Adaptive coherent interferometric imaging for sensor networks

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(Dated: December 25, 2007)

A multi-step adaptive algorithm for imaging with distributed sensor networks is presented. The sensors record the impulse response of the unknown background medium with and without the scatterers to be imaged. The signal difference is used to image the scatterers with a three step algorithm. First, the decoherence frequency is estimated using adaptive coherent interferometry (CINT)\textsuperscript{1}, with uniform illumination. CINT is a smoothed and stable form of travel-time migration, and performs well in an unknown complex background in which the difference traces have a lot of delay spread. CINT depends on the decoherence frequency $\Omega_d$ of the data, the reciprocal of the delay spread. It is estimated by optimizing the quality of the images, as measured with the bounded variation (BV) norm. Second, the CINT images are optimized with a selective subspace algorithm based on the singular value decomposition, as is done in Ref.1 for arrays. Third, the strength of the illumination by the sensors is optimized. In array imaging, optimal illumination has a relatively small effect on CINT images. It is seen here that optimal illumination is essential in distributed sensor imaging. The algorithm is explored with numerical simulation in 2D, in a region with many background scatterers.

I. INTRODUCTION

Localization and imaging with distributed sensors is of considerable interest in many different applications, including wireless sensor networks\textsuperscript{2} and structural health monitoring\textsuperscript{3}. In the microwave regime of wireless communications, positioning and navigation from channel state information at fixed or mobile stations is important in complex, indoor environments and elsewhere where GPS is not available or not provided by un-cooperative users. In structural health monitoring with ultrasound the aim is to be able to detect and locate, and if possible, image, defects in structures from information gathered by embedded active sensors. Most location methods used in wireless involve some form of arrival time estimation\textsuperscript{4}. Arrival time analysis and triangulation is also used in structural health monitoring\textsuperscript{5}, but because of the dispersive nature of the propagation environment migration methods are also used\textsuperscript{6}.

In this paper we formulate a general, distributed sensor imaging problem that includes these two applications, and present a three-step adaptive algorithm for locating and imaging small objects in a complex environment. We consider here the more involved case of active sensors and passive objects that are to be imaged. The algorithm can also be applied to the simpler case of active source imaging by passive sensors. A key feature of this imaging algorithm is the use of adaptive interferometry\textsuperscript{7}, which makes it robust to multiple scattering effects in an unknown, complex background. A time-reversal algorithm for distributed sensor imaging when the background is known is considered in Ref.8.

The sensors record an approximate impulse response of the background medium with and without the scattering objects to be imaged. The difference of the recorded time signals is used to image the scatters. The algorithm is as follows. First we estimate the decoherence frequency of the data using adaptive coherent interferometry (CINT)\textsuperscript{7}, with uniform illumination. CINT is a smoothed and stabilized form of travel-time migration, and performs well in an unknown complex background, in which the difference traces have a lot of delay spread. The CINT images depend on the decoherence frequency $\Omega_d$ of the data, which is the reciprocal of the delay spread, is not known, and is characteristic of scattering complexity of the background. It is estimated by optimizing the quality of the images measured with the bounded variation (BV) norm. Second, we enhance the CINT images with an optimizing selective subspace algorithm that is based on the singular value decomposition, as was done in Ref.1 for arrays. Third, we enhance the images further by optimizing with respect to the strength of the illumination by the sensors. The way in which optimizing subspace selection and optimizing illumination must be iterated, in order to get a clearer image of each scatterer, is considered in detail in this paper.

In array imaging, optimal illumination has a relatively small effect on CINT images. We see here that optimal illumination is an essential feature of distributed sensor imaging. We carry out numerical experiments, in 2D, in a region with many background scatterers. We image point-like scatterers using several isotropic sensors.

Other than arrival time analysis and triangulation, the most commonly used algorithm for imaging, especially in array imaging, is with the Kirchhoff or travel time migra-
tion (KM) functional and its variants\textsuperscript{9–11}. This only requires knowledge of the background speed of propagation in order to compute travel times. However, it does not perform well with distributed sensors in a complex, unknown, background when an average background speed is used. The recorded traces have a lot of delay spread, which introduces dense speckles in the images. Coherent interferometric imaging was introduced in Ref.\textsuperscript{12–14} in order to deal with this issue in Kirchhoff migration. CINT for array imaging is a smoothed and stabilized version of KM because it migrates the local space-time, or equivalently space-frequency, cross correlations of the traces, rather than the traces themselves. In distributed sensor imaging, computing local space correlations may or may not be important, depending on the overall setup, but in this study all sensor cross correlations are used. Correlations is the frequency domain are important only when frequencies are less than $\Omega_d$ apart. This decoherence frequency is not known and is estimated adaptively by optimizing the quality of the image using the bounded variation norm\textsuperscript{7}.

The singular value decomposition of the response matrix, at each frequency, is used extensively in order to image selectively well separated scatterers\textsuperscript{15,16}. It was introduced as the DORT (decomposition of operators of reversal of time) method in physical time reversal\textsuperscript{17–19}, and is based on the observation that scattering from well separated scatterers can be analyzed one at a time using the singular vectors. However, this one-to-one correspondence between scatterers and singular vectors will not work when, for example, the scatterers are close to each other, or when the background is complex and the scatterers are then effectively close to each other. In the selective subspace imaging algorithm a convex linear combination of the leading singular vectors is chosen to optimally image each scatterer\textsuperscript{1}. The optimization algorithm uses the bounded variation norm of the image that is being formed.

With optimal illumination we assign a set of weights to each source in order to enhance the image of each scatterer obtained with subspace selection. The optimal weights are also computed by minimizing the BV norm of the image. Optimal illumination is an important part of distributed sensor imaging because the resolution depends sensitively on the way the scatterers are illuminated.

The paper is organized as follows. In the next section we present the numerical setup that is used. The CINT and KM imaging algorithm are described in section III. In subsection III.A we give a brief review of Kirchhoff migration. In subsection III.B we present the adaptive CINT algorithm and compare it to Kirchhoff migration. The optimal subspace selection and illumination algorithms are described in section IV. The implementation and performance of the algorithms is illustrated and discussed in section IV.C. In section V we give a brief summary and conclusions. In the appendix A we explain in a simple setup why optimal illumination is important in distributed sensor imaging.

\section{II. Setup for Simulations}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(color online) The sensor configuration (blue crosses and small figure on right), background scatterers (brown), and scatterers to be imaged (green). The 2D region in which the simulations are carried out is a square $50\lambda$ by $50\lambda$ where $\lambda$ is the central wavelength of the probing pulse. The background has a large scatterer of size $3\lambda$ on the top right, 25 small scatterers of size $\lambda/2$ on a a vertical line, and 20 small scatterers of size $\lambda/4$ that are randomly distributed, all shown in brown color. All dimensions are given in units of $\lambda$. The computational region is surrounded by a perfectly matched layer (PML), shown as a pink border, so as not to have boundary reflections. We want to image four point-like scatterers, shown by green dots. In all images shown in this paper, the imaging function is computed in a square of size $10\lambda$ with a grid resolution of $\lambda/4$, shown with dashed blue lines.\label{fig1}}
\end{figure}

We consider the distributed sensor simulation setup shown in Fig. 1, where all dimensions are in units of the central wavelength $\lambda$ of the probing pulse sent by the sensors. It is a 2D region of size $50\lambda$ by $50\lambda$. For microwave sensors in the wireless communications regime, the central frequency is 2.4GHz so that $\lambda = 12\text{cm}$. We use an ultra-wideband signal with 130\% bandwidth. Bandwidth plays an important role in imaging in complex media\textsuperscript{20,21}. For ultrasound sensors used in structural health imaging a typical central frequency is 350KHz so that $\lambda = 1.4\text{cm}$ at a speed of 5km/sec, which is that of the lowest symmetric propagating Lamb mode in a 1mm thick aluminum plate.

In the simulations the background is not homogeneous. We have a fixed, large square scatterer of size $3\lambda$ (top right), and 25 vertically aligned smaller scatterers of size $\lambda/2$. We have, in addition, 20 even smaller scatterers, of size $\lambda/4$, that are randomly placed. There is no intrinsic absorption but there is loss of energy due to radiation out of the region, which is surrounded by a perfectly matched layer (pink in Fig. 1) to simulate an infinite medium.

The four scatterers that we want to image are point-like and located at (8, 16), (9, 18), (12,17) and (13,15) in $\lambda$ units, shown in green. The closest scatterers are about $2\lambda$ apart from each other. All the scatterers, in the background and to be imaged, are simulated numerically as perfect reflectors with Dirichlet boundary conditions. Other boundary conditions can, of course, also be used.
For imaging we use $N = 12$ regularly distributed isotropic point sensors, shown in blue in Fig. 1. The rows of sensors are separated by $10\lambda$ and in each row the sensors are $20\lambda$ apart. The location of the sensors is denoted by $x_s$ or by $x_r$, $1 \leq s, r \leq N$ since they both send and receive signals. The probing pulse that we use is the second derivative of a Gaussian given by

$$f(t) = (2[\alpha(t - t_0)]^2 - 1) \exp(-[\alpha(t - t_0)]^2)$$

where $\alpha = \pi\nu$, $\nu$ is the frequency, and $t_0$ is a translation of the time origin. The central frequency for the ultrasound regime is 350KHz, with an approximate bandwidth $[220, 850]$KHz at −6dB, which gives an 130% relative bandwidth. The frequency band is denoted $B$ and its length, the bandwidth, by $|B|$.

The wave equation in two dimensions is solved with a numerical method based on the discretization of the mixed velocity-pressure formulation for acoustics. For the spatial discretization we use a finite element method which is compatible with mass-lumping$^{22,23}$, that is, which leads to a diagonal mass matrix so that explicit time discretization schemes can be used. For the time discretization we use an explicit second order centered finite difference scheme. In the simulations, the point-like scatterers are modeled by small squares whose side is given by the space step of the grid, which is, $\lambda/32$. An infinite medium is simulated by embedding the computational domain into a perfectly matched absorbing layer$^{24,25}$.

For the given arrangement of sensors, the response matrix of the background medium is computed in the time domain. Each sensor located at $x_r, s = 1 \ldots N$ emits a pulse into the medium and the signals are received at all sensors located at $x_s, r = 1 \ldots N$. This response matrix is denoted $P^0(t) = (P^0_{sr}(t))_{s,r=1 \ldots N}$. It is symmetric because of reciprocity. The response matrix in the same background but including the scatterers to be imaged is computed with the same configuration of sensors and is denoted by $P^d(t)$. The difference between response matrices with and without the scatterers is denoted by $P_{sr}(t) = P^d_{sr}(t) - P^0_{sr}(t), s, r = 1 \ldots N$. Henceforth we call $P(t)$ the response matrix. It contains information from the scatterers that we want to image and from multiple scattering between them. It also contains multiple scattering with the scatterers in the background.

Since the sensors in array imaging are close to each other, the direct arrivals are in the early part of the received signals. They can, therefore, be removed by cutting off that part of the signal, since the region to be imaged is at some distance from the array. In distributed sensor imaging, however, the direct arrivals cannot be removed by a simple cut-off. This is one reason why knowledge of the response of the background is needed when imaging with distributed sensors; it provides information about it.

The sixth column of each of the matrices $P^0(t), P^d(t)$ and $P(t)$ is shown from left to right in Fig. 2. These are the signals recorded at all 12 sensors when sensor #6 is firing. All signals shown are normalized so that their maximum is one. We do not consider signal-to-noise ratio issues here, assuming that SNR is very high or infinite. The amplitude of the signals coming from the scatterers to be imaged (in $P(t)$) is approximately 100 times smaller than that of the direct arrivals (in $P^0(t)$ and $P^d(t)$). The difference signals are due to the scatterers to be imaged and it is clear that the direct arrivals have been taken out. Because of multiple scattering, however, there is also no clear arrival time associated with the scatterers, and there is considerable delay spread on the right in Fig. 2.

In Fig. 3 we show signals from the same configuration as in Fig. 2 but now without the 20 randomly placed small scatterers in the background (brown dots in Fig. 1). The difference traces look similar in structure. However, the delay spread of the ones with the random scatterers, in Fig. 2, is somewhat longer, as expected. We only show imaging results when the small random scatterers are included in the background because there is not much difference with the optimal CINT images when they are not included.

Because we are dealing with cross correlation of the data, the imaging functionals we consider in this paper are conveniently formulated in the frequency domain. We denote by $\hat{P}_{sr}(\omega)$ the Fourier transform of the response matrix. It is computed from $P_{sr}(t)$ with the discrete Fourier transform.
III. Imaging Algorithms

A. Kirchhoff Migration

Kirchhoff migration is the simplest way to image with distributed sensors, without having to estimate arrival times and triangulate. It is used extensively in seismic array imaging. It uses travel times and requires only knowledge of the smooth (or piecewise smooth) part of the propagation velocity of the background, which is assumed constant here. Given the data \( \hat{P}_{sr}(\omega) \) and a background propagation speed \( c_0 \), we compute at each search point \( y^S \) in the region we want to image the functional

\[
T_{KM}(y^S) = \int d\omega \sum_s \sum_r \hat{P}_{sr}(\omega)e^{-i\omega(\tau_s(y^S) + \tau_r(y^S))},
\]

(2)

Here \( \tau_s(y^S) = |x_s - y^S|/c_0 \) is the travel time from \( x_s \) to \( y^S \) and \( \tau_r(y^S) \) is defined similarly.

The imaging functional \( T_{KM}(y^S) \) can be viewed as a correlation between the data and the model of the data given by single scattering (Born approximation) from an isotropic point-like scatterer located at \( y^S \).

In the time domain the KM functional has the form

\[
T_{KM}(y^S) = \sum_s \sum_r P_{sr}(\tau_s(y^S) + \tau_r(y^S)).
\]

(3)

If \( y^S \) is near a scatterer then the trace \( P_{sr}(t) \) will have a peak at \( t = \tau_s(y^S) + \tau_r(y^S) \). The imaging functional \( T_{KM}(y^S) \) sums coherently these peak values for the different sources and receivers, producing a local maximum or minimum. If \( y^S \) is far from a scatterer then the traces will be added incoherently and \( |T_{KM}(y^S)| \) will be small.

For the setup of Fig. 1, the image obtained with Kirchhoff Migration functional in a \( 10 \times 10A \) window about the scatterers that we want to image is shown on the right in Fig. 4. It finds roughly the location of the scatterers that we want to image is shown on the right.

B. Coherent Interferometric with distributed sensors

The Coherent Interferometric imaging (CINT) functional was introduced in28 for array imaging. The idea is to migrate cross correlations of the traces, rather that the traces themselves, while exploiting both their spatial and frequency coherence to localize properly the computation of correlations. In addition to the search point \( y^S \) and the background speed \( c_0 \), CINT depends on two additional parameters: the decoherence length \( X_d \) and the decoherence frequency \( \Omega_d \). The traces are cross correlated in space and frequency with thresholding using these parameters. For the simulations carried out here, we do not have to take into consideration spatial decoherence because it is weak. Therefore we use the simpler CINT following functional

\[
\hat{T}_{CINT}(y^S; \Omega_d) = \int d\omega d\omega' \sum_{r, r'} \hat{Q}_{sr}(\omega; y^S)\overline{\hat{Q}_{sr'}(\omega'; y^S)}, \quad (4)
\]

where \( \hat{Q}_{sr}(\omega; y^S) \) is the frequency domain data migrated to the search point \( y^S \):

\[
\hat{Q}_{sr}(\omega; y^S) = \hat{P}_{sr}(\omega)e^{-i\omega(\tau_s(y^S) + \tau_r(y^S))}.
\]

(5)

The decoherence frequency is in the interval \( \Omega_d \in (0, |B|) \) and the restriction on the difference frequency smooths the image. When \( \Omega_d = |B| \), there is no smoothing and CINT reduces to the square of the KM functional. When \( \Omega_d = 0 \), then CINT, suitably interpreted, reduces to matched field imaging, which in the present context can be considered as incoherent interferometric imaging.

To order to image with CINT we must first estimate the decoherence frequency \( \Omega_d \). This can be done adaptively7 by minimizing the bounded variation norm of the image produced with trial values of \( \Omega_d \).

We compute

\[
\Omega_d^* = \arg\min_{\Omega_d} \left( \|\hat{T}_{CINT}(\cdot; \Omega_d)\|_{BV} \right)
\]

(6)

where the BV norm of a function \( u(x) \) over the imaging domain \( D \) is

\[
\|u\|_{BV} = \int_D (|u(x)| + \alpha |\nabla u(x)|)dx.
\]

(7)

\[
\hat{T}_{CINT}(\cdot; \Omega_d) = \max_{y^S \in D} \left( \frac{\hat{T}_{CINT}(y^S; \Omega_d)}{\hat{T}_{CINT}(y^S; g)} \right).
\]

(8)

To avoid degeneracies, the image must be normalized so that its maximum is one. The parameter \( \alpha \) in Eq. (7) is chosen so that the two integral terms in the BV norm balance. In our simulations \( \alpha \) equal to one works well.

The total variation norm (TV), which is the integral of the gradient in Eq. (7), is widely used in image denoising because it tends to preserve large scale features while minimizing small scale speckles of the image. In the BV norm the smoothing is limited by the \( L^1 \) norm of the image, which tends to reduce its support and therefore the blurring.

We compute the CINT image for all decoherence frequencies in the bandwidth \( -B, B \), plot its BV norm versus \( \Omega_d \), and then pick the value at which it is minimum. The BV norm versus decoherence frequency \( \Omega_d \) is shown in Fig. 5. A clear minimum appears near \( \Omega_d^* = 31\% \cdot |B| \). This adaptive estimate of the decoherence frequency depends on the background and is a measure of the reciprocal of the delay spread in Fig. 2. Henceforth all CINT images that we show are computed with this estimated value \( \Omega_d^* \).

In Fig. 4 we show the square root of CINT images for 3 different values of \( \Omega_d \). On the left \( \Omega_d = 0 \) (single integral in Eq. (4)), which is matched field imaging.
is clearly not a good image because all frequency coherence of the data is lost. On the right $\Omega_d = |B|$, which is KM as noted above. In the middle is the image obtained using the optimal decoherence frequency $\Omega_d = \Omega^*_d$, computed with Eq. (6). It is better than KM because the speckles have been smoothed and the scatterers are more clearly visible. The adaptive CINT images are not only smoother. They are also more stable with respect to variations in the unknown background that have the same statistics. This can also be seen in the images obtained with the DORT method below, in Fig. 8.

**IV. ADAPTIVE IMAGING**

**A. The DORT method**

The singular value decomposition (SVD) of the response matrix at each frequency can often be used to image each small scatterer separately. This is the DORT method (French acronym for decomposition of operator of time reversal). If there are $M$ point scatterers, located at position $y^*_k$, $k = 1 \ldots M$, then the response matrix at frequency $\omega$ is given by (see e.g.\(^{10}\)):

$$\tilde{P}^{mod}(\omega) = \hat{f}(\omega) \sum_{k=1}^{M} \xi_k(\omega) \hat{g}_k(\omega) \hat{g}_k^T(\omega),$$

where the superscript $T$ denotes transpose. Here multiple reflections between the scatterers are neglected, $\xi_k(\omega)$ is the scattering amplitude of the $k^{th}$ scatterer, $\hat{f}(\omega)$ is the Fourier Transform of the probing pulse Eq. (1). We denote by $\hat{g}_k(\omega) = \left[ \hat{G}(y^*_k, x_1, \omega), \hat{G}(y^*_k, x_2, \omega), \ldots, \hat{G}(y^*_k, x_N, \omega) \right]^T$ the illuminating vector at the sensors when a source is at $y^*_k$, with $\hat{G}(y, x_r, \omega)$ the outgoing, time harmonic Green’s function of the background medium.

The SVD of $P(\omega)$ is given by

$$\tilde{P}(\omega) = \sum_{k=1}^{N} \sigma_k(\omega) \tilde{U}_k(\omega) \tilde{V}^*_k(\omega),$$

where $\star$ denotes the hermitian transpose. The vectors

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**FIG. 4.** (color online) Square root of CINT images for 3 values of $\Omega_d$, with uniform illumination. Left: $\Omega_d = 0$, which is incoherent interferometry or matched field imaging. Center: $\Omega_d = \Omega^*_d$, which is coherent interferometry (CINT) with adaptively estimated decoherence frequency. Right: $\Omega_d = |B|$, which is Kirchhoff migration.

**FIG. 5.** (color online) Bounded Variation (BV) norm of the CINT functional versus relative decoherence frequency $\Omega_d/|B|$.

**FIG. 6.** (color online) The first 5 singular values of the response matrix $\tilde{P}(\omega)$ versus relative frequency, that is, frequency divided by the central frequency.
and KM for the subspaces corresponding to the first four singular values, from left to right. Clearly we are not in the case of well separated scatterers. The CINT images have reduced speckles and the scatterers are more clearly visible in them but they are not isolated by the subspace selection. Uniform illumination is used in both top and bottom images.

B. Optimal illumination and selective subspace algorithm

We present here the algorithm introduced in Ref. 1 that uses two different optimizations for array imaging in clutter, and then use it with distributed sensors. We first determine weights assigned to each source in order to select those which illuminate best one of the scatterers. This is an optimal illumination step in the algorithm. With this illumination fixed, we then use the singular value decomposition of the response matrix to image selectively each scatterer. This is a subspace projection step that generalizes the DORT method. We shall first describe these two optimization steps in the algorithm and then we will use them.

a. Optimal illumination We introduce normalized weights associated with each source located at \( x_s \), denoted \( (w_s)_{1 \leq s \leq N} \), which are in

\[
W = \{ w_s, s = 1 \ldots N, \text{ s.t. } w_s \geq 0, \text{ and } \sum_{s=1}^{N} w_s = 1 \}. 
\]

For a given set of weights \( w \in W \) we compute the CINT image

\[
\mathcal{I}^{\text{CINT}}(y^S; \Omega_d; (w_s)) = \int \int \sum_{|\omega - \omega'| \leq \Omega_d} w_s w_{s'} \hat{Q}_{sr}(\omega; y^S) \hat{Q}_{sr'}(\omega'; y^S), \tag{14}
\]

where \( \hat{Q}_{sr}(\omega; y^S) \) is the data migrated to the search point \( y^S \), defined in Eq. (5). We note that in Eq. (14) we can replace the response matrix by any of the projected response matrices \( \hat{P}^{(k)}(\omega) \) defined by Eq. (12), as well as by a convex linear combination of them as described below (see Eq. (16)).

We determine the weights \( w \in W \) by minimizing the objective of the normalized image

\[
\mathcal{O}(w) = \| \mathcal{I}^{\text{CINT}}(y^S; \Omega_d; (w_s)) \|_{BV} \max_{y^S \in D} \mathcal{I}^{\text{CINT}}(y^S; \Omega_d; (w_s)). \tag{15}
\]

over \( W \).

b. Selective subspace imaging. We image with filtered data that are constructed with convex linear combinations of the data projected on each singular vector, which generalizes DORT. We introduce the filtered matrix

\[
\hat{D}(\omega; \{ \hat{d}_j \}) = \sum_{j=1}^{M} \hat{d}_j(\omega) \hat{P}^{(j)}(\omega), \tag{16}
\]

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where the weights \( \{ \hat{d} \} = \{ \hat{d}_j(\omega) \} \) are normalized filters which belong to
\[
\mathcal{F} = \{ \hat{d}_j(\omega), j = 1 \ldots M, \omega \in \mathcal{B} \},
\] (17)
such that
\[
\hat{d}_j(\omega) \geq 0 \quad \text{and} \quad \sum_{j=1}^{M} \int_{\mathcal{B}} \hat{d}_j(\omega) d\omega = 1.
\] (18)
We only consider projection on the \( M \) leading singular vectors.

For a given set of filters \( \{ \hat{d} \} \in \mathcal{F} \) and weights \( (w_s) \in \mathcal{W} \), we form the CINT image
\[
I_{\text{CINT}}(y^S; \Omega_d; (w_s), \{ \hat{d} \}) = \max_{y^S \in \mathcal{D}} |I_{\text{CINT}}(y^S; \Omega_d; \{ \hat{d} \}, (w_s))|
\] (21)
over \( \mathcal{F} \). This selective subspace optimization is done with the following successive steps.

1. We first search the filter distribution in \( \mathcal{F} \) which minimizes the objective Eq. (21). Denote the minimizer by \( \{ \hat{d}^{(1)} \} \).
2. In the second step we look for filters that are approximately orthogonal to \( \{ \hat{d}^{(1)} \} \). That is, we look now for filters such that
\[
(\{ \hat{d} \}, \{ \hat{d}^{(1)} \}) = \sum_{j=1}^{M} \hat{d}_j(\omega) \hat{d}_j^{(1)}(\omega) e^{-i(\tau_s(y^S) + \tau_r(y^S))} = 0, \forall \omega \in \mathcal{B}
\] (22)
holds approximately in the following sense. At each frequency, we only keep the largest coefficient of

select only the \( k \)th projected matrix
\[
d_j(\omega) = \frac{\delta_{jk}}{|\mathcal{B}|}.
\] (20)
where \( |\mathcal{B}| \) is the bandwidth.

Let \( (w_s) \in \mathcal{W} \) be a given set of weights. In order to determine the filters \( \{ \hat{d} \} \in \mathcal{F} \) we minimize the objective

\[
O(\{ \hat{d} \}) = \frac{I_{\text{CINT}}(y^S; \Omega_d; \{ \hat{d} \}, (w_s))}{\max_{y^S \in \mathcal{D}} |I_{\text{CINT}}(y^S; \Omega_d; \{ \hat{d} \}, (w_s))|}
\] (21)
over \( \mathcal{F} \). This selective subspace optimization is done with the following successive steps.

1. We first search the filter distribution in \( \mathcal{F} \) which minimizes the objective Eq. (21). Denote the minimizer by \( \{ \hat{d}^{(1)} \} \).
2. In the second step we look for filters that are approximately orthogonal to \( \{ \hat{d}^{(1)} \} \). That is, we look now for filters such that
\[
(\{ \hat{d} \}, \{ \hat{d}^{(1)} \}) = \sum_{j=1}^{M} \hat{d}_j(\omega) \hat{d}_j^{(1)}(\omega) e^{-i(\tau_s(y^S) + \tau_r(y^S))} = 0, \forall \omega \in \mathcal{B}
\] (22)
holds approximately in the following sense. At each frequency, we only keep the largest coefficient of

We note that the DORT image associated to the \( k \)th singular vector is obtained by choosing the uniform illumination \( w_s = 1/N, s = 1 \ldots N \) and the filters which
In each subsequent step we add the new constraint
\[ \forall j = 1 \ldots M, \quad \hat{d}_j^{(1)}(\omega) = \max_j \hat{d}_j^{(1)}(\omega). \] Then we let
\[ \hat{d}_j^{(1)}(\omega) = \left[ \hat{d}_{j_0}^{(1)}(\omega) \right] \delta_{jj_0} \tag{23} \]
Now we search the filters \( \{\hat{d}\} \) which solve the constrained minimization problem
\[ \min_{\hat{d} \in \mathcal{F}; \{\hat{d}(\omega)\} = 0} \mathcal{O}(\hat{d}). \tag{24} \]
Denote \( \{\hat{d}^{(2)}\} \) the optimizer.

3. In each subsequent step we add the new constraint of approximate orthogonality \( \{\hat{d}_i, \{\hat{d}^{(2)}\}\} = 0 \), interpreted as above. The relaxed filters \( \{\hat{d}^{(2)}\} \) are constructed in the same way as \( \{\hat{d}^{(1)}\} \). The process can, in principle, be iterated up to \( M \) times.

C. Iterating between optimal illumination and subspace selection

We describe next how optimal illumination and subspace selection can be used in imaging with distributed sensors. The algorithm is a succession of the two optimization steps, described in the previous section, for the selection of \( (w_s) \) and \( \{\hat{d}\} \). Because in array imaging all the sensors are closely spaced, illumination plays a different role than with distributed sensors. It is therefore natural in array imaging to start with the selective subspace imaging algorithm, using uniform illumination, as is done in Ref.1. We determine first the convex linear combination with weights \( \{d\} \) associated to each of the \( M \) scatterers. Then optimal illumination is a way of enhancing the quality of the images.

Imaging with distributed sensors is different because good illumination is important from the beginning. This is illustrated in a simple setup in appendix A. If we start with the optimal subspace selection step and with uniform illumination, then the images tend to show only the strongest scatterers as they are seen by the (distributed) sensors. We have to start with an illumination that is adapted to the scatterers. Put in another way, we need to decide first which are the most appropriate sensors to be used as illuminators in order to image the scatterers whose location is not known, and in a background that is not known.

We therefore have to start with an optimal illumination step. The proposed algorithm is as follows.

1. Choose a singular vector \( k \) associated with a significant singular value. Determine the optimal illumination with weight distribution \( (w_s) \) when only the \( k \)th subspace is selected, as in DORT. Denote this optimal illumination by \( (w_s^{(0)}) \). This step tends to focus the illumination on the chosen scatterer. It is well illustrated in Fig. 9, top two figures, where the first singular vector is chosen. On the top right we show the optimal illumination and on the top left the corresponding CINT image. The optimal illumination is biased towards the sensors on the left and is very different from uniform illumination. The image picks roughly the top left scatterer (see also Fig. 1). Similarly, in Fig. 11, top two figures, the fourth singular vector is chosen. On the top right we show the optimal illumination and on the top left the corresponding CINT image. The opti-
FIG. 10. (color online) Top: Image (top left) with adaptive CINT and optimal subspace selection that is approximately orthogonal to the one in Fig. 9, middle right, and illumination as in Fig. 9, top right. The optimal subspace weights \( \{ \hat{d}^{(2)}(\omega) \} \) as functions of the frequency are shown in the middle right (first singular vector weight is blue, second is green, third is red, and forth is cyan). Bottom: Image (bottom left) as in the top left but with optimal illumination obtained after the optimal subspace selection. The new optimal illumination weights, \( (w_s^{(2)}) \), are shown in the bottom right. They differ from the the top right illumination in Fig. 9. The bottom left image picks the bottom left scatterer while the bottom left image in Fig. 9 picks the top left scatterer.

FIG. 11. (color online) Same as Fig. 9 except that the illumination at the top right is the optimal one obtained using only the subspace corresponding to the fourth singular vector. The bottom left image, obtained with optimal subspace weights shown in the middle right and with optimal illumination shown in the bottom right, picks essentially the bottom right scatterer.

2. Using the optimal illumination \( (w_s^{(0)}) \), apply the optimal subspace selection steps to determine the optimal weights \( \hat{d}^{(0)} \). In Fig. 9, middle right, we show the optimal weights after the first subspace selection, as functions of frequency. In the middle left we show the corresponding CINT image. It is clearly much better focused on the top left scatterer that we have chosen. In Fig. 10, top, we show the results of the second subspace selection. The weights on the top right are quite different from the ones in Fig. 9, middle right, as they must be approximately orthogonal. The image in Fig. 10, top left, picks now the bottom left scatterer. Similarly with Fig. 11, middle row, and Fig. 12, top row, where the algorithm starts with the fourth singular vector. The image in Fig. 11, middle left, after the first subspace selection picks the bottom right scatterer. After the second subspace selection, Fig. 12, top left, it picks the top right scatterer.

3. We now we re-optimize the illumination after fixing a subspace selection in the previous step. The new illuminations are not very different from the previous ones, in each case, but they do clean up the image. This is seen in Fig. 9, bottom row, and Fig. 10, bottom row, when the algorithm starts with the first singular vector. When it starts with the fourth singular vector, the corresponding results are shown in Fig. 11, bottom row, and Fig. 12, bottom row.

We have shown results from our numerical simulations when the proposed algorithm is used starting with the first and fourth singular vectors, and doing two succes-
sive subspace selections in each. Doing further optimal subspace selection does not improve the images. The proposed algorithm can, of course, be used starting with the second and third singular vectors. The results are quite similar and are therefore not shown.

Computations are done with Matlab, using the fmincon routine. The filters $d(\omega)$ are discretized with piecewise constant functions over the bandwidth using $N_\omega = 7$ regular intervals. Since there are $M = 4$ leading singular vectors, the filters have 28 degrees of freedom.

V. SUMMARY AND CONCLUSION

This paper is a companion to Ref.1. Here we formulate and explore with numerical simulations distributed sensor imaging, instead of array imaging, using adaptive coherent interferometry (CINT), optimal illumination, and optimal subspace selection. CINT produces smoothed and stable images, which are of considerably better quality that Kirchhoff migration (KM) images. Similarly, optimal subspace selection works much better than single subspace selection (as in DORT). The main result of this paper is the identification of optimal illumination as an essential step in the imaging algorithm with distributed sensors.

The numerical setup that we used in this paper simulates better structural health monitoring in the ultrasonic regime than localizing and imaging with microwaves in the wireless communications regime. Realistic simulation in the microwave regime are much more computationally demanding. Another issue that needs to be addressed is imaging with distributed sensors that have directional properties. One possibility is to use small, distributed clusters of sensors and to incorporate their directivity properties into the imaging algorithms proposed in this paper.

Acknowledgments

The work of G. Deraveaux and G. Papanicolaou was partially supported by US Army grant W911NF-07-2-0027-1, ONR grant N00014-02-1-0088, and DARPA/ARO grant 02-SC-ARO-1067.

APPENDIX A: RESOLUTION-CONTROLLED ILLUMINATION WITH DISTRIBUTED AND ARRAY SENSORS

In this appendix we illustrate with a simple example why good illumination is an important issue in distributed sensor imaging.

We recall briefly the geometrical interpretation of the Kirchhoff migration functional in a homogeneous background. If there is only one illuminating source located at $x_s$, in the time domain this imaging functional is

$$T^{KM}(y^S) = \sum_r P_r(\tau_r(y^S) + \tau_r(y^S)),$$  \hspace{1cm} (A1)

where $P_r(t)$ is the time trace recorded at receiver $x_r$, $r = 1 \ldots N$ and $\tau_r(y^S)$ is the travel time from point $x_s$ to point $y^S$, as in section III.A. We assume that the probing pulse is very narrow in time and denote by $T$ its width.

In a homogeneous background, a single point-like scatterer located at $y^*$ will act as a secondary source when it encounters an incident wave. This happens at time $\tau_r(y^*)$. The trace recorded at the receiver located at $x_r$ will have a peak at time $\tau_r(y^*) + \tau_r(y^*)$. Consider a source-receiver pair in the sum Eq. (A1). The peaks will contribute to it when the search points $y^S$ are at the same distance from the pair $(x_s, x_r)$ as $y^*$. That is, all points $y^S$ such that

$$|y^S - x_s| + |y^S - x_r| = |y^* - x_s| + |y^* - x_r|$$ \hspace{1cm} (A2)

contribute significantly in Eq. (A1). The locus of points $y^S$ that satisfy Eq. (A2) is an ellipse whose foci are at $x_s$ and $x_r$ and goes through $y^*$ (see Fig. 13).

The point $y^S$ where the maximum is achieved for the sum over receivers in Eq. (A1) can be interpreted geometrically as the intersection of all ellipses whose foci are at

![FIG. 12. (color online) This figure corresponds to Fig. 11 in the same way as Fig. 10 corresponds to Fig. 9. Top: Image (top left) with adaptive CINT and optimal subspace selection that is approximately orthogonal to the one in Fig. 11, middle right, and illumination as in Fig. 11, top right. The optimal subspace weights as functions of the frequency are shown in the middle right (first singular vector weight is blue, second is green, third is red, and forth is cyan). Bottom: Image (bottom left) as in the top left but with optimal illumination obtained after the optimal subspace selection. The new optimal illumination weights are shown in the bottom right. They differ from the the top right illumination in Fig. 11. The bottom left image picks the top right scatterer while the bottom right image in Fig. 11 picks the bottom right scatterer.](image-url)
FIG. 13. (color online) Locus of the search points $y^S$ that are at the same distance to the source-receiver pair as the scatterer located at $y^*$. It is an ellipse whose foci are at $x_s$ and $x_r$, a fixed $x_s$ and at $x_r$, $r = 1 \ldots N$, and passing through $y^*$. This is why migration is simply related to triangulation. We distinguish two different situations as follows.

- If $x_s$ and $x_r$ are on the same side of the scatterer $y^*$ (acute angle between $(x_s, y^*, x_r)$), which is typically the case in array imaging, then the ellipse has small eccentricity. When $x_s = x_r$, as in synthetic aperture imaging, the ellipse is a circle. If the receivers are very close to each other and the array is narrow, then the circles will be almost tangent and the cross range resolution is not good. If the receivers are far apart, then the circles will intersect sharply and the cross range resolution will be good. Range resolution is controlled by the width of the circles, which is the travel time resolution times the speed of propagation, $c_0 T$.

- If the scatterer $y^*$ is between $x_s$ and $x_r$ (obtuse angle between $(x_s, y^*, x_r)$ with the worst case being when the three points are aligned), then the ellipse tends to be flat. If we intersect only ellipses of this kind, then they will have in common all the points that are between $x_s$ and $y^*$. Such a configuration of sources and receivers gives a bad image.

To illustrate these two different situations, we show in Fig. 14 the optimal illumination weights and the associated KM images computed in three simple cases. We want to image a point-like scatterer located at $(0, 10)$ shown by the green dot in the figures. We use three sensors that are placed in different locations in each case considered. They are shown with blue crosses. The sensor traces are computed analytically in the frequency domain, using the Born approximation. Optimal illumination weights are then computed following the algorithm described in section IV. The optimal weights at each sensor are shown with red circles whose radius is proportional to the optimal weight.

1. At the top in Fig. 14 the three sensors are aligned, $\lambda/2$ apart, which constitutes a small array of size $\lambda$. In order to get as much cross range resolution as possible the optimal weights algorithm takes out the sensor at the center. This agrees with the optimal illumination theory for arrays\(^3\).

2. In the middle in Fig. 14 the three sensors are aligned, but the outer sensors are now $6\lambda$ apart. The third one is close to one of them, at $\lambda/2$. This time, the optimal weight algorithm keeps them all. Note the improvement in cross range resolution.

3. At the bottom in Fig. 14 the sensors are on a relatively flat triangle with the scatterer inside. This is a bad situation because the sensor which is at the tip of the triangle will produce flat ellipses. The optimal illumination algorithm removes this sensor.

FIG. 14. (color online) Illustration of optimal illumination with three different configurations of sensors.

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