the power spectral density $\Lambda(t, \mathbf{x}, \omega)$ of the signal received at $\mathbf{x}$ at time t when the medium is excited by a point source may be calculated from the transport equation (3.14) and integral (3.29). We emphasize this dependence by writing $\Lambda(t, \mathbf{x}, \omega; c_1(\cdot), \alpha(\cdot))$. Now let us suppose that power spectral estimates $\hat{\Lambda}(t_i, \mathbf{x}_j, \omega_l)$ have been constructed from data at positions $\mathbf{x}_1, \dots, \mathbf{x}_{N_r}$, and for frequencies $\omega_1, \dots, \omega_{N_f}$. If the receivers are sparsely spaced, i.e., $|\mathbf{x}_{j_1} - \mathbf{x}_{j_2}|$ are much greater than a wavelength for all j_1, j_2 , then the estimation errors at the receivers are uncorrelated. Hence we may write the likelihood function for $c_1(\cdot), \alpha(\cdot)$ as

$$(6.30) \quad P\{\hat{\Lambda}(t_i, \mathbf{x}_j, \omega_l) | i = 1, \dots, N_t; j = 1, \dots, N_r; l = 1, \dots, N_f | c_1(\cdot), \alpha(\cdot)\} \\ = \prod_{i=1}^{N_t} \prod_{j=1}^{N_r} \prod_{l=1}^{N_f} \frac{e^{-\hat{\Lambda}(t_i, \mathbf{x}_j, \omega_l) / \Lambda(t_i, \mathbf{x}_j, \omega_l; c_1(\cdot), \alpha(\cdot))}}{\Lambda(t_i, \mathbf{x}_j, \omega_l; c_1(\cdot), \alpha(\cdot))}.$$

Let us discretize space at grid points $z_1, z_2, \dots, z_{N_d}$ and put $c^j = c_1(z_j)$ and $\alpha^j = \alpha(z_j)$. If $c_1(\cdot)$ and $\alpha(\cdot)$ are considered piecewise linear between gridpoints, then these functions may be optimally determined from the data by maximizing the expression (6.30) over all vectors $(c^1, c^2, \dots, c^{N_d}, \alpha^1, \alpha^2, \dots, \alpha^{N_d})^T$. This global optimization can be accomplished by numerical methods in $2N_d$ -dimensional space, and the result is our best guess at the inversion.

It remains to determine that $\alpha(\cdot)$ and $c_1(\cdot)$ are indeed theoretically recoverable from the data. It appears that data from at least two receivers is necessary for a full point source inversion, analogous to the necessity of at least two incidence angles for the plane wave inversion problem. To see what is recoverable from data at a single receiver, we compute the low-frequency part of the spectrum by taking $\omega \rightarrow 0$ in (3.14). We take the matched case so that ${}_I(0) = {}_I(-L) = 0$.

First, observe that, from equation (3.14),

$$(6.31) \quad \frac{\partial W^0}{\partial z} = 0$$

so that

$$(6.32) \quad W^0(z, t, \mathbf{x}, \omega) = \delta(t)\delta(\mathbf{x})$$

We now expand

$$(6.33) \quad W^N = W^{N,0} + \omega^2 W^{N,1} + \dots$$

as $\omega \rightarrow 0$ to get

$$(6.34) \quad \frac{\partial W^{N,0}}{\partial z} + \frac{2N}{(1 - c_1^2 \kappa^2)^{1/2}} \left\{ \frac{1}{c_1} \frac{\partial W^{N,0}}{\partial t} + c_1 \boldsymbol{\kappa} \cdot \nabla W^{N,0} \right\} = 0, \\ W^{N,0} |_{z=-L} = 0 \text{ for } N = 1, 2, \dots$$

Therefore

$$(6.35) \quad W^{N,0} = 0 \text{ for } N = 1, 2, \dots$$

The equation for $W^{1,1}$ becomes

$$(6.36) \quad \frac{\partial W^{1,1}}{\partial z} + \frac{2}{(1 - c_1^2 \kappa^2)^{1/2}} \left\{ \frac{1}{c_1} \frac{\partial W^{1,1}}{\partial t} + c_1 \boldsymbol{\kappa} \cdot \nabla W^{1,1} \right\} - \frac{2\alpha}{c_1^2 (1 - c_1^2 \kappa^2)} W^{0,0} = 0,$$

The solution of (6.36) is written in terms of the characteristic curves. Let

$$(6.37) \quad \bar{t}(z, s, \kappa) = \int_s^z \frac{2ds'}{c_1(s')(1 - \kappa^2 c_1^2(s'))^{1/2}},$$

$$(6.38) \quad \bar{\mathbf{x}}(z, s, \kappa) = \int_s^z \frac{2c_1(s')ds'}{(1 - \kappa^2 c_1^2(s'))^{1/2}} \boldsymbol{\kappa}.$$

Then

$$(6.39) \quad W^{1,1} = \int_{-L}^z ds' \frac{2\alpha}{c_1^2(1 - c_1^2 \kappa^2)} \delta(t - \bar{t}(z, s', \kappa)) \delta(\mathbf{x} - \bar{\mathbf{x}}(z, s', \kappa)).$$

It is apparent that, because of the δ -function, the solution is nonzero only when

$$(6.40) \quad \frac{x_1}{x_2} = \frac{\kappa_1}{\kappa_2}.$$

Equations (6.37), (6.38), (6.40) have a geometric interpretation; (6.37), (6.38) are merely the ray equations for a ray with initial slowness vector $\boldsymbol{\kappa}$, and traveling for only half the time $t/2$. Equation (6.40) expresses the fact that, for a layered medium, source, receiver, and the initial direction of the ray reflected back to the receiver must lie in a single plane.

Substitution of (6.39) into (3.29) now yields, after evaluation of the various Jacobians

$$(6.41)$$

$$\lim_{\omega \downarrow 0} \frac{\Lambda(t, \mathbf{x}, \omega)}{\omega^2} = \frac{\alpha}{2c_1(1 - \kappa^2 c_1^2) \int_z^0 \frac{c_1(s)ds}{(1 - \kappa^2 c_1^2(s))^{3/2}} \int_z^0 \frac{2c_1(s)ds}{(1 - \kappa^2 c_1^2(s))^{1/2}}} \Big|_{z=\tilde{z}(t, \mathbf{x}), \kappa=\tilde{\kappa}(t, \mathbf{x})},$$

where $\tilde{z}(t, \mathbf{x})$ and $\tilde{\kappa}(t, \mathbf{x})$ are the inverses of (6.37) and (6.38) when $\boldsymbol{\kappa}$ is constrained by (6.40). In (6.41) we have assumed that t is bigger than the first ray arrival at $\mathbf{x}$.

7. Conclusions. We have presented a comprehensive theory for multiple scattering of waves in randomly layered media. With a unified mathematical approach based on asymptotics for stochastic equations we have calculated from first principles statistical properties of reflected and transmitted fields generated by incident pulsed or time harmonic, plane or spherical waves. The main assumption in our theory is the separation of scales in the variations of the medium properties: a random microstructure superimposed on a macroscopic background profile. The wavelengths are assumed to be intermediate between these two scales. This theory encompasses both geometrical acoustics relative to the macroscales and universal statistical behavior of localization phenomena relative to the microscales. Within this theory we have formulated and solved a number of statistical inverse problems where we recover the background medium properties from multiply scattered signals. The scope of the theory has been explored with extensive numerical simulations.

Our approach relies on an invariant embedding formulation of the scattering problems, principally a stochastic Riccati equation for the reflection coefficient. Moments of reflection coefficients are computed by asymptotic methods for stochastic equations. This results in a system of transport equations. We show here that nearly all important scattering phenomena can be analyzed by studying these transport equations.

Although the hypothesis of plane parallel layering is essential for the theory presented here, it can be relaxed in an important way. We can extend this theory to locally layered media, that is, media with bulk modulus of the form

$$K^{-1} = K_1^{-1}(\mathbf{x}) \left(1 + \nu(\phi(\mathbf{x})/\varepsilon^2)\right)$$

in the layered region, where $\phi(\mathbf{x}) = \zeta$ are the layering surfaces which need not be planes. Moreover, the mean bulk modulus $K_1(\mathbf{x})$ can, within certain limits, be a general function of $\mathbf{x} = (x, y, z)$. This extension greatly expands the applicability of the theory, for instance in geophysical problems. However, the analogue of the W -equations is now much more complicated so the theory is not easy to use.

When, however, the one-dimensional random fluctuations are weak and there are weak three-dimensional fluctuations which dominate, then there is no localization and radiative transport theory [88]-[90] may be appropriate. Our transport equations, the W -equations, are essentially different from radiative transport equations.

Sheng and coworkers [91], [92] (cf. also [93]) allow for three-dimensional, isotropic random perturbations to coexist with randomly layered perturbations and they have shown that there is a sharp transition from essentially one-dimensional localized behavior to three-dimensional transport (absence of localization). Thus our theory, which is based on a purely layered medium, will not change qualitatively when small three-dimensional random perturbations are added.

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A. Plane wave amplitudes for a point source. In the homogeneous half space $z > 0$, starting from the Fourier transform in time of (2.1) for p and $\mathbf{u}$, we can

eliminate $\mathbf{u}$ to obtain

$$\Delta p + \left(\frac{\omega}{\varepsilon c_0}\right)^2 p = \varepsilon^{3/2} \hat{f}(\omega) \delta'(z - z_s),$$

where c_0 is a constant. Let us first consider the usual point source problem

$$\Delta \phi + \left(\frac{\omega}{c}\right)^2 \phi = -\delta(x)\delta(y)\delta(z).$$

Then its solution is given by the Weyl integral [79]

$$\phi = \frac{i\omega}{8\pi^2} \iint \frac{e^{i\omega S}}{S_z} d\boldsymbol{\kappa},$$

where

$$S = \boldsymbol{\kappa} \cdot \mathbf{x} + \left(\frac{1}{c^2} - \kappa^2\right)^{1/2} |z|$$

with $\mathbf{x} = (x, y)$, $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$, and $\text{Im } \omega (1/c^2 - \kappa^2)^{1/2} > 0$. Thus the solution for the time-transformed pressure $\tilde{p}$ will be

$$\begin{aligned} \tilde{p} &= -\varepsilon^{3/2} \hat{f}(\omega) \frac{\partial}{\partial z} \frac{i\omega}{8\pi^2 \varepsilon} \iint \frac{e^{i\omega S/\varepsilon}}{S_z} d\boldsymbol{\kappa} + \tilde{p}_{\text{refl}} \\ &= -\varepsilon^{-1/2} \frac{\hat{f}(\omega) \omega^2}{8\pi^2} \iint e^{i\omega S/\varepsilon} d\boldsymbol{\kappa} + \tilde{p}_{\text{refl}} \end{aligned}$$

for $0 < z < z_s$, where

$$S = \boldsymbol{\kappa} \cdot \mathbf{x} + c_0^{-1} \left(1 - c_0^2 \kappa^2\right)^{1/2} |z - z_s|.$$

Compare this with the decomposition of the pressure into upgoing and downgoing parts (2.29). Recall that $\tilde{p}_{\text{refl}}$ corresponds to the upgoing A -term. Then the equivalent plane wave amplitude, B , must be

$$(A.1) \quad B(\omega, \boldsymbol{\kappa}) = \varepsilon^{3/2} \frac{\hat{f}(\omega)}{2} (\zeta_0)^{-1/2} e^{i\omega c_0^{-1} (1 - \kappa^2 c_0^2)^{1/2} z_s / \varepsilon},$$

where ζ_0 is the impedance defined by (2.25).

B. A brief review of asymptotics for stochastic equations. Equations (2.27) for the wave amplitudes A , B , the Riccati equation (2.32) for the reflection coefficient $\tilde{r}$, and the hierarchy of moment equations (3.5) for $\tilde{\gamma}^{NM}$ are examples of stochastic equations with rapidly oscillating coefficients. More precisely, equations (2.27) are a linear, two-point boundary value problem with $\nu = \nu(z/\varepsilon^2)$ (through n given by (2.28)) a coefficient that is a rapidly varying stochastic process. They also contain rapidly varying deterministic coefficients through the phase factor $e^{\pm i\tau(z)/\varepsilon}$. The Riccati equations (2.32) are a nonlinear stochastic initial value problem with rapidly oscillating deterministic and random coefficients and they are equivalent to the A, B equations (2.27). The system (3.5) is an infinite-dimensional linear initial value problem with rapidly oscillating deterministic and random coefficients, a system

of partial differential equations, and what we have referred to in (3.5) as “negligible terms” includes higher t and x derivatives. System (3.5) gives us directly the quantities of interest, the two-frequency two-slowness moment hierarchy of the reflection coefficient (3.2) (suitably transformed using (3.4)); it is the most complex problem of the three stated above because it involves generalized functionals of frequencies and slownesses. The locally plane layered problem, not discussed here, cannot be formulated at all as a basic ordinary differential equation whose associated integral functionals are the quantities of interest. Therefore we need a method that handles systems like (3.5) directly, in a suitable weak formulation that makes the analysis tractable.

We begin by reviewing the elementary theory [32], [45] that deals with equations like (2.32) for x , y , with ω and κ fixed. They have the form²⁰

$$(B.1a) \quad \frac{dx^\varepsilon}{dz} = \frac{1}{\varepsilon} F \left(x^\varepsilon(z), y^\varepsilon(z), \frac{\tau(z)}{\varepsilon}, z \right) + G \left(x^\varepsilon(z), y^\varepsilon(z), \frac{\tau(z)}{\varepsilon}, z \right), \quad z > 0,$$

$$(B.1b) \quad x^\varepsilon(0) = x_0,$$

where $x^\varepsilon(z)$ takes values in some $\mathbf{R}^d$, $y^\varepsilon(z)$ represents the rapidly varying random coefficients,²¹ and dependence on $\tau(z)/\varepsilon$ and z represents deterministic fast and slow variations, respectively. The coefficient process $y^\varepsilon(z)$ has the form $y(z/\varepsilon^2)$, where $y(z)$ is a given stationary process taking values in some set $\mathcal{S}$ (a subset of some $\mathbf{R}^m$, for example) and is assumed to have strong ergodic properties, such as rapidly decaying correlations. The functions F and G map $\mathbf{R}^d \times \mathcal{S} \times [0, \infty)^2$ into $\mathbf{R}^d$ and are assumed to be smooth in $x \in \mathbf{R}^d$ and periodic in τ (or almost periodic in τ , uniformly in the other variables). We also assume that

$$(B.2) \quad \frac{d\tau(z)}{dz} \neq 0 \quad \text{in } 0 \leq z \leq T \leq \infty$$

and

$$(B.3) \quad E \{ F(x, y(z), \tau, z) \} = 0,$$

where $[0, T]$ is the interval over which we seek the approximation of the process $x^\varepsilon(z)$. Then it is easy to see, as is done in detail in the references cited above, that $x^\varepsilon(z)$ converges weakly as $\varepsilon \rightarrow 0$ to a diffusion Markov process $x(z)$ whose infinitesimal generator has the form

$$(B.4) \quad \begin{aligned} \mathcal{L}_z f(x) = & \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L d\tau \int_0^\infty ds E \{ F(x, y(s), \tau, z) \cdot \nabla_x [F(x, y(0), \tau, z) \cdot \nabla_x f(x)] \} \\ & + \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L d\tau E \{ G(x, y(s), \tau, z) \cdot \nabla_x f(x) \} \end{aligned}$$

for f a smooth and bounded function in $\mathbf{R}^d$. Note in particular the averaging over the rapid deterministic variations of τ in (B.4). This is a simple add-on in the elementary theory that is easily accommodated. It is very important in the wave propagation applications because it carries the geometrical optics (acoustics) approximations.

²⁰ In this section we do not use boldface for vectors, as is customary in the literature on stochastic differential equations

²¹ The process $\nu(z/\varepsilon^2)$, which enters through n in (2.28).

In the appendix of [10] we give a simple, multiple-scales derivation of (B.4) without technical details. If $f(x)$ is a smooth and bounded function in $\mathbf{R}^d$, let

$$(B.5) \quad u^\varepsilon(z, \zeta, x) = E \{ f(x^\varepsilon(z)) \},$$

where $x^\varepsilon(z)$ for $\zeta \leq z \leq T$ is the process defined by (B.1) with $x^\varepsilon(\zeta) = x$. Then as ε tends to zero, u^ε tends to $u(z, \zeta, x)$ that satisfies the backward Kolmogorov equation

$$(B.6a) \quad \frac{\partial u}{\partial \zeta} + \mathcal{L}_\zeta u = 0, \quad \zeta \leq z,$$

$$(B.6b) \quad u(z, z, x) = f(x).$$

The adjoint equation

$$(B.7a) \quad \frac{\partial P}{\partial z} = \mathcal{L}_z^* P, \quad z \geq \zeta,$$

$$(B.7b) \quad P(\zeta, \zeta, x, x_0) = \delta(x - x_0)$$

is the Fokker-Planck equation for the transition probability density $P(z, \zeta, x, x_0)$ of the limiting diffusion process $x(z)$.

Let us now consider systems of the form (3.5) which are linear but infinite-dimensional because they involve partial derivatives and infinite sets of indices M and N . Assuming for the moment that the terms we have grouped under “negligible terms” are absent, system (3.5) has the form

$$(B.8a) \quad \frac{dU^\varepsilon}{dz} = \frac{1}{\varepsilon} A \left(z, \frac{\tau(z)}{\varepsilon}, \frac{z}{\varepsilon^2} \right) U^\varepsilon + B \left(z, \frac{\tau(z)}{\varepsilon}, \frac{z}{\varepsilon^2} \right) U^\varepsilon,$$

$$(B.8b) \quad U^\varepsilon(0) = U_0,$$

where A and B are operators that depend on z/ε^2 through ν (cf. (2.28)) and on z through τ_z . Assume further that the system is finite dimensional which means that A and B are matrices. Then under hypotheses identical to the ones that led to (B.4) we show easily that

$$(B.9) \quad \lim_{\varepsilon \downarrow 0} E \{ U^\varepsilon(z) \} = \overline{U}(z)$$

satisfies the linear system

$$(B.10a) \quad \frac{d\overline{U}(z)}{dz} = \overline{A}A(z)\overline{U}(z) + \overline{B}(z)\overline{U}(z),$$

$$(B.10b) \quad \overline{U}(0) = U_0,$$

where

$$(B.11) \quad \overline{A}A(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \int_0^\infty ds E \{ A(z, \tau, s) A(z, \tau, 0) \}$$

and

$$(B.12) \quad \overline{B}(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \, E \{B(z, \tau, \zeta)\}.$$

In Appendix C we derive transport equations for reflected waves when the fluctuations in the sound speed of the layered medium are weak. In that case the scaled stochastic equations that must be analyzed are different from (B.8) and have the form

$$(B.13a) \quad \frac{dU^\varepsilon}{dz} = \frac{1}{\varepsilon} A \left(z, \frac{z}{\varepsilon^2} \right) U^\varepsilon + B \left(z, \frac{z}{\varepsilon^2} \right) U^\varepsilon,$$

$$(B.13b) \quad U^\varepsilon(0) = U_0,$$

where A and B are operators that depend on $\frac{z}{\varepsilon^2}$ through ν and through oscillatory terms of the form $e^{\pm 2i\omega\tau(z)/\varepsilon^2}$; see (C.15) and (C.6). This is because the pulse width is now comparable to the size of the layers. The limit (B.9) satisfies again equations (B.10) with

$$(B.14a) \quad \overline{AA}(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \int_0^\tau ds \, E \{A(z, \tau)A(z, s)\},$$

$$(B.14b) \quad \overline{B}(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \, E \{B(z, \tau)\}.$$

For finite-dimensional systems the validity of (B.9) and the form of the limit operators (B.11), (B.12), or (B.14) is an easy consequence of the more general result (B.4) [32]. We can also prove directly results like (B.11), (B.12) or (B.14) under suitable hypotheses [83]. A quick way to derive W -equations formally as in §3 is by using (B.11), (B.12).

A simple, heuristic way to see how formulas like (B.14) arise, for example, is the following. Assume for simplicity that B in (B.13) is identically zero and that $A = A(\frac{z}{\varepsilon^2})$ which means that A does not depend on the slow scale z . Write the simplified (B.13) as an integral equation

$$(B.15) \quad U^\varepsilon(z) = U_0 + \frac{1}{\varepsilon} \int_0^z A \left(\frac{s}{\varepsilon^2} \right) U^\varepsilon(s) ds.$$

Iterate once, take expectations, and use the mean zero property of A

$$(B.16) \quad E \{U^\varepsilon(z)\} = U_0 + \frac{1}{\varepsilon^2} \int_0^z \int_0^s E \left\{ A \left(\frac{s}{\varepsilon^2} \right) A \left(\frac{\sigma}{\varepsilon^2} \right) U^\varepsilon(\sigma) \right\} d\sigma ds.$$

For small ε the expectation is approximately a product of expectations because correlations in A die fast. This is the most difficult step to justify in the analysis. Thus, retaining the equality sign after the approximation, we have

$$(B.17) \quad E \{U^\varepsilon(z)\} = U_0 + \frac{1}{\varepsilon^2} \int_0^z \int_0^s E \left\{ A \left(\frac{s}{\varepsilon^2} \right) A \left(\frac{\sigma}{\varepsilon^2} \right) \right\} E \{U^\varepsilon(\sigma)\} d\sigma ds.$$

A further approximation is to pull $E \{U^\varepsilon(\sigma)\}$ outside the integral as $E \{U^\varepsilon(s)\}$ because it is slowly varying. Hence

$$(B.18) \quad E \{U^\varepsilon(z)\} = U_0 + \int_0^z \tilde{A}^\varepsilon(s) E \{U^\varepsilon(s)\} ds,$$

where

$$\tilde{A}^\varepsilon(s) = \frac{1}{\varepsilon^2} \int_0^s E \left\{ A \left(\frac{s}{\varepsilon^2} \right) A \left(\frac{\sigma}{\varepsilon^2} \right) \right\} d\sigma.$$

We get the final form of the approximation by replacing $\tilde{A}^\varepsilon$ by its average, which is the usual averaging method in ordinary differential equations

$$(B.19) \quad \begin{aligned} \overline{AA} &= \lim_{\varepsilon \downarrow 0} \int_0^1 \tilde{A}^\varepsilon(s) ds \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_0^s E \{A(s)A(\sigma)\} d\sigma ds. \end{aligned}$$

We now turn to the question of how to deal properly with system (3.5) or the more complex locally layered case [51]. Even though we can get the W -equations formally by using (B.11) and (B.12), it is important to know in what sense

$$(B.20) \quad \lim_{\varepsilon \downarrow 0} E \left\{ \tilde{\cdot}^{NM} \right\} = \tilde{W}^{NM},$$

where W^{NM} (obtained from $\tilde{W}^{NM}$ by (3.11)) satisfy (3.12). Moreover, there are different ways in which (B.20) can be interpreted and the results are different.

We begin by noting that (3.4) is not well defined as it stands. It is a formal expression because the Fourier integrals with respect to h and λ must be interpreted properly. We know from (2.32) that $|\cdot| \leq 1$ so $|\cdot|^{NM} \leq 1$. Let $\mathcal{D}$ denote the space of complex valued infinitely differentiable functions $f = f(t, \omega, \mathbf{x}, \boldsymbol{\kappa})$ with compact support in $\mathbf{R}^6$ and let

$$(B.21) \quad \begin{aligned} \mathcal{D}_H &= \left\{ \lambda = (\lambda^{NM}), N \geq 0, M \geq 0 \mid (\lambda^{NM})^* = \lambda^{MN}, \lambda^{MN} \in \mathcal{D}, \right. \\ &\quad \left. \lambda^{NM} \equiv 0 \text{ for } N \geq p, M \geq p, \text{ and some } p \geq 0 \right\}, \end{aligned}$$

which is the space of finite-dimensional Hermitian matrices (of arbitrary dimension p) whose entries are infinitely differentiable functions of compact support in $\mathbf{R}^6$. This space is endowed with the convergence

$$\lambda_n \rightarrow \lambda \text{ in } \mathcal{D}_H \text{ as } n \rightarrow \infty$$

if there exists a compact set $K \subset \mathbf{R}^6$ and a nonnegative integer p such that (i) for all n large enough $\lambda_n^{NM}(t, \omega, \mathbf{x}, \boldsymbol{\kappa})$ are in $C^\infty(\mathbf{R}^6)$ with support inside K and $\lambda_n^{NM}(t, \omega, \mathbf{x}, \boldsymbol{\kappa}) \equiv 0$ for $N > p, M > p$ and (ii) $\lambda_n^{NM}(t, \omega, \mathbf{x}, \boldsymbol{\kappa})$ converges uniformly in K to $\lambda^{NM}(t, \omega, \mathbf{x}, \boldsymbol{\kappa})$ along with derivatives of all orders for each $N \leq p, M \leq p$. If on $\mathcal{D}_H$ we introduce the bilinear form

$$(B.22) \quad \langle \lambda_1, \lambda_2 \rangle = \int_{\mathbf{R}^6} dt d\omega d\mathbf{x} d\boldsymbol{\kappa} \sum_{N, M=0}^{\infty} \lambda_1^{NM} \lambda_2^{MN}$$

then the space of distributions $\mathcal{D}'_H$ can be identified with the dual of $\mathcal{D}_H$, as usual.

From the formal expression (3.4) we see that if $\lambda \in \mathcal{D}_H$ then

$$\begin{aligned}
 (B.23) \quad \langle \cdot, \lambda \rangle &= \int_{\mathbf{R}^6} dt d\omega d\mathbf{x} d\boldsymbol{\kappa} \sum_{N,M=0}^{\infty} \langle \cdot, \lambda^{MN} \rangle \\
 &= \int_{\mathbf{R}^6} dt d\omega d\mathbf{x} d\boldsymbol{\kappa} \sum_{N,M=0}^{\infty} \lambda^{MN}(t, \omega, \mathbf{x}, \boldsymbol{\kappa}) \frac{\omega^2}{(2\pi)^3} \int dh d\boldsymbol{\lambda} e^{iht - i\omega \boldsymbol{\lambda} \cdot \mathbf{x}} \\
 &\quad \langle \cdot, \lambda^N(z, \omega - \frac{\varepsilon h}{2}, \mathbf{x}, \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2}) \rangle \langle \cdot, \lambda^M(z, \omega + \frac{\varepsilon h}{2}, \mathbf{x}, \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2}) \rangle \\
 &= \sum_{N,M=0}^{\infty} \int_{\mathbf{R}^6} \hat{\lambda}^{MN}(h, \omega, \omega \boldsymbol{\lambda}, \boldsymbol{\kappa}) \\
 &\quad \langle \cdot, \lambda^N(z, \omega - \frac{\varepsilon h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon \boldsymbol{\lambda}}{2}) \rangle \langle \cdot, \lambda^M(z, \omega + \frac{\varepsilon h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon \boldsymbol{\lambda}}{2}) \rangle dh d\omega d\boldsymbol{\lambda} d\boldsymbol{\kappa}
 \end{aligned}$$

is well defined. Here $\hat{\lambda}^{MN}(h, \omega, \omega \boldsymbol{\lambda}, \boldsymbol{\kappa})$ is the Fourier transform of $\lambda^{MN}(t, \omega, \mathbf{x}, \boldsymbol{\kappa})$ with respect to t and $\mathbf{x}$. Therefore, starting with the solution $\langle \cdot, \lambda(z, \omega, \boldsymbol{\kappa}) \rangle$, $-L \leq z \leq 0$ of (2.32) we can define a family of stochastic processes $\langle \cdot, \lambda^\varepsilon(z) \rangle$ with values in $\mathcal{D}'_H$. That is, for each $\varepsilon > 0$, we have a probability measure P^ε on the space of continuous functions on $-L \leq z \leq 0$ with values in $\mathcal{D}'_H$. The continuity of $\langle \cdot, \lambda^\varepsilon(z) \rangle$ as a function of z when ε is positive follows from (2.32).

Assume that the process ν in (2.28) is bounded, Markovian,²² and strongly ergodic and let P^ε be the probability measure on $C(-L \leq z \leq 0; \mathcal{D}'_H)$ induced by the process defined by (2.32). Then as $\varepsilon \rightarrow 0$ the laws P^ε converge weakly to the deterministic law concentrated on the distribution solution W^{MN} of (3.12) [51]. The existence and uniqueness of solutions for (3.12) is outlined in §3.5 and is given in [51]. Thus, the convergence in (B.20) when interpreted in the sense of distributions is valid in probability.

The smoothing that the distribution interpretation introduces kills the fluctuations in the limit. However, this means that we do not have convergence of the moment functionals pointwise in t . Pointwise in t convergence is important in applications to inverse problems and to get it we would have to introduce a smoothing of λ^{MN} that is not fixed as $\varepsilon \rightarrow 0$ but changes in such a way such as to localize the time variable. There are many indications, especially from numerical simulations, [11], and Fig. E.5, that the limit law is related to a Gaussian law but this has not been proven yet.

C. Small fluctuation theory. In this appendix we will derive transport equations for the local power spectral density of reflected signals when the fluctuations in the medium parameters are weak but the pulse width is not necessarily large compared to the characteristic length (for example, the correlation length) of the inhomogeneities (cf. §2.3). If $\varepsilon > 0$ is the small parameter proportional to the standard deviation of the fluctuations in the compressibility K^{-1} we now have that

$$(C.1) \quad \rho(z) = \begin{cases} \rho_0, & z > 0, \\ \rho_1, & -L < z < 0, \\ \rho_2, & z < -L \end{cases}$$

²² This is not an essential hypothesis and merely serves to simplify the argument.

and

$$(C.2) \quad K^{-1}(z) = \begin{cases} K_0^{-1}, & z > 0, \\ K_1^{-1} \left(1 + \varepsilon \nu\left(\frac{z}{\varepsilon^2}\right)\right), & -L < z < 0, \\ K_2^{-1}, & z < -L, \end{cases}$$

and the scaled acoustic equations are

$$(C.3a) \quad \rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p = \varepsilon f\left(\frac{t}{\varepsilon^2}\right) \delta(x) \delta(y) \delta(z - z_s) \mathbf{e},$$

$$(C.3b) \quad K^{-1} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0.$$

Here we have again assumed for simplicity that the density ρ is a constant. We have also assumed that macroscopic distances are large, of order ε^{-2} compared to the correlation length, because it is for such long propagation distances that the weak fluctuation effects accumulate and make an order 1 contribution to field intensities. Note that the pulse width in (C.3) is of the same order as the scale of the inhomogeneities but its strength is now proportional to ε and not $\sqrt{\varepsilon}$ as in (2.17). Since the problem is linear, this simply adjusts the strength of the source so that the total energy released does not depend on ε and is of order 1. Note finally that we have assumed that K_1 in (C.2) is constant, independent of z , which is an essential difference with (2.16) and the rest of our work, inverse problems (§6) in particular. In the weak fluctuation scaling there is no nice way to allow for large scale variations of medium properties, where the geometrical acoustics approximation and the asymptotics for stochastic equations enter in a canonical way, as in the scaling of §2.3. There are several ways of putting in large scale variations approximately but we will not discuss them here. The weak fluctuations asymptotic limit of this appendix is the main approximation in nearly all other works in waves in random media, although this is rarely stated explicitly or treated in a systematic way. In the remainder of the appendix we will assume, as in §3, that the source is at the origin so that $z_s = 0$ and that the source directivity vector is in the z direction, $\mathbf{e} = (0, 0, 1)$.

We proceed as in §2.4 to get an integral representation of the reflected pressure, accounting for the new scaling. Let

$$(C.4a) \quad \hat{p}(\boldsymbol{\kappa}, \omega, z) = \iiint e^{i\omega(t - \boldsymbol{\kappa} \cdot \mathbf{x})/\varepsilon^2} p(t, \mathbf{x}, z) dt d\mathbf{x},$$

$$(C.4b) \quad \hat{\mathbf{u}}(\boldsymbol{\kappa}, \omega, z) = \iiint e^{i\omega(t - \boldsymbol{\kappa} \cdot \mathbf{x})/\varepsilon^2} \mathbf{u}(t, \mathbf{x}, z) dt d\mathbf{x},$$

where $\mathbf{x} = (x, y)$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ is the horizontal slowness vector. Analogously to (2.22) we now have for $z < 0$ the equations

$$(C.5) \quad \frac{d\hat{p}}{dz} = \frac{i\omega\rho_1}{\varepsilon^2} \hat{u}_3,$$

$$\frac{d\hat{u}_3}{dz} = \frac{i\omega}{\varepsilon^2\rho_1} \left(\frac{\rho_1}{K} - \kappa^2 \right) \hat{p}.$$

Let

$$(C.6) \quad c_1 = \sqrt{K_1/\rho_1}, \quad \zeta_1 = \frac{\rho_1 c_1}{\sqrt{1 - \kappa^2 c_1^2}}, \quad \tau(z, \kappa) = \frac{\sqrt{1 - \kappa^2 c_1^2}}{c_1} z.$$

In terms of these variables, we transform to upgoing and downgoing wave amplitudes A and B as in (2.26),

$$(C.7a) \quad \hat{p} = (\zeta_1)^{1/2} \left[A e^{i\omega\tau/\varepsilon^2} - B e^{-i\omega\tau/\varepsilon^2} \right],$$

$$(C.7b) \quad \hat{u}_3 = (\zeta_1)^{-1/2} \left[A e^{i\omega\tau/\varepsilon^2} + B e^{-i\omega\tau/\varepsilon^2} \right].$$

Then $\hat{p} = A/B$ satisfies the stochastic Riccati equation

$$(C.8) \quad \frac{d\hat{p}}{dz} = -\frac{i\omega}{\varepsilon} n \left\{ e^{-2i\omega\tau/\varepsilon^2} - 2, + e^{2i\omega\tau/\varepsilon^2}, 2 \right\}, \quad -L < z < 0,$$

$$\hat{p}|_{z=-L} = e^{-i2\omega\tau(-L)/\varepsilon^2}, \quad I(-L),$$

where the initial condition in (C.8) is just the continuity condition for $\hat{p}$ and $\hat{u}_3$ at $z = -L$ and

$$(C.9) \quad n\left(\frac{z}{\varepsilon^2}, \kappa\right) = \frac{\nu(z/\varepsilon^2)}{2c_1(1 - c_1^2\kappa^2)^{1/2}}.$$

The integral representation of the reflected pressure, the analogue of (2.35), is now

$$(C.10) \quad p_{\text{refl}}(t, \mathbf{x}, 0) = \frac{-1}{2(2\pi\varepsilon)^3} \iiint \hat{f}(\omega) \omega^2 e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - t)/\varepsilon^2}, \quad (0^+, \omega, \boldsymbol{\kappa}) \, d\omega \, d\boldsymbol{\kappa}.$$

The two-point intensity function is defined by

$$(C.11) \quad I(t, \mathbf{x}, \bar{t}, \bar{\mathbf{x}}) = \lim_{\varepsilon \downarrow 0} \left\langle p_{\text{refl}} \left(t + \frac{\varepsilon^2 \bar{t}}{2}, \mathbf{x} + \frac{\varepsilon^2 \bar{\mathbf{x}}}{2}, 0 \right) p_{\text{refl}} \left(t - \frac{\varepsilon^2 \bar{t}}{2}, \mathbf{x} - \frac{\varepsilon^2 \bar{\mathbf{x}}}{2}, 0 \right) \right\rangle.$$

We now use (2.33) in (C.10), expand the fraction in a geometric series, multiply two expressions (C.10), and do a change of variables like (2.39), with the obvious modifications dictated by (C.11). This gives the analogue of (2.40):

$$(C.12) \quad I(t, \mathbf{x}, \bar{t}, \bar{\mathbf{x}}) = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{4(2\pi)^3} \iiint d\boldsymbol{\kappa} d\omega |\hat{f}(\omega)|^2 \omega^2 e^{i\omega(\boldsymbol{\kappa} \cdot \mathbf{x} - \bar{t})} \\ \cdot \left[,_I^{N+M} \tilde{W}^{N+1, M+1} - ,_I^{M+N+1} \tilde{W}^{N, M+1} \right. \\ \left. - ,_I^{M+N+1} \tilde{W}^{N+1, M} + ,_I^{M+N+2} \tilde{W}^{NM} \right],$$

where $,_I = ,_I(0)$ and $\tilde{W}^{NM}$ is defined by

$$(C.13) \quad \tilde{W}^{NM}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) = \lim_{\varepsilon \downarrow 0} \frac{\omega^2}{(2\pi)^3} \iiint dh d\boldsymbol{\lambda} e^{iht - i\omega \boldsymbol{\lambda} \cdot \mathbf{x}} \\ \cdot \left\langle ,^N(z, \omega - \frac{\varepsilon^2 h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon^2 \boldsymbol{\lambda}}{2}) ,^M(z, \omega + \frac{\varepsilon^2 h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon^2 \boldsymbol{\lambda}}{2}) \right\rangle.$$

The arguments of $\tilde{W}$ in (C.13) are evaluated at

$$z = 0, \quad t \rightarrow t - \boldsymbol{\kappa} \cdot \mathbf{x}, \quad \mathbf{x}, \quad \boldsymbol{\kappa}, \quad \text{and } \omega$$

and we will assume again that the interface reflection coefficients, r_I are real. They depend in general on the slowness vector $\boldsymbol{\kappa}$ and are complex for evanescent modes but the evanescent spectrum contribution to the reflected signal is negligible in the asymptotic limit we consider here, just as it was in §3.

The transport equations for $\tilde{W}^{NM}$ are derived the same way as in §§3.1 and 3.2, which we review briefly, noting the differences. Let

$$(C.14) \quad \tilde{r}^{NM} = \tilde{r}^N \left(z, \omega - \frac{\varepsilon^2 h}{2}, \boldsymbol{\kappa} - \frac{\varepsilon^2 \boldsymbol{\lambda}}{2} \right), \quad \tilde{r}^M \left(z, \omega + \frac{\varepsilon^2 h}{2}, \boldsymbol{\kappa} + \frac{\varepsilon^2 \boldsymbol{\lambda}}{2} \right),$$

which is the analogue of (3.2). Let $\tilde{r}^{NM}$ be defined by (3.4) with the $\tilde{r}^{NM}$ given by (C.14). Then the equation for $\tilde{r}^{NM}$ analogous to (3.5) is

$$(C.15) \quad \begin{aligned} & \frac{\partial \tilde{r}^{NM}}{\partial z} + (N + M) \tau_z \left\{ \frac{\partial \tilde{r}^{NM}}{\partial t} - \frac{\nabla_{\boldsymbol{\kappa}} \tau_z}{\tau_z} \cdot \nabla \tilde{r}^{NM} \right\} \\ &= \frac{i\omega}{\varepsilon} 2n(N - M) \tilde{r}^{NM} - \frac{i\omega}{\varepsilon} n \left\{ N e^{-2i\omega\tau/\varepsilon^2} \tilde{r}^{N-1,M} + N e^{2i\omega\tau/\varepsilon^2} \tilde{r}^{N+1,M} \right. \\ & \quad \left. - M e^{2i\omega\tau/\varepsilon^2} \tilde{r}^{N,M-1} - M e^{-2i\omega\tau/\varepsilon^2} \tilde{r}^{N,M+1} \right\}. \end{aligned}$$

with initial conditions at $z = -L$,

$$\tilde{r}^{NM} |_{z=-L} = (r_I(-L))^N (\tilde{r}_I(-L))^M e^{-2i\omega\tau(-L)(N-M)/\varepsilon^2} \delta(t) \delta(\mathbf{x}).$$

We now apply the limit theorem with oscillatory phases of order ε^{-2} , (B.14), to get the transport equations for the expectation of $\tilde{r}^{NM}$ in the limit $\varepsilon \rightarrow 0$, which is $\tilde{W}^{NM}$. If we define

$$(C.16) \quad W^{NM}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega) = e^{2i\omega\tau(-L)(N-M)/\varepsilon^2} \tilde{W}^{NM}(z, t - \boldsymbol{\kappa} \cdot \mathbf{x}, \mathbf{x}, \boldsymbol{\kappa}, \omega)$$

analogously to (3.11), then $W^{NM}(z, t, \mathbf{x}, \boldsymbol{\kappa}, \omega)$ satisfies the transport equations

$$(C.17) \quad \begin{aligned} & \frac{\partial W^{NM}}{\partial z} + (N + M) \left\{ \frac{1}{c_1 (1 - c_1^2 \kappa^2)^{1/2}} \frac{\partial W^{NM}}{\partial t} + \frac{c_1 \boldsymbol{\kappa}}{(1 - c_1^2 \kappa^2)^{1/2}} \cdot \nabla W^{NM} \right\} \\ & - \frac{2\omega^2 \alpha_R N M}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ W^{N+1, M+1} - 2W^{NM} + W^{N-1, M-1} \right\} \\ & + \frac{2\omega^2}{c_1^2 (1 - c_1^2 \kappa^2)} \left\{ (2\alpha_0 + \alpha_R)(N - M)^2 + i\alpha_I(N - M) \right\} W^{NM} = 0, \end{aligned}$$

$$W^{NM} |_{z=-L} = (r_I(-L))^N (\tilde{r}_I(-L))^M \delta(t) \delta(\mathbf{x}),$$

where $N, M \geq 0$, $z < 0$, $t, \omega \in \mathbb{R}^1$, $\mathbf{x} \in \mathbb{R}^2$ and

$$(C.18) \quad \begin{aligned} \alpha_0 &= \frac{1}{4} \int_0^\infty E \{ \nu(s) \nu(0) \} ds, \\ \alpha_R &= \frac{1}{4} \int_0^\infty E \{ \nu(s) \nu(0) \} \cos \left(2\omega c_1^{-1} \sqrt{1 - c_1^2 \kappa^2} s \right) ds, \\ \alpha_I &= \frac{1}{4} \int_0^\infty E \{ \nu(s) \nu(0) \} \sin \left(2\omega c_1^{-1} \sqrt{1 - c_1^2 \kappa^2} s \right) ds. \end{aligned}$$

We can now compare the transport equations (C.17) to (3.12) which are obtained under the separation-of-scales hypotheses of §2.3. Clearly the small fluctuation theory agrees with the previous results if $\alpha_0 = \alpha_R = \alpha$ and $\alpha_I = 0$ which is true if $\omega = 0$ in (C.18), the low frequency limit of the small fluctuation theory. The most prominent difference is that for $\alpha_I \neq 0$ a complex term is introduced in (C.17) which affects only phases because it is zero when $N = M$. In most applications where only the diagonal part of the equations matters, the equations for $W^{NN} = W^N$ are the same as (3.14) except that α is replaced by α_R (which is ω and κ dependent) in (C.18).

D. The O'Doherty-Anstey theory. This theory [70] describes the behavior of a pulse advancing in a random medium with weak fluctuations. It predicts that if the pulse is observed in a Lagrangean frame that moves with the random acoustic velocity then the pulse appears to retain its shape up to a slow spreading. A rather convincing heuristic explanation of this phenomenon is given in §2 of [18]. The O'Doherty-Anstey theory is analyzed in detail in [18] for a special class of discrete media, the Goupillaud media, and further comments and results can be found in [69]. In this appendix we will show how the O'Doherty-Anstey theory comes out of the weak fluctuation framework of Appendix C, for a plane wave normally incident on a random half space with uniform background and weak random fluctuations in the sound speed.

The normal velocity component $u(t, z)$ and the pressure $p(t, z)$ satisfy the equations

$$(D.1a) \quad \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = \frac{1}{\varepsilon} f \left(\frac{t + c_1^{-1} z_s}{\varepsilon^2} \right) \delta(z - z_s),$$

$$(D.1b) \quad \frac{1}{K} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} = 0,$$

where $\rho(z) = \rho_1$ everywhere,

$$(D.2) \quad K^{-1}(z) = \begin{cases} K_1^{-1}, & z > 0, \\ K_1^{-1} \left(1 + \varepsilon \nu \left(\frac{z}{\varepsilon^2} \right) \right), & z < 0, \end{cases}$$

and

$$(D.3) \quad c(z) = \left(\frac{K(z)}{\rho(z)} \right)^{1/2}$$

is the local sound speed. We assume for simplicity that the mean sound speed is equal to $c_1 = \sqrt{K_1/\rho_1}$ throughout all of space so that there are no discontinuities in the average medium properties. The source is located at $z_s > 0$ and the pulse is shifted in time so that it arrives at the interface $z = 0$ at $t = -c_1^{-1} z_s$. The scaling factor ε^{-1} in the source term in (D.1) makes the energy of the incident pulse independent of ε . As before, the pulse shape function f is assumed smooth and vanishing for $t < 0$ and for large t . In the homogeneous halfspace $z > 0$ the sound speed is constant and equal to c_1 .

The scaled Fourier transforms in time of p and u are:

$$(D.4a) \quad \hat{p}(z, \omega) = \int e^{i\omega t/\varepsilon^2} p(t, \omega) dt,$$

$$(D.4b) \quad \hat{u}(z, \omega) = \int e^{i\omega t/\varepsilon^2} u(t, \omega) dt.$$

Let

$$(D.5) \quad \tau(z) = \int_0^z \frac{1}{c(s)} ds$$

be the travel time, let $\xi(\tau)$ be its inverse, and let

$$(D.6) \quad \zeta(z) = (\rho(z)K(z))^{1/2}$$

be the acoustic impedance²³. Define up- and downgoing wave amplitudes by

$$(D.7a) \quad \hat{p} = \zeta^{1/2} \left[A e^{i\omega\tau/\varepsilon^2} - B e^{-i\omega\tau/\varepsilon^2} \right],$$

$$(D.7b) \quad \hat{u} = \zeta^{-1/2} \left[A e^{i\omega\tau/\varepsilon^2} + B e^{-i\omega\tau/\varepsilon^2} \right].$$

Then, $A(z, \omega)$ and $B(z, \omega)$ satisfy the differential equations

$$(D.8a) \quad \frac{dA}{dz} = \frac{d}{dz} \left(\ln \zeta^{1/2}(z) \right) e^{-2i\omega\tau/\varepsilon^2} B,$$

$$(D.8b) \quad \frac{dB}{dz} = \frac{d}{dz} \left(\ln \zeta^{1/2}(z) \right) e^{2i\omega\tau/\varepsilon^2} A.$$

Following the ideas of §3.14 regarding transmitted fields we introduce $\hat{\cdot}$ and Z by

$$(D.9) \quad \hat{\cdot}(z, \omega) = \frac{A(z, \omega)}{B(z, \omega)}, \quad Z(z, \omega) = \frac{\hat{p}(-L, \omega)}{B(z, \omega)}.$$

They satisfy the equations

$$(D.10a) \quad \frac{d\hat{\cdot}}{dz} = \frac{d}{dz} \left(\ln \zeta^{1/2}(z) \right) \left[e^{-2i\omega\tau(z)/\varepsilon^2} - \hat{\cdot}^2 e^{2i\omega\tau(z)/\varepsilon^2} \right],$$

$$(D.10b) \quad \frac{dZ}{dz} = -\frac{d}{dz} \left(\ln \zeta^{1/2}(z) \right) e^{2i\omega\tau(z)/\varepsilon^2} Z$$

in $-L < z < 0$, with $\hat{\cdot}(-L) = 0$ and $Z(-L) = \zeta^{1/2}(-L) e^{-2i\omega\tau(-L)/\varepsilon^2}$. From the continuity of $\hat{u}$ and $\hat{p}$ at $z = 0$ we find that

$$(D.11) \quad B(0^-, \omega) = \frac{\varepsilon}{2} (\rho_1 c_1)^{-1/2} \hat{f}(\omega)$$

and hence from (D.9)

$$\hat{p}(-L, \omega) = \frac{\varepsilon}{2} (\rho_1 c_1)^{-1/2} \hat{f}(\omega) Z(0, \omega)$$

and

$$(D.12) \quad p(-L, t) = \frac{1}{4\pi\varepsilon} (\rho_1 c_1)^{-1/2} \int \hat{f}(\omega) Z(0, \omega) e^{-i\omega t/\varepsilon^2} d\omega.$$

Note that $Z(0, \omega)$ depends on $-L$, which is the initial point for (D.10). Note also that in order to calculate the properties of the advancing pulse using initial value problems

²³ Note that the travel time is negative when z is negative and that both it and the acoustic impedance are random processes with rapid spatial variations.

(the invariant imbedding method) we must analyze the pulse at the instant it emerges from the random medium into a fictitious uniform medium located below the depth at which we make the observations.

We want to calculate the pressure as a function travel time which means that we are interested in $p(\xi(\tau), t)$ as $\varepsilon \rightarrow 0$. For this purpose we let τ be the independent variable in (D.10) and let σ , (τ, ω) , $Z(\tau, \omega)$, and $\zeta(\tau)$ stand for σ , $(\xi(\tau), \omega)$, $Z(\xi(\tau), \omega)$, and $\zeta(\xi(\tau))$, respectively. Then, equations (D.10) become

$$(D.13a) \quad \frac{d}{d\sigma} = \frac{d}{d\sigma} \left(\ln \zeta^{1/2}(\sigma) \right) \left[e^{-2i\omega\sigma/\varepsilon^2} - e^{2i\omega\sigma/\varepsilon^2} \right],$$

$$(D.13b) \quad \frac{dZ}{d\sigma} = -\frac{d}{d\sigma} \left(\ln \zeta^{1/2}(\sigma) \right) e^{2i\omega\sigma/\varepsilon^2}, \quad Z$$

in $-\tau < \sigma < 0$, with $\sigma(-\tau) = 0$ and $Z(-\tau) = \zeta^{1/2}(-\tau)e^{-2i\omega\tau/\varepsilon^2}$. When we expand the impedance ζ in powers of ε and include the equation for $\xi(\tau)$, from (D.5), we find that equations (D.13) become

$$(D.14a) \quad \frac{d}{d\sigma} = \left[\frac{1}{\varepsilon} \mu \left(\frac{\xi(\sigma)}{\varepsilon^2} \right) + q \left(\frac{\xi(\sigma)}{\varepsilon^2} \right) \right] \left[e^{-2i\omega\sigma/\varepsilon^2} - e^{2i\omega\sigma/\varepsilon^2} \right] + \dots,$$

$$(D.14b) \quad \frac{dZ}{d\sigma} = - \left[\frac{1}{\varepsilon} \mu \left(\frac{\xi(\sigma)}{\varepsilon^2} \right) + q \left(\frac{\xi(\sigma)}{\varepsilon^2} \right) \right] e^{2i\omega\sigma/\varepsilon^2}, \quad Z + \dots,$$

$$(D.14c) \quad \frac{d\xi}{d\sigma} = c_1 \left[1 - \frac{\varepsilon}{2} \nu \left(\frac{\xi(\sigma)}{\varepsilon^2} \right) + \frac{3\varepsilon^2}{8} \nu^2 \left(\frac{\xi(\sigma)}{\varepsilon^2} \right) \right] + \dots,$$

where

$$(D.15) \quad \mu(z) = -\frac{c_1}{4} \frac{d\nu(z)}{dz}, \quad q(z) = -\frac{c_1}{8} \nu(z) \frac{d\nu(z)}{dz}.$$

This system is different from (C.8) not only because we have in addition to σ , the variable Z , as we do in §3.14, but primarily because the argument of the random processes μ , q , and ν is not σ/ε^2 but $\xi(\sigma)/\varepsilon^2$, where $\xi(\sigma)$ is the inverse travel time and satisfies the third equation in (D.14). We will not give the details here but it can be shown²⁴ that the asymptotic behavior of (D.14) as $\varepsilon \rightarrow 0$ is the same as when we set $\xi(\sigma) = c_1\sigma$ in the first two equations in (D.14). This is not an obvious statement because the scaled inverse travel time $\xi(\sigma)/\varepsilon^2$ is affected by the higher order corrections to $\xi = c_1\sigma$. In fact, this asymptotic equivalence is not true in general and is true here because of the special form of μ in (D.15). It is the derivative of ν which is also the first correction to $d\xi/d\sigma$.

Using this asymptotic equivalence we can proceed much as we did in §3.14, allowing for the necessary modifications given in Appendix C for the small fluctuation scaling that we now have. We find that,

$$(D.16) \quad \langle p(\xi(\tau), t) \rangle \sim \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(t-|\tau|)/\varepsilon^2} e^{i\omega R(0)|\tau|/8} e^{-\omega^2(\alpha_R + i\alpha_I)|\tau|} d\omega,$$

where

$$(D.17) \quad R(s) = E\{\nu(s)\nu(0)\},$$

²⁴ By the generator perturbation method in the appendix of [10]

and

$$(D.18) \quad \alpha_R = \frac{1}{4c_1} \int_0^\infty R(s) \cos\left(\frac{2\omega s}{c_1}\right) ds, \quad \alpha_I = \frac{1}{4c_1} \int_0^\infty R(s) \sin\left(\frac{2\omega s}{c_1}\right) ds.$$

From (D.16) we see that the mean pulse advances into the medium with a retardation given by $\varepsilon^2 R(0)/8$ and with its shape convoluted by the complex Gaussian whose Fourier transform is $\exp\{-\omega^2(\alpha_R + i\alpha_I)|\tau|\}$. This description is valid when α_R and α_I are approximately constant as ω varies over the support of $\hat{f}$. As in §3.3 we can also calculate from (D.16) the asymptotic behavior of the square of the coherent pulse. We find that

$$(D.19) \quad \langle p(\xi(\tau), t) \rangle^2 \sim \delta(t - |\tau|) \frac{1}{8\pi} \int_{-\infty}^\infty |\hat{f}(\omega)|^2 e^{-2\omega^2 \alpha_R |\tau|} d\omega.$$

What is remarkable and characteristic of the O'Doherty-Anstey theory, however, is that not only $\langle p(\xi(\tau), t) \rangle$ behaves like (D.16) but also $p(\xi(\tau), t)$. This means that the fluctuations are negligible when the advancing pulse is observed relative to its travel time. To prove this we must show that near the front $t = |\tau|$

$$(D.20) \quad \langle p(\xi(\tau), t)^2 \rangle \sim \langle p(\xi(\tau), t) \rangle^2,$$

which we do by calculating explicitly $\langle p(\xi(\tau), t)^2 \rangle$ as we did in §3.14. We find that it behaves exactly like (D.19) and hence (D.20) holds. Property (D.20) is not true if we observe the pulse in the neighborhood of the mean distance travelled by time t , which is $c_1 t$.

Is the O'Doherty-Anstey property (D.20) valid when the fluctuations in the medium properties are large, that is, with the scaling of §2.3? It is clear from the results of the numerical experiments shown in Figs. 1.1a and 1.3b that it should be valid but this has not been shown yet. Property (D.20) must be shown to be valid near the front but the simple description (D.16) and (D.19) of the advancing pulse will not be true unless the fluctuations are weak.

E. A windowed Fourier transform for the local power spectral density.

In the numerical simulation of reflected signals in §4 and in the implementation of inversion algorithms in §6, we saw that it is important to have an accurate and robust method for estimating local power spectral densities. In this appendix we introduce such a method using the windowed Fourier transform.

First we review briefly the windowed Fourier transform for functions $f(t) \in L^2(\mathbf{R})$ [84], [85]. Let $g(t) \in C_0^\infty(\mathbf{R})$ be the localizing or window function and choose it so that

$$(E.1) \quad \text{support } g = \frac{2\pi}{\omega_0},$$

$$(E.2) \quad \pi < \omega_0 t_0 < 2\pi,$$

$$(E.3) \quad \sum_{n=-\infty}^{\infty} |g(t - nt_0)|^2 = \frac{1}{t_0},$$

$$(E.4) \quad \int_{-\infty}^{\infty} g^2(t) dt = 1.$$

An example of such a g is

$$(E.5) \quad g(t) = \begin{cases} t_0^{-1/2} \sin \left[\frac{\pi}{2} \nu \left(\lambda^{-1} \left(\frac{\pi}{\omega_0} + t \right) \right) \right] & \text{for } \frac{-\pi}{\omega_0} \leq t \leq \frac{-\pi}{\omega_0} + \lambda, \\ t_0^{-1/2} & \text{for } \frac{-\pi}{\omega_0} + \lambda \leq t \leq \frac{\pi}{\omega_0} - \lambda, \\ t_0^{-1/2} \cos \left[\frac{\pi}{2} \nu \left(\lambda^{-1} \left(t - \frac{\pi}{\omega_0} + \lambda \right) \right) \right] & \text{for } \frac{\pi}{\omega_0} - \lambda \leq t \leq \frac{\pi}{\omega_0}, \end{cases}$$

where $\nu(t)$ is smooth, $\nu(t) = 0$ for $t \leq 0$, $\nu(t) = 1$ for $t \geq 1$ and $\lambda = \frac{2\pi}{\omega_0} - t_0$. Figure E.1a shows this function whose squares form a partition of unity (property (E.3)). To define the local or windowed Fourier transform let

$$(E.6) \quad g_{mn}(t) = e^{-i\omega_0 m t} g(t - nt_0),$$

and for $f \in C_0^\infty(\mathbf{R})$ let

$$(E.7) \quad f_{mn} = (f, g_{mn}) = \int_{-\infty}^{\infty} f(t) \overline{g_{mn}(t)} dt$$

be the local Fourier coefficients. The index n specifies the location of the window center at $t = nt_0$ and the index m specifies the harmonic at radian frequency $\omega_0 m$. Note that (E.2) restricts the sampling frequency ω_0 and the window spacing t_0 , so that $T_0 = 2\pi/\omega_0$ is bigger than t_0 . This means that the function f can be replaced by a periodic extension of period T_0 outside windows spaced by $t_0 < T_0$ and no information is lost or altered. The lower inequality in (E.2) is not needed in general but is required in the example (E.5). The main theorem is now that $f(t)$ can be recovered from its local Fourier coefficients

$$(E.8) \quad f(t) = \frac{\omega_0 t_0}{2\pi} \sum_{m,n=-\infty}^{\infty} f_{mn} g_{mn}(t)$$

and we have the Parseval relation

$$(E.9) \quad \int_{-\infty}^{\infty} f^2(t) dt = \frac{\omega_0 t_0}{2\pi} \sum_{m,n=-\infty}^{\infty} |f_{mn}|^2.$$

From (E.9) it follows that the representation (E.8) extends to all of $L^2(\mathbf{R})$.

The proof of (E.8) is quite simple and is based on the Poisson summation formula [86], which we write in the form

$$(E.10) \quad \frac{1}{2\pi} \sum_m e^{i\omega_0 m \sigma} = \frac{1}{\omega_0} \sum_m \delta \left(\sigma + \frac{2\pi m}{\omega_0} \right).$$

We have that for $f \in C_0^\infty(\mathbf{R})$

$$\begin{aligned} & \frac{\omega_0 t_0}{2\pi} \sum_{m,n=-\infty}^{\infty} g_{mn}(t) \int_{-\infty}^{\infty} f(s) \overline{g_{mn}(s)} ds \\ &= \frac{\omega_0 t_0}{2\pi} \int_{-\infty}^{\infty} ds f(s) \sum_n g(t - nt_0) g(s - nt_0) \sum_m e^{i\omega_0 m(s-t)} \\ &= t_0 \int_{-\infty}^{\infty} ds f(s) \sum_n g(t - nt_0) g(s - nt_0) \sum_m \delta \left(s - t + \frac{2\pi m}{\omega_0} \right) \\ &= f(t) t_0 \sum_n |g(t - nt_0)|^2 = f(t), \end{aligned}$$

where we have used (E.6) and (E.10) on the second line, (E.1) and (E.2) on the third line, and (E.3) on the last line.

We now return to our reflected signals, for example, $p_{\text{ref}}(t, \mathbf{x}, 0)$ given by (2.35), and consider the estimation of its spectrum at a given location $\mathbf{x}$ on the surface. Let $R_f^\varepsilon(t)$ denote the signal with f the fixed pulse shape function of the source (cf. (2.17)) and ε the pulse width. We know that as ε tends to zero, $R_f^\varepsilon(t)$ tends to a locally stationary process. Let

$$(E.11) \quad C_f^\varepsilon(t, \sigma) = E \left\{ R_f^\varepsilon \left(t - \frac{\varepsilon\sigma}{2} \right) R_f^\varepsilon \left(t + \frac{\varepsilon\sigma}{2} \right) \right\}$$

be the local or windowed correlation function of the signal. Then

$$C_f^\varepsilon(t, \sigma) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\sigma} |\hat{f}(\omega)|^2 \Lambda(t, \omega) d\omega$$

for ε small where Λ is determined by solving a transport equation, as we saw in §3. We may therefore consider the expression

$$(E.12) \quad \int_{-M}^M e^{-i\omega\sigma} C_f^\varepsilon(t, \sigma) d\sigma$$

for some $M < \infty$ as an approximation for $|\hat{f}(\omega)|^2 \Lambda(t, \omega)$, the local power spectral density. We note first that although (E.12) is well defined for $\varepsilon > 0$ and $M < \infty$, it is not a nonnegative function. This is well known in signal analysis [87]. Second, if we try to use (E.12) as an estimate for $|\hat{f}(\omega)|^2 \Lambda(t, \omega)$ it will depend sensitively on the truncation M . With enough experimenting with the data we can make a suitable version of (E.12) provide useful estimates for $\Lambda(t, \omega)$. This was done in [11] and in [14] and [15] with data generated by direct numerical simulation of the scattering problem. It was actually done in a somewhat different way which is more efficient computationally and is described in detail in [11]. The question is, then: What is a good way to estimate $\Lambda(\omega, t)$ from $R_f^\varepsilon(t)$ when we only know ε , the pulse width? By good we mean primarily robust, an algorithm that does not need fine tuning each time we look at a different set of data. The estimation must also be unbiased and asymptotically consistent. For reflected signals like $R_f^\varepsilon(t)$, the local Fourier transform works exceptionally well because knowing the parameter ε , we also know that the interval of stationarity of the signal and its correlation time are of order ε . To make this precise, we assume that $C_f^\varepsilon(t, \sigma)$ satisfies the conditions

$$(E.13) \quad \overline{\lim}_{\varepsilon \downarrow 0} \sup_{|\sigma| \leq M} \sup_{|t-\tau| \leq \varepsilon M} \left| C_f^\varepsilon(t, \sigma) - C_f^\varepsilon(\tau, \sigma) \right| = 0$$

for each $M < \infty$,

$$(E.14) \quad \lim_{M \uparrow \infty} \overline{\lim}_{\varepsilon \downarrow 0} \sup_{t \leq K} \int_{|\sigma| > M} \left| C_f^\varepsilon(t, \sigma) \right| d\sigma = 0$$

and

$$(E.15) \quad \lim_{\varepsilon \downarrow 0} C_f^\varepsilon(t, \sigma) = C_f(t, \sigma)$$

pointwise in t and σ . The first condition says that $R_f^\varepsilon(t)$ is approximately stationary on intervals of length ε and the second that the correlation length is of order ε . We can now estimate the local power spectral density as follows.

Let $g^\delta(t)$ be a window function like g in (E.1)-(E.4) with $\omega_0 \mapsto \omega_0/\delta$ and $t_0 \mapsto \delta t_0$. This means that $g^\delta(t)$ has support proportional to δ and that the spacing of the g 's in (E.3) is of order δ . In fact, $g^\delta(t) = \delta^{-1/2}g(t\delta^{-1})$. Let

$$(E.16) \quad g_{mn}^\delta(t) = e^{-i\omega_0 m t / \delta} g^\delta(t - n\delta t_0)$$

and

$$(E.17) \quad R_{mn}^{\delta \varepsilon} = (R_f^\varepsilon, g_{mn}^\delta) = \int_{-\infty}^{\infty} R_f^\varepsilon(t) \overline{g_{mn}^\delta(t)} dt.$$

Then

$$(E.18) \quad |\hat{f}(\omega)|^2 \Lambda(t, \omega) = \lim_{\gamma \uparrow \infty} \lim_{\varepsilon \downarrow 0} \sum_{mn} E \left\{ |R_{mn}^{\gamma \varepsilon}|^2 \right\} G_{mn}^{\gamma \varepsilon}(t, \omega),$$

where

$$(E.19) \quad G_{mn}^{\delta \varepsilon}(t, \omega) = \frac{\omega_0 t_0}{2\pi} \int_{-\infty}^{\infty} e^{-i(\omega - \omega_0 m \varepsilon \delta^{-1})\sigma} g^\delta\left(t - n\delta t_0 - \varepsilon \frac{\sigma}{2}\right) g^\delta\left(t - n\delta t_0 + \varepsilon \frac{\sigma}{2}\right) d\sigma.$$

The meaning of (E.18) is that given the reflected signal $R_f^\varepsilon(t)$, we compute its local Fourier coefficients $R_{mn}^{\delta \varepsilon}$ relative to the cutoff function g^δ with δ a multiple of ε , say $\delta = 3\varepsilon$, so that $\gamma = 3$. Then with t of the form $t = N\delta t_0$ and ω of the form $\omega_0 M \varepsilon \delta^{-1}$ we estimate the local power spectral density by

$$(E.20) \quad |\hat{f}(\omega)|^2 \Lambda(t, \omega) \sim |R_{MN}^{\gamma \varepsilon}|^2 G_0^\varepsilon + |R_{M+1N}^{\gamma \varepsilon}|^2 G_1^\varepsilon + |R_{M-1N}^{\gamma \varepsilon}|^2 G_{-1}^\varepsilon,$$

where

$$(E.21) \quad G_M^\varepsilon = \frac{\omega_0 t_0}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_0 M \sigma / \gamma} g^\varepsilon\left(-\varepsilon \frac{\sigma}{2}\right) \overline{g^\varepsilon\left(\varepsilon \frac{\sigma}{2}\right)} d\sigma.$$

We plot G_M^ε as a function of M in Fig. E.1b with $\gamma = 1$. Note that G_0^ε dominates and also that G_1^ε and G_{-1}^ε are small negative numbers. We only take three terms in the sum over m in (E.20) because G_M^ε decays very rapidly with M . Also, when t and ω take the above special values then only one term in the sum over n enters. A naive estimate for $|\hat{f}(\omega)|^2 \Lambda(\omega, t)$ would be a multiple of $|R_{MN}^{\gamma \varepsilon}|^2$, which is comparable to the periodogram for stationary processes. Our estimate is a very significant improvement in practice.

The proof of (E.18) is not difficult, given (E.13), (E.14) and the properties (E.1)-(E.4) of $g^\delta(t)$ (with $\omega_0 \mapsto \omega_0/\delta$ and $t_0 \mapsto \delta t_0$). Conditions (E.13), (E.14), and (E.15) are however difficult to check for our reflected signals and they have not yet been proven. Before giving the proof (E.18), we note that it is not the only way to estimate $|\hat{f}(\omega)|^2 \Lambda(\omega, t)$. We also have

$$(E.22) \quad |\hat{f}(\omega)|^2 \Lambda(t, \omega) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{mn} E \left\{ |R_{mn}^{\delta \varepsilon}|^2 \right\} G_{mn}^{\delta \varepsilon}(t, \omega).$$

This result is valid with condition (E.14) and the following two weaker conditions which have been proven for reflected signals in [51]:

$$(E.23) \quad \lim_{\varepsilon \downarrow 0} \iint \phi(\tau, \sigma) C_f^\varepsilon(\tau, \sigma) d\tau d\sigma = \iint \phi(\tau, \sigma) C_f(\tau, \sigma) d\tau d\sigma$$

for each smooth function ϕ of compact support and

$$(E.24) \quad \lim_{\delta \downarrow 0} \int_{|\tau-t| \leq \delta} d\tau \int d\sigma |C_f(\tau, \sigma) - C_f(t, \sigma)| = 0$$

for each t . Moreover, while the mean of the square of the difference between the two sides in (E.20)

$$E \left\{ \left(|\hat{f}(\omega)|^2 \Lambda(t, \omega) - \left(|R_{MN}^{\gamma \varepsilon \varepsilon}|^2 G_0^\varepsilon + |R_{M+1N}^{\gamma \varepsilon \varepsilon}|^2 G_1^\varepsilon + |R_{M-1N}^{\gamma \varepsilon \varepsilon}|^2 G_{-1}^\varepsilon \right) \right)^2 \right\}$$

is of order one as ε goes to zero, when ε goes to zero and then δ goes to zero the fluctuation also goes to zero [51]. We will not discuss fluctuations in this Appendix however.

We prove (E.22) first. Note that

$$\begin{aligned} E\{|R_{mn}^{\delta \varepsilon}|^2\} &= \iint E\{R_f^\varepsilon(t) \overline{R_f^\varepsilon(s)}\} g_{mn}^\delta(t) \overline{g_{mn}^\delta(s)} dt ds \\ &= \varepsilon \iint C_f^\varepsilon(\tau, \sigma) e^{-i\omega_0 m \varepsilon \sigma / \delta} g^\delta\left(\tau - n\delta t_0 - \frac{\varepsilon \sigma}{2}\right) g^\delta\left(\tau - n\delta t_0 + \frac{\varepsilon \sigma}{2}\right) d\tau d\sigma, \end{aligned}$$

where we have made the change of variables

$$t = \tau - \varepsilon \frac{\sigma}{2} \text{ and } s = \tau + \varepsilon \frac{\sigma}{2}$$

Thus

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty} E\{|R_{m,n}^{\delta \varepsilon}|^2\} G_{mn}^{\delta \varepsilon}(t, \omega) &= \frac{\varepsilon \omega_0 t_0}{2\pi} \sum_{m,n=-\infty}^{\infty} \iint d\tau d\sigma d\tilde{\sigma} C_f^\varepsilon(\tau, \sigma) e^{-i\omega \tilde{\sigma}} e^{i(\omega_0 \tilde{\sigma} - \omega_0 \sigma) m \varepsilon / \delta} \\ &\quad \cdot g^\delta\left(\tau - n\delta t_0 - \frac{\varepsilon \sigma}{2}\right) g^\delta\left(t - n\delta t_0 + \frac{\varepsilon \tilde{\sigma}}{2}\right) g^\delta\left(t - n\delta t_0 - \frac{\varepsilon \tilde{\sigma}}{2}\right) g^\delta\left(\tau - n\delta t_0 + \frac{\varepsilon \sigma}{2}\right). \end{aligned}$$

We now use the Poisson summation formula (E.10) to get

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty} E\{|R_{m,n}^{\delta \varepsilon}|^2\} G_{mn}^{\delta \varepsilon}(t, \omega) &= \iint d\tau d\sigma C_f^\varepsilon(\tau, \sigma) e^{-i\omega \sigma} \\ &\quad \cdot \delta t_0 \sum_{m=-1}^1 e^{2\pi i m \delta / \omega_0 \varepsilon} \sum_{n=-\infty}^{\infty} g^\delta\left(\tau - n\delta t_0 - \frac{\varepsilon \sigma}{2}\right) g^\delta\left(\tau - n\delta t_0 + \frac{\varepsilon \sigma}{2}\right) \\ &\quad \cdot g^\delta\left(t - n\delta t_0 + \frac{\varepsilon \sigma}{2} + \frac{m\delta \pi}{\omega_0}\right) g^\delta\left(t - n\delta t_0 - \frac{\varepsilon \sigma}{2} - \frac{m\delta \pi}{\omega_0}\right). \end{aligned}$$

When the Poisson summation formula is used, only the terms $m = 0, +1, -1$ in the sum contribute because of the support properties (E.1) and (E.2) of g^δ and this gives

the above equality. Again because of the support properties of g^δ , the τ integration is only over the interval $|t - \tau| \leq 3\pi\delta\omega_0^{-1}$ and the σ integration over $|\sigma| \leq 3\pi\delta/\varepsilon\omega_0$. Using (E.13) and (E.14) we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \sum_{m,n=-\infty}^{\infty} E\{|R_{mn}^{\delta\varepsilon}|^2\} G_{mn}^{\delta\varepsilon}(t, \omega) \\ = \iint d\tau d\sigma C_f(\tau, \sigma) e^{-i\omega\sigma} \delta t_0 \sum_{n=-\infty}^{\infty} |g^\delta(\tau - n\delta t_0)|^2 |g^\delta(t - n\delta t_0)|^2. \end{aligned}$$

The terms $m = \pm 1$ in the sum vanish because of the support properties (E.1) and (E.2) of g^δ and only the term $m = 0$ survives. Since $C_f(\tau, \sigma)$ is continuous in τ in the sense of (E.24) and in view of (E.3) and (E.4)

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{m,n=-\infty}^{\infty} E\{|R_{mn}^{\delta\varepsilon}|^2\} G_{mn}^{\delta\varepsilon}(t, \omega) = \int d\sigma C_f(\tau, \sigma) e^{-i\omega\sigma},$$

which is what we wanted to prove.

To prove (E.18) we note that with $\delta = \gamma\varepsilon$ we let ε go to zero so that $t = N\gamma\varepsilon t_0$ with t and t_0 fixed and $N = 1, 2, 3, \dots$. Then we have

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty} E\{|R_{mn}^{\gamma\varepsilon}|^2\} G_{mn}^{\gamma\varepsilon}(t, \omega) &= \iint d\tau d\sigma C_f^\varepsilon(\tau, \sigma) e^{-i\omega\sigma} \\ &\cdot \gamma\varepsilon t_0 \sum_{m=-1}^1 e^{2\pi i m \gamma / \omega_0} \sum_{n=-\infty}^{\infty} g^{\gamma\varepsilon} \left(\tau - n\varepsilon\gamma t_0 - \frac{\varepsilon\sigma}{2} \right) g^{\gamma\varepsilon} \left(\tau - n\varepsilon\gamma t_0 + \frac{\varepsilon\sigma}{2} \right) \\ &\cdot g^{\gamma\varepsilon} \left((N - n)\varepsilon\gamma t_0 + \frac{\varepsilon\sigma}{2} + \frac{m\varepsilon\gamma\pi}{\omega_0} \right) g^{\gamma\varepsilon} \left((N - n)\varepsilon\gamma t_0 - \frac{\varepsilon\sigma}{2} - \frac{m\varepsilon\gamma\pi}{\omega_0} \right). \end{aligned}$$

The σ integral is decomposed into two parts, one over $|\sigma| \leq M$ and one over $|\sigma| > M$. The part over $|\sigma| > M$ is arbitrarily small uniformly in ε and γ if M is large enough because of (E.14) and also because $g^\delta(t) = \delta^{-1/2} g(t\delta^{-1})$ where $g(t)$ is a nonnegative function of compact support given explicitly by (E.5). In the integral over $|\sigma| \leq M$ we add and subtract the term $C_f^\varepsilon(t, \sigma)$. By the uniform continuity condition (E.13) and the convergence (E.15), we see that we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \sum_{m,n=-\infty}^{\infty} E\{|R_{mn}^{\gamma\varepsilon}|^2\} G_{mn}^{\gamma\varepsilon}(t, \omega) &= \lim_{M \rightarrow \infty} \lim_{\gamma \rightarrow \infty} \int_{|\sigma| \leq M} d\sigma C_f(t, \sigma) e^{-i\omega\sigma} \\ &\cdot t_0 \sum_{m=-1}^1 e^{2\pi i m \gamma / \omega_0} \sum_{\tilde{n}=-\infty}^{\infty} \int ds g \left(s - \tilde{n}t_0 - \frac{\sigma}{2\gamma} \right) g \left(s - \tilde{n}t_0 + \frac{\sigma}{2\gamma} \right) \\ &\cdot g \left(\tilde{n}t_0 + \frac{\sigma}{2\gamma} + \frac{m\pi}{\omega_0} \right) g \left(\tilde{n}t_0 - \frac{\sigma}{2\gamma} - \frac{m\pi}{\omega_0} \right), \end{aligned}$$

which is equal to

$$\begin{aligned} \lim_{M \rightarrow \infty} \int_{|\sigma| \leq M} d\sigma C_f(t, \sigma) e^{-i\omega\sigma} \\ \cdot t_0 \sum_{\tilde{n}=-\infty}^{\infty} \int ds g^2(s - \tilde{n}t_0) g(\tilde{n}t_0) g(\tilde{n}t_0). \end{aligned}$$

Here the index $\tilde{n} = N - n$, the sum is over a finite number of terms, we have used again $g^\delta(t) = \delta^{-1/2}g(t\delta^{-1})$, and we have changed variables in the τ integration by letting $s = (\tau - t)(\varepsilon\gamma)^{-1}$. Now only the term $m = 0$ survives in the sum because of the support properties of g , and the result (E.18) follows from properties (E.3) and (E.4).

As we noted above, however, the necessary conditions on the local covariance (E.13), (E.14), and (E.15) have not been proven yet for reflected signals. Moreover, contrary to (E.22), the variance of the estimate (E.20) does not go to zero as ε goes to zero when $\delta = \gamma\varepsilon$.

We have done extensive numerical experiments with normally incident plane wave pulses in a uniform background for which the local power spectral density has the form [10]

$$(E.25) \quad \Lambda(\omega, t) = \frac{\alpha\omega^2}{(1 + \alpha\omega^2t)^2}$$

with α a positive constant and we have obtained excellent results. We used the layer code described in §4 to generate reflected signals. In Fig. E.2a we compare (E.25) with estimates obtained using (E.20) with $\gamma = 3$, with 800 realizations and with a t_0 that is as big as can be chosen compatible with (E.2), for a given ω_0 that is fixed. The parameter γ is chosen so that if l is a characteristic decay length of $C_f(t, \sigma)$, then $l\omega_0 < 2\pi\gamma$. In Fig. E.2b we see that choosing a smaller t_0 and different ω_0 's in the range (E.2) does not affect the results. In Fig. (E.3) we examine the dependence of the estimate (E.20) on ε for a fixed, small number of realizations (seven). It is clear that the fluctuations do not die out with ε since (E.20) is analogous to a periodogram estimate for a stationary process. In Fig. E.4 we plot the estimate of $\Lambda(t, \omega)$ as a function of ω for fixed t and compare it with the theoretical result (E.25). With only 59 realizations we get much better agreement with the theory than in our earlier work [14] where the windowing was done heuristically.²⁵ Finally, in Fig. E.5 we see that the estimate $\hat{\Lambda}(t, \omega)$ of (E.20) has an exponential law, a fact that we used in our inversions in §6.

²⁵ The main point of [14] is, however, the localization length scaling, an idea that extends the validity of the W equations to narrower pulses.

FIG. E.1A. *Window function $g(t)$.*

FIG. E.1B. *The function G_M^i .*

FIG. E.2A. *Local power spectral density estimate with minimal overlap.* FIG. E.2B. *Local power spectral density estimate with extended overlap.*

 FIG. E.3. *Effect of ε on local power spectral density.*

 FIG. E.4. *Frequency plot of local power spectral density.*

FIG. E.5. The exponential law for the local power spectral density.

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