Transport Equations for Elastic and Other Waves in Random Media

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May 24, 1996

Abstract

We derive and analyze transport equations for the energy density of waves of any kind in a random medium. The equations take account of nonuniformities of the background medium, scattering by random inhomogeneities, polarization effects, coupling of different types of waves, etc. We also show that diffusive behavior occurs on long time and distance scales and we determine the diffusion coefficients. The results are specialized to acoustic, electromagnetic, and elastic waves. The analysis is based on the governing equations of motion and uses the Wigner distribution.

Contents

1 Introduction and Summary

1.1 Radiative Transport Equations

The theory of radiative transport was originally developed to describe how light energy propagates through a turbulent atmosphere. It is based upon a linear transport equation for the angularly resolved energy density and was first derived phenomenologically at the beginning of this century [1,2]. We shall show how this theory can be derived from the governing equations for light and for other waves of any type, in a randomly inhomogeneous medium. Our results take into account nonuniformity of the background medium, scattering by random inhomogeneities, the effect of polarization, the coupling of different types of waves, etc. The main new application is to elastic waves, in which shear waves exhibit polarization effects while the compressional waves do not, and the two types of waves are coupled. We also analyze solutions of the transport equations at long times and long distances and show that they have diffusive behavior.

Transport equations arise because a wave with wave vector $\mathbf{k}'$ at a point $\mathbf{x}$ in a randomly inhomogeneous medium can be scattered into any direction $\hat{\mathbf{k}}$ with wave vector $\mathbf{k}$, where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Therefore one must consider the angularly resolved, wave vector dependent, scalar energy density $a(t,\mathbf{x},\mathbf{k})$ defined for all $\mathbf{k}$ at each point $\mathbf{x}$ and time t. For scalar waves, energy conservation is expressed by the transport equation

$$\begin{aligned}\frac{\partial a(t,\mathbf{x},\mathbf{k})}{\partial t} &+ \nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}a(t,\mathbf{x},\mathbf{k}) - \nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}a(t,\mathbf{x},\mathbf{k}) \\ &= \int_{R^3}\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')a(t,\mathbf{x},\mathbf{k}')d\mathbf{k}' - \Sigma(\mathbf{x},\mathbf{k})a(t,\mathbf{x},\mathbf{k}).\end{aligned} \tag{1.1}$$

Here $\omega(\mathbf{x},\mathbf{k})$ is the frequency at $\mathbf{x}$ of the wave with wave vector $\mathbf{k}$, $\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')$ is the differential scattering cross-section -the rate at which energy with wave vector $\mathbf{k}'$ is converted to wave energy with wave vector $\mathbf{k}$ at position $\mathbf{x}$- and

$$\int\sigma(\mathbf{x},\mathbf{k}',\mathbf{k})d\mathbf{k}' = \Sigma(\mathbf{x},\mathbf{k}) \tag{1.2}$$

is the total scattering cross-section. Both σ and Σ are nonnegative and σ is usually symmetric in $\mathbf{k}$ and $\mathbf{k}'$. For acoustic waves $\omega(\mathbf{x},\mathbf{k}) = v(\mathbf{x})|\mathbf{k}|$, with v the sound speed (3.36), and the differential scattering cross-section is given by

$$\begin{aligned}\sigma(\mathbf{x},\mathbf{k},\mathbf{k}') = \frac{\pi v^2(\mathbf{x})|\mathbf{k}|^2}{2}&\{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \\ &\hat{R}_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\}\cdot\delta(v(\mathbf{x})|\mathbf{k}| - v(\mathbf{x})|\mathbf{k}'|).\end{aligned} \tag{1.3}$$

Here $\hat{R}_{\rho\rho}$, $\hat{R}_{\rho\kappa}$ and $\hat{R}_{\kappa\kappa}$ are the power spectra of the fluctuations of the density ρ and compressibility κ defined by (4.3) and (4.37). The left side of (1.1) is the total time derivative of $a(t,\mathbf{x},\mathbf{k})$ at a point moving along a ray in phase space $(\mathbf{x},\mathbf{k})$, with the frequency adjusting to the appropriate local value. The right side of (1.1) represents the effects of scattering.

The transport equation (1.1) is conservative when (1.2) holds because then

$$\int\int a(t,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} = \text{const}$$

independent of time. For simplicity we will assume that there is no intrinsic attenuation. However, it is accounted for easily by letting the total scattering cross-section be the sum of two terms

$$\Sigma(\mathbf{x},\mathbf{k}) = \Sigma_{sc}(\mathbf{x},\mathbf{k}) + \Sigma_{ab}(\mathbf{x},\mathbf{k})$$

where $\Sigma_{sc}(\mathbf{x}, \mathbf{k})$ is the total scattering cross-section given by (1.2) and $\Sigma_{ab}(\mathbf{x}, \mathbf{k})$ is the intrinsic attenuation rate.

The reason that the power spectral densities of the inhomogeneities determine the scattering cross-section (1.3) is seen most easily from a Born expansion of the wave solution for weak inhomogeneities. The single scattering approximate solution of (1.1) and the second moments of the single scattering approximate solution for the underlying wave equation must be the same. The latter are determined by the power spectra of the inhomogeneities. The same considerations explain the appearance of the delta function in the scattering cross-section (1.3) when the random inhomogeneities do not depend on time, for then the frequency is unchanged by scattering. The transport equation (1.1) arises also when the waves are scattered by discrete scatterers that are randomly distributed in the medium. In this case the scattering cross-section (1.3) is the same as the cross-section of a single scatterer times the density of scatterers. We will deal only with continuous random media.

Equation (1.1) has been derived from equations governing particular wave motions by various authors, such as Stott [3], Watson et.al. [4], [5], [6], [7], Barabanenkov et.al. [8], Besieris and Tappert [9], Howe [10], Ishimaru [11] and Besieris et. al. [12] with a recent survey presented in [13]. These derivations also determine the functions $\omega(\mathbf{x}, \mathbf{k})$ and $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ and show how a is related to the wave field. We shall derive (1.1) and these functions as a special case of our more general theory.

We expect that radiative transport equations will provide a good description of wave energy transport when (i) typical wavelengths are short compared to macroscopic features of the medium (high frequency case), (ii) correlation lengths of the inhomogeneities are comparable to wavelengths and (iii) the fluctuations of the inhomogeneities are weak. Condition (ii) is important because it allows strong interaction between the waves and the inhomogeneities, which is the most interesting and difficult case to analyze. In addition to these three conditions, the inhomogeneities must not be too anisotropic because in layered random media wave localization occurs even with weak fluctuations, instead of transport [14]. When the fluctuations are strong, wave localization can occur even when the inhomogeneities are isotropic [15], [16].

We shall also analyze the diffusive behavior of solutions of (1.1) which emerges at times and distances that are long compared to a typical transport mean free time $1/\Sigma$ and a typical transport mean free path $|\nabla_{\mathbf{k}}\omega|/\Sigma$. In this regime the phase space energy density $a(t, \mathbf{x}, \mathbf{k})$ is approximately independent of the direction of the wave vector $\mathbf{k}$, $a(t, \mathbf{x}, \mathbf{k}) \sim \bar{a}(t, \mathbf{x}, |\mathbf{k}|)$. In the simplest, spatially homogeneous case, $\bar{a}$ satisfies the diffusion equation

$$\frac{\partial \bar{a}}{\partial t} = \nabla_{\mathbf{x}} \cdot (D \nabla_{\mathbf{x}} \bar{a}) \tag{1.4}$$

with a constant diffusion coefficient $D = D(|\mathbf{k}|)$, (5.13, 5.14), determined by the differential scattering cross-section σ. Diffusion approximations for scalar transport equations are well known [17], including their behavior near boundaries [18], [19]. Our results show that diffusion approximations are also valid for the more general transport equations that arise for electromagnetic and elastic waves.

1.2 Transport Theory for Electromagnetic Waves

To describe electromagnetic waves in isotropic media we must know their state of polarization. Therefore the radiative transport theory of electromagnetic waves must account for energy transport in different states of polarization. Such transport equations were first proposed by Chandrasekhar [1]. They are a coupled system of transport equations for the Stokes parameters I, Q, U, V as functions of time, position and wave number [20]. The Stokes vector is related to the coherence matrix $W(t, \mathbf{x}, \mathbf{k})$ by

$$W(t, \mathbf{x}, \mathbf{k}) = \frac{1}{2} \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}. \tag{1.5}$$

In terms of W, which is Hermitian and positive definite, Chandrasekhar's transport equation is

$$\begin{aligned} &\frac{\partial W}{\partial t} + \nabla_{\mathbf{k}} \omega(\mathbf{x}, \mathbf{k}) \cdot \nabla_{\mathbf{x}} W - \nabla_{\mathbf{x}} \omega(\mathbf{x}, \mathbf{k}) \cdot \nabla_{\mathbf{k}} W + W N(\mathbf{x}, \mathbf{k}) - N(\mathbf{x}, \mathbf{k}) W \\ &\quad = \int_{R^3} \sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')[W(t, \mathbf{x}, \mathbf{k}')] d\mathbf{k}' - \Sigma(\mathbf{x}, \mathbf{k}) W(t, \mathbf{x}, \mathbf{k}). \end{aligned} \tag{1.6}$$

Here $\omega(\mathbf{x}, \mathbf{k}) = v(\mathbf{x})|\mathbf{k}|$ is the local frequency and $v(\mathbf{x}) = (\epsilon(\mathbf{x})\mu(\mathbf{x}))^{-1/2}$ is the local speed of light. The differential scattering cross-section $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is a tensor. In the simplest case of isotropic random inhomogeneities, without fluctuations in the magnetic permeability μ, it has the form

$$\begin{aligned} \sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')[W(t, \mathbf{x}, \mathbf{k}')] = &\frac{\pi v^2(\mathbf{x})|\mathbf{k}|^2 \hat{R}_{\epsilon\epsilon}(|\mathbf{k} - \mathbf{k}'|)}{2} T(\mathbf{k}, \mathbf{k}') W(t, \mathbf{x}, \mathbf{k}') T(\mathbf{k}', \mathbf{k}) \\ &\cdot \delta(v(\mathbf{x})|\mathbf{k}| - v(\mathbf{x})|\mathbf{k}'|). \end{aligned} \tag{1.7}$$

Here $\hat{R}_{\varepsilon\varepsilon}(\mathbf{k})$ is the power spectrum of the dimensionless fluctuations of the relative dielectric permittivity. The total scattering cross-section $\Sigma(\mathbf{x}, \mathbf{k})$ is given by

$$\Sigma(\mathbf{x}, \mathbf{k}) = \frac{\pi^2 |\mathbf{k}|^4 v(\mathbf{x})}{2} \int_{-1}^{1} \hat{R}_{\epsilon\epsilon}(|\mathbf{k}|\sqrt{2 - 2\eta})(1 + \eta^2) d\eta. \tag{1.8}$$

The differential scattering cross-section σ and the total scattering cross-section Σ are related by the matrix analog of (1.2)

$$\int_{R^3} \sigma(\mathbf{k}', \mathbf{k})[I] d\mathbf{k}' = \Sigma(\mathbf{k}) I, \tag{1.9}$$

where I is 2×2 identity matrix.

To define T and N, which occur in (1.7) and (1.6), respectively, we let $(\hat{\mathbf{k}}, \mathbf{z}^{(1)}(\mathbf{k}), \mathbf{z}^{(2)}(\mathbf{k}))$ be the orthonormal propagation triple consisting of the direction of propagation $\hat{\mathbf{k}}$ and two transverse unit vectors $\mathbf{z}^{(1)}(\mathbf{k}), \mathbf{z}^{(2)}(\mathbf{k})$. In polar coordinates they are

$$\hat{\mathbf{k}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \ \mathbf{z}^{(1)}(\mathbf{k}) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \ \mathbf{z}^{(2)}(\mathbf{k}) = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{1.10}$$

Then the 2×2 matrix T is given by

$$T_{ij}(\mathbf{k}, \mathbf{k}') = \mathbf{z}^{(i)}(\mathbf{k}) \cdot \mathbf{z}^{(j)}(\mathbf{k}') \tag{1.11}$$

and in polar coordinates it has the form

$$T(\mathbf{k}, \mathbf{k}') = \begin{pmatrix} \cos\theta\cos\theta'\cos(\phi-\phi') + \sin\theta\sin\theta' & \cos\theta\sin(\phi-\phi') \\ \cos\theta'\sin(\phi'-\phi) & \cos(\phi-\phi') \end{pmatrix}.$$

The coupling matrix N is given by

$$N(\mathbf{x}, \mathbf{k}) = \sum_{i=1}^{3} \frac{\partial v(\mathbf{x})}{\partial x^i} |\mathbf{k}| \mathbf{z}^{(1)}(\mathbf{k}) \cdot \frac{\partial \mathbf{z}^{(2)}(\mathbf{k})}{\partial k_i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.12}$$

Chandrasekhar considered a homogeneous background only, in which case the speed of light v is a constant so that $\nabla_{\mathbf{x}}\omega = 0$ and $N = 0$. Law and Watson [6] derived (1.6) in general, from Maxwell's equations in a random medium, as was done also in [21].

We will now explain the physical meaning of the matrices T and N, which do not appear in the scalar transport equation (1.1). The 2×2 matrix $T(\mathbf{k}, \mathbf{k}')$ involves the angles between the directions transverse to $\mathbf{k}'$ before the scattering and the directions transverse to $\mathbf{k}$ after the scattering. Thus T_{ij} is the fraction of wave amplitude going in the direction $\mathbf{k}'$ and polarized along the transverse direction $\mathbf{z}^{(j)}(\mathbf{k}')$ that scatters into a wave going in the direction $\mathbf{k}$ and polarized along the transverse direction $\mathbf{z}^{(i)}(\mathbf{k})$. Since the coherence matrix W is related to the mean square of the wave amplitudes (see sections 3.3 and 4.4), the transformation matrix T acts on W twice in (1.7). The coupling matrix $N(\mathbf{x}, \mathbf{k})$, defined by (1.12), arises from the slow variations of the background because the rays in inhomogeneous media are curved, and this leads to rotation of the polarization vector around the ray as the wave propagates (Lewis [22]). This rotation corresponds to parallel transport along the ray in the metric $v^{-1}(\mathbf{x})ds$ where $v(\mathbf{x})$ is the propagation speed. The coherence matrix $W(t, \mathbf{x}, \mathbf{k})$ captures this behavior of polarization for quantities quadratic in the electromagnetic field through the matrix N. In the absence of scattering, so that the right side of (1.6) is zero, the solution of (1.6) with geometrical optics initial conditions (see (2.4) and (3.58)) is the coherence matrix of Lewis' solution.

When the transport mean free path is small compared to the overall propagation distance, there is a diffusion approximation for Chandrasekhar's equation (1.6). The coherence matrix W is approximated by $\bar{\phi}(t, \mathbf{x}, |\mathbf{k}|)I$ with I the 2×2 identity matrix and $\bar{\phi}$ the solution of a diffusion equation (see section 5.2). In this approximation the coherence matrix is independent of the direction of the wave vector $\mathbf{k}$ and is completely depolarized since it is proportional to the identity. In section 5.2 we give an explicit formula (5.30) for the diffusion coefficient $D(|\mathbf{k}|)$.

1.3 Transport Theory for Elastic Waves

Radiative transport theory was used in seismology by Wesley [23], Nakamura [24], Dainty and Toksöz [25], Wu [26] and others. The stationary, scalar transport equation was used to successfully assess scattering and intrinsic attenuation (the albedo) [27], [28], [29], [30], [31], [32] and the time dependent scalar transport equation was used by Zeng, Su and Aki [33], Zeng [34] and Hoshiba [35]. In all these papers the vector nature of the underlying elastic wave motion was not taken into consideration. Mode conversion for surface waves was considered in a phenomenological way by Chen and Aki [36] and general mode conversion between longitudinal compressional or P waves and transverse shear or S waves was considered by Sato [37] and by Zeng [38]. However, the transport equations proposed phenomenologically in [37], [38] do not account for polarization of the shear waves. Starting from the elastic wave equations in a random medium we derive a system of transport equations that accounts correctly for P to S mode conversion and for polarization effects.

Longitudinal or P waves propagate with local speed $v_P(\mathbf{x}) = \sqrt{(2\mu(\mathbf{x}) + \lambda(\mathbf{x}))/\rho(\mathbf{x})}$ and transverse shear or S waves propagate with local speed $v_S(\mathbf{x}) = \sqrt{\mu(\mathbf{x})/\rho(\mathbf{x})}$. The corresponding dispersion relations are $\omega_P = v_P|\mathbf{k}|$ and $\omega_S = v_S|\mathbf{k}|$, respectively. Here λ and μ are the Lame parameters. The P and S wave modes interact in an inhomogeneous medium because a P wave with wavenumber $\mathbf{k}$ can scatter into an S wave with wavenumber $\mathbf{p}$ with the same frequency; that is, with $v_P(\mathbf{x})|\mathbf{k}| = v_S(\mathbf{x})|\mathbf{p}|$, and vice versa. Therefore the transport equations for P and S wave energy densities must be coupled. The transport equation for the P wave should be a scalar equation similar to (1.1) with an additional term that accounts for S to P conversion. Similarly, the transport equation for the S wave coherence matrix should be like Chandrasekhar's equation (1.6) with an additional term that accounts for P to S conversion. We show in section 4.5 that this is indeed the case and we determine explicitly the form of the scattering cross-sections in terms of the power spectral densities of the material inhomogeneities.

The coupled radiative transport equations for the P wave energy density $a^P(t, \mathbf{x}, \mathbf{k})$ and the 2×2 coherence matrix $W^S(t, \mathbf{x}, \mathbf{k})$ for the S waves have the forms

$$\begin{aligned} \frac{\partial a^P}{\partial t} &+ \nabla_{\mathbf{k}} \omega^P \cdot \nabla_{\mathbf{x}} a^P - \nabla_{\mathbf{x}} \omega^P \cdot \nabla_{\mathbf{k}} a^P \\ &= \int \sigma^{PP}(\mathbf{k}, \mathbf{k}') a^P(\mathbf{k}') d\mathbf{k}' - \Sigma^{PP}(\mathbf{k}) a^P(\mathbf{k}) \\ &+ \int \sigma^{PS}(\mathbf{k}, \mathbf{k}')[W^S(\mathbf{k}')] d\mathbf{k}' - \Sigma^{PS}(\mathbf{k}) a^P(\mathbf{k}) \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \frac{\partial W^S}{\partial t} &+ \nabla_{\mathbf{k}} \omega^S \cdot \nabla_{\mathbf{x}} W^S - \nabla_{\mathbf{x}} \omega^S \cdot \nabla_{\mathbf{k}} W^S + WN - NW \\ &= \int \sigma^{SS}(\mathbf{k}, \mathbf{k}')[W^S(\mathbf{k}')] d\mathbf{k}' - \Sigma^{SS}(\mathbf{k}) W^S(\mathbf{k}) \\ &+ \int \sigma^{SP}(\mathbf{k}, \mathbf{k}')[a^P(\mathbf{k}')] d\mathbf{k}' - \Sigma^{SP}(\mathbf{k}) W^S(\mathbf{k}). \end{aligned} \tag{1.14}$$

The coupling matrix N is the same as (1.12) for electromagnetic waves except that the speed v is now the shear speed $v_S(\mathbf{x}) = \sqrt{\mu(\mathbf{x})/\rho(\mathbf{x})}$. The differential scattering cross-section $\sigma^{PP}(\mathbf{k}, \mathbf{k}')$ for P to P scattering is similar to (1.3) for scattering of scalar waves and the differential scattering tensor $\sigma^{SS}(\mathbf{k}, \mathbf{k}')$ is similar to Chandrasekhar's tensor (1.7). They have the forms

$$\sigma^{PP}(\mathbf{k}, \mathbf{k}') = \sigma_{pp}(\mathbf{k}, \mathbf{k}') \delta(v_P|\mathbf{k}| - v_P|\mathbf{k}'|) \tag{1.15}$$

and

$$\begin{aligned} \sigma^{SS}(\mathbf{k}, \mathbf{k}')[W(\mathbf{k}')] = \{ \; & \sigma_{ss}^{TT} T(\mathbf{k}, \mathbf{k}') W(\mathbf{k}') T(\mathbf{k}', \mathbf{k}) + \sigma_{ss}^{\Gamma\Gamma} \, , (\mathbf{k}, \mathbf{k}') W(\mathbf{k}') \, , (\mathbf{k}', \mathbf{k}) \\ & + \sigma_{ss}^{\Gamma T} [T(\mathbf{k}, \mathbf{k}') W(\mathbf{k}') \, , (\mathbf{k}', \mathbf{k}) + \, , (\mathbf{k}, \mathbf{k}') W(\mathbf{k}') T(\mathbf{k}', \mathbf{k})] \} \\ & \cdot \; \delta(v_S|\mathbf{k}| - v_S|\mathbf{k}'|). \end{aligned} \tag{1.16}$$

The 2×2 matrix $, (\mathbf{k}, \mathbf{k}')$ is similar to T and is defined by

$$, _{ij}(\mathbf{k}, \mathbf{k}') = (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')(\mathbf{z}^{(i)}(\mathbf{k}) \cdot \mathbf{z}^{(j)}(\mathbf{k}')) + (\hat{\mathbf{k}} \cdot \mathbf{z}^{(j)}(\mathbf{k}'))(\hat{\mathbf{k}}' \cdot \mathbf{z}^{(i)}(\mathbf{k})) \tag{1.17}$$

while σ_{pp} and σ_{ss} are scalar functions given in terms of power spectral densities of the inhomogeneities by (4.54) and (4.55). The total scattering cross-sections Σ^{PP} and Σ^{SS} are the integrals of the corresponding differential scattering cross-sections, as in (1.2) and (1.9).

The scattering cross-sections for the S to P and P to S coupling terms, σ^{PS} and σ^{SP}, respectively, have the forms

$$\sigma^{PS}(\mathbf{k},\mathbf{k}')[W^S(\mathbf{k}')] = \mathrm{Tr}[\sigma_{ps}(\mathbf{k},\mathbf{k}')G(\mathbf{k},\mathbf{k}')W^S(\mathbf{k}')]\delta(v_P|\mathbf{k}| - v_S|\mathbf{k}'|) \tag{1.18}$$

$$\sigma^{SP}(\mathbf{k},\mathbf{k}')[a^P(\mathbf{k}')] = \sigma_{ps}(\mathbf{k}',\mathbf{k})G(\mathbf{k}',\mathbf{k})a^P(\mathbf{k}')\delta(v_S|\mathbf{k}| - v_P|\mathbf{k}'|) \tag{1.19}$$

with the 2×2 matrix G given by

$$G_{ij}(\mathbf{k},\mathbf{k}') = (\hat{\mathbf{k}} \cdot \mathbf{z}^{(i)}(\mathbf{k}'))(\hat{\mathbf{k}} \cdot \mathbf{z}^{(j)}(\mathbf{k}')). \tag{1.20}$$

The scalar function σ_{ps} is given explicitly in terms of power spectral densities of the inhomogeneities by (4.56). The scattering operator on the right side of (1.13) and (1.14) is symmetric in a^P, W^S and conservative. This implies in particular that

$$\Sigma^{SP}(\mathbf{k}) = \int \sigma_{ps}(\mathbf{k}',\mathbf{k})G(\mathbf{k}',\mathbf{k})\delta(v_S|\mathbf{k}| - v_P|\mathbf{k}'|)d\mathbf{k}'. \tag{1.21}$$

with

$$\Sigma^{PS}(\mathbf{k}) = \int \sigma_{ps}(\mathbf{k},\mathbf{k}')\mathrm{Tr}G(\mathbf{k},\mathbf{k}')\delta(v_S|\mathbf{k}'| - v_P|\mathbf{k}|)d\mathbf{k}'. \tag{1.22}$$

The geometrical meaning of the 2×2 matrices T, , and G that appear in the differential scattering cross-sections (1.16) and (1.18) is similar to that of T in the electromagnetic case (1.7). They arise from a single scattering event of P or S waves with wave vector $\mathbf{k}'$ that scatter to P or S waves with wave vector $\mathbf{k}$, and from the fact that the transport equations deal with quadratic field quantities. In the analysis given in sections 3.4 and 4.5 this is captured in the structure of the eigenvalues and eigenvectors of the dispersion matrix L (3.84) for the elastic wave equations.

As for the scalar transport equation (1.1) and Chandrasekhar's equation (1.6), the elastic transport equations (1.13) and (1.14) simplify considerably in the regime where the diffusion approximation is valid. This occurs when the scattering mean free path is small compared to the propagation distance. In this regime the P wave energy density $a^P(t,\mathbf{x},\mathbf{k})$ and the S wave coherence matrix $W^S(t,\mathbf{x},\mathbf{k})$ are independent of the direction of the wave vector $\mathbf{k}$. Furthermore, W^S is proportional to the identity matrix

$$a^P(t,\mathbf{x},\mathbf{k}) \sim \phi(t,\mathbf{x},|\mathbf{k}|), \quad W^S(t,\mathbf{x},\mathbf{k}) \sim w(t,\mathbf{x},|\mathbf{k}|)I \tag{1.23}$$

and the equipartition relation

$$\phi(t, \mathbf{x}, |\mathbf{k}|) = w(t, \mathbf{x}, \frac{v_P|\mathbf{k}|}{v_S}) \tag{1.24}$$

holds with ϕ satisfying the diffusion equation (1.4). The diffusion coefficient $D(|\mathbf{k}|)$ is given by (5.46).

When integrated over $\mathbf{k}$, the equipartition relation (1.24) is

$$\mathcal{E}_P(t, \mathbf{x}) = \frac{v_S^3}{2v_P^3}\mathcal{E}_S(t, \mathbf{x}) \tag{1.25}$$

where $\mathcal{E}_P$ and $\mathcal{E}_S$ are the P and S wave spatial energy densities. They are related to a^P and W^S by

$$\mathcal{E}_P(t, \mathbf{x}) = \int a^P(t, \mathbf{x}, \mathbf{k})d\mathbf{k}$$

and

$$\mathcal{E}_S(t, \mathbf{x}) = \int \mathrm{Tr}W^S(t, \mathbf{x}, \mathbf{k})d\mathbf{k},$$

respectively. From the point of view of seismological applications of transport theory, relation (1.25) is important because it predicts universal behavior of the P to S wave energy ratio in the diffusive regime. This ratio is independent of the details of the multiple scattering process and of the source distribution. When we use the typical S to P wave speed ratio of 1 to 1.7, relation (1.25) predicts $\mathcal{E}_S/\mathcal{E}_P \sim 10$. This is in general agreement with seismological data and it would be interesting to identify cases where $\mathcal{E}_S/\mathcal{E}_P$ stabilizes. This stabilization, which is derived here from first principles, is reminiscent of the important empirical observation of Hansen, Ringdal and Richards [39] regarding the stabilization of crustal waveguide mode energy ratios.

1.4 Brief Outline

In section 2, to motivate the phase space setup, we analyze the Schrödinger wave equation, which is relatively simple. The Wigner distribution is introduced and its usefulness for energy calculations is shown. The analysis of scattering in random media is given in section 2.3, with some of the details relegated to the Appendix. In section 3 we present the high frequency approximation for general symmetric hyperbolic systems and the equations of acoustic, electromagnetic and elastic waves, in particular. We do this in phase space using the Wigner distribution, and show the connection with the standard high frequency approximation. In section 4 we derive the transport equations, first for general symmetric hyperbolic systems, sections 4.1 and 4.2, and then for the equations of acoustic, electromagnetic and elastic waves in sections 4.3, 4.4 and 4.5, respectively. We rely here on the formalism explained in detail for the Schrödinger equation in section 2.3 and in the Appendix. The diffusion approximation is analyzed in detail in section 5. The energy equipartition results for elastic waves are discussed in section 5.3.

2 Radiative Transport Theory for the Schrödinger Equation

2.1 High Frequency Asymptotics

It is convenient to introduce the derivation of radiative transport theory in a simple setting, that of the Schrödinger or parabolic wave equation, before considering systems of wave equations (hyperbolic systems). This will also allow us to introduce the Wigner distribution (section 2.2) which plays an important role in the analysis.

The Schrödinger equation

$$\begin{aligned} & i\frac{\partial \phi}{\partial t} + \frac{1}{2}\Delta\phi - V(\mathbf{x})\phi = 0 \\ & \phi(0, \mathbf{x}\ ) = \phi_0(\mathbf{x}) \end{aligned} \tag{2.1}$$

arises not only in quantum mechanics but also in many other wave propagation problems. It describes, in particular, an approximate plane wave propagating primarily in one direction and can be derived from the Helmholtz equation as a paraxial approximation. In this case t is distance in the direction of propagation, $\mathbf{x}$ stands for the two-dimensional transverse coordinates and the potential is related to the index of refraction and will depend on t, in general. An important property of (2.1) is that the L_2-norm of the solution is conserved

$$\int_{R^3} |\phi(t,\mathbf{x})|^2 d\mathbf{x} = \int_{R^3} |\phi_0(\mathbf{x})|^2 d\mathbf{x}. \tag{2.2}$$

We consider high frequency asymptotics which concerns approximate solutions of (2.1) that are good approximations to oscillatory solutions. For such solutions the propagation distance is long compared to the wavelength, the propagation time is large compared to the period and the potential $V(\mathbf{x})$ varies slowly. To make this precise we introduce slow time and space variables $t \to t/\varepsilon$, $\mathbf{x} \to \mathbf{x}/\varepsilon$ and the scaled wave function $\phi^\varepsilon(t,\mathbf{x}) = \phi(t/\varepsilon, \mathbf{x}/\varepsilon)$ which satisfies the scaled Schrödinger equation

$$i\varepsilon\phi_t^\varepsilon + \frac{\varepsilon^2}{2}\Delta\phi^\varepsilon - V(\mathbf{x})\phi^\varepsilon = 0. \tag{2.3}$$

In the standard high frequency approximation [40] we consider initial data of the form

$$\phi^\varepsilon(0,\mathbf{x}) = e^{iS_0(\mathbf{x})/\varepsilon} A_0(\mathbf{x}) \tag{2.4}$$

with a smooth, real valued initial phase function $S_0(\mathbf{x})$ and a smooth compactly supported complex valued initial amplitude $A_0(\mathbf{x})$. We then look for an asymptotic solution of (2.3) in the same form as the initial data (2.4), with evolved phase and amplitude

$$\phi^\varepsilon(t,\mathbf{x}) \sim e^{iS(t,\mathbf{x})/\varepsilon} A(t,\mathbf{x}). \tag{2.5}$$

Inserting this form into (2.3) and equating the powers of ε we get evolution equations for the phase and amplitude

$$S_t + \frac{1}{2}|\nabla S|^2 + V(\mathbf{x}) = 0, \qquad S(0,\mathbf{x}) = S_0(\mathbf{x}) \tag{2.6}$$

and

$$(|A|^2)_t + \nabla \cdot (|A|^2 \nabla S) = 0, \qquad |A(0,\mathbf{x})|^2 = |A_0(\mathbf{x})|^2. \tag{2.7}$$

The phase equation (2.6) is called the *eiconal* and the amplitude equation (2.7) the *transport* equation. The terminology for the latter is standard in the high frequency approximation but should not be confused with the radiative transport equation that will be derived later. These equations can be rewritten using the Hamiltonian ω of the Schrödinger equation

$$\omega(\mathbf{x},\mathbf{k}) = \frac{1}{2}\mathbf{k}^2 + V(\mathbf{x}). \tag{2.8}$$

Then the eiconal equation (2.6) is

$$S_t + \omega(\mathbf{x}, \nabla S) = 0 \tag{2.9}$$

and the transport equation (2.7) is

$$(|A|^2)_t + \nabla \cdot (|A|^2 \nabla_{\mathbf{k}} \omega(\mathbf{x},\mathbf{k})|_{\mathbf{k}=\nabla S}) = 0. \tag{2.10}$$

This form of the eiconal and transport equations is general and remains valid in the case of symmetric hyperbolic systems (section 3.2).

The eiconal equation (2.6) is nonlinear and its solution exists in general only up to some time t^* that depends on the initial phase $S_0(\mathbf{x})$ and $V(\mathbf{x})$. This solution can be constructed by the method of characteristics and singularities form when these characteristics (rays) cross.

2.2 The Wigner distribution

An essential step in our approach to deriving radiative transport equations from wave equations is the introduction of the Wigner distribution [41]. For any smooth function ϕ, rapidly decaying at infinity, the Wigner distribution is defined by

$$W(\mathbf{x},\mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int_{R^d} e^{i\mathbf{k}\cdot\mathbf{y}}\phi(\mathbf{x}-\frac{\mathbf{y}}{2})\overline{\phi}(\mathbf{x}+\frac{\mathbf{y}}{2})d\mathbf{y} \tag{2.11}$$

where $\bar{\phi}$ is the complex conjugate of ϕ and the dimension $d = 2$ or 3. The Wigner distribution is defined on phase space and has many important properties. It is real and its $\mathbf{k}$-integral is the modulus square of the function ϕ,

$$\int_{R^d} W(\mathbf{x},\mathbf{k})d\mathbf{k} = |\phi(\mathbf{x})|^2, \tag{2.12}$$

so we may think of $W(\mathbf{x},\mathbf{k})$ as wave number resolved energy density. This is not quite right though because $W(\mathbf{x},\mathbf{k})$ is not always positive but it does become positive in the high frequency limit. The energy flux is expressed through $W(\mathbf{x},\mathbf{k})$ by

$$\boldsymbol{\mathcal{F}} = \frac{1}{2i}(\phi\nabla\overline{\phi} - \overline{\phi}\nabla\phi) = \int_{R^d} \mathbf{k}W(\mathbf{x},\mathbf{k})d\mathbf{k} \tag{2.13}$$

and its second moment in $\mathbf{k}$ is

$$\iint |\mathbf{k}|^2 W(\mathbf{x},\mathbf{k})d\mathbf{k}d\mathbf{x} = \int |\nabla\phi(\mathbf{x})|^2 d\mathbf{x}. \tag{2.14}$$

The Wigner distribution posesses an important $\mathbf{x}$-to-$\mathbf{k}$ duality given by the alternative definition

$$W(\mathbf{x},\mathbf{k}) = \int e^{i\mathbf{p}\cdot\mathbf{x}}\hat{\phi}(-\mathbf{k}-\frac{\mathbf{p}}{2})\overline{\hat{\phi}(-\mathbf{k}+\frac{\mathbf{p}}{2})}d\mathbf{p}. \tag{2.15}$$

where $\hat{\phi}$ is the Fourier transform of ϕ

$$\hat{\phi}(\mathbf{k}) = \frac{1}{(2\pi)^d}\int e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x})d\mathbf{x}.$$

These properties make the Wigner distribution a good candidate for analyzing the evolution of wave energy in phase space.

Given a wave function of the form (2.5), that is, inhomogeneous wave with phase $S(t,\mathbf{x})/\varepsilon$, its scaled Wigner distribution has the weak limit

$$W^\varepsilon(\mathbf{x},\mathbf{k}) = \frac{1}{\varepsilon^d}W(\mathbf{x},\frac{\mathbf{k}}{\varepsilon}) \to |A(\mathbf{x})|^2\delta(\mathbf{k}-\nabla S(\mathbf{x})), \tag{2.16}$$

as a generalized function as $\varepsilon \to 0$. Here $d = 2$ or 3 is the dimension of the space. This suggests that the correct scaling for the high frequency limit is

$$W^\varepsilon(t,\mathbf{x},\mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int e^{i\mathbf{k}\cdot\mathbf{y}}\phi^\varepsilon(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\overline{\phi^\varepsilon(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})}d\mathbf{y}. \tag{2.17}$$

where ϕ^ε satisfies (2.3). From (2.16) we conclude that as $\varepsilon \to 0$ the scaled Wigner distribution of the solution $\phi^\varepsilon(t, \mathbf{x})$ of (2.3) with initial data (2.4) is given by

$$W(t, \mathbf{x}, \mathbf{k}) = |A(t, \mathbf{x})|^2 \delta(\mathbf{k} - \nabla S(t, \mathbf{x})), \tag{2.18}$$

where $S(t, \mathbf{x})$ and $A(t, \mathbf{x})$ are solutions of the eiconal and transport equations (2.6) and (2.7), respectively.

We will now derive the high frequency approximation of the scaled Wigner distribution directly from the differential equations. Let us assume that the initial Wigner distribution $W_0^\varepsilon(\mathbf{x}, \mathbf{k})$ tends to a smooth function $W_0(\mathbf{x}, \mathbf{k})$ that decays at infinity. Note that this is not the case with the Wigner function corresponding to $\phi^\varepsilon(0, \mathbf{x})$ given by (2.4) but it is the case for random initial wave functions. The evolution equation for $W^\varepsilon(t, \mathbf{x}, \mathbf{k})$ corresponding to the Schrödinger equation (2.3) is the Wigner equation

$$W_t^\varepsilon + \mathbf{k} \cdot \nabla_{\mathbf{x}} W^\varepsilon + \mathcal{L}^\varepsilon W^\varepsilon = 0. \tag{2.19}$$

Here the operator $\mathcal{L}^\varepsilon$ is defined by

$$\mathcal{L}^\varepsilon Z(\mathbf{x}, \mathbf{k}) = i \int_{R^d} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{V}(\mathbf{p}) \frac{1}{\varepsilon} \left[Z(\mathbf{x}, \mathbf{k} + \frac{\varepsilon \mathbf{p}}{2}) - Z(\mathbf{x}, \mathbf{k} - \frac{\varepsilon \mathbf{p}}{2}) \right] d\mathbf{p} \tag{2.20}$$

on any smooth function Z in phase space. The Fourier transform is denoted by a hat

$$\hat{V}(\mathbf{p}) = \frac{1}{(2\pi)^d} \int e^{i\mathbf{p}\cdot\mathbf{x}} V(\mathbf{x}) d\mathbf{x}. \tag{2.21}$$

From (2.20) we can find easily the limit of the operator $\mathcal{L}^\varepsilon$ as $\varepsilon \to 0$. For any smooth and decaying function $Z(\mathbf{x}, \mathbf{k})$ we have

$$\mathcal{L}^\varepsilon Z(\mathbf{x}, \mathbf{k}) \to -\nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{k}} Z. \tag{2.22}$$

Thus, the limit Wigner equation is the Liouville equation in phase space

$$W_t + \mathbf{k} \cdot \nabla_{\mathbf{x}} W - \nabla V \cdot \nabla_{\mathbf{k}} W = 0 \tag{2.23}$$

with the initial condition $W(0, \mathbf{x}, \mathbf{k}) = W_0(\mathbf{x}, \mathbf{k})$. This is a linear partial differential equation that can be solved by characteristics. When the initial Wigner distribution has the high frequency form

$$W_0(\mathbf{x}, \mathbf{k}) = |A_0(\mathbf{x})|^2 \delta(\mathbf{k} - \nabla S_0(\mathbf{x})) \tag{2.24}$$

then it is easy to see that the solution of (2.23) is given by

$$W(t, \mathbf{x}, \mathbf{k}) = |A(t, \mathbf{x})|^2 \delta(\mathbf{k} - \nabla S(t, \mathbf{x})) \tag{2.25}$$

where $S(t, \mathbf{x})$ and $A(t, \mathbf{x})$ are solutions of the eiconal and transport equations (2.6) and (2.7), respectively. We see, therefore, that from the Wigner distribution we can recover all the information in the standard high frequency approximation. In addition, it provides flexibility to deal with initial data that is not of the form (2.24).

2.3 Random Potential and the Transport Equations

We now consider small random perturbations of the potential $V(\mathbf{x})$. It is well known that in one space dimension, waves in a random medium get localized even when the random perturbations are small [16], so our analysis is restricted to three dimensions. We could treat two-dimensional problems with time dependent perturbations but we do not consider this case here. We assume that the correlation length of the random perturbation is of the same order as the wavelength, so the potential has the form

$$V(\mathbf{x}) = V_0(\mathbf{x}) + V_1(\frac{\mathbf{x}}{\varepsilon}). \tag{2.26}$$

Here $V_0(\mathbf{x})$ is the slowly varying background and $V_1(\mathbf{y})$ is a mean zero, stationary random function with correlation length of order one. This scaling allows the random potential to interact fully with the waves. We shall also assume that the fluctuations are space- homogeneous and isotropic so that

$$< V_1(\mathbf{x})V_1(\mathbf{y}) >= R(|\mathbf{x} - \mathbf{y}|), \tag{2.27}$$

where $<,>$ denotes statistical averaging and $R(|\mathbf{x}|)$ is the covariance of random the fluctuations. The power spectrum of the fluctuations is defined by

$$\hat{R}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int e^{i\mathbf{k}\cdot\mathbf{y}} R(\mathbf{x}) d\mathbf{k}. \tag{2.28}$$

When (2.27) holds the fluctuations are isotropic and $\hat{R}$ is a function of $|\mathbf{k}|$ only. Moreover,

$$< \hat{V}(\mathbf{p})\hat{V}(\mathbf{q}) >= \hat{R}(\mathbf{p})\delta(\mathbf{p} + \mathbf{q}). \tag{2.29}$$

If the amplitude of these fluctuations is strong then scattering will dominate and waves will be localized [15]. This means that we cannot assume that the fluctuations in the random potential $V_1(\mathbf{y})$ are large. If the random fluctuations are too weak they will not affect energy transport at all. In order that the scattering produced by the random potential and the influence of the slowly varying background affect energy transport in comparable ways the fluctuations in the random potential must be of order $\sqrt{\varepsilon}$. Then equation (2.3) becomes

$$i\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta\phi^\varepsilon - (V_0(\mathbf{x}) + \sqrt{\varepsilon}V_1(\frac{\mathbf{x}}{\varepsilon}))\phi^\varepsilon = 0$$
$$\phi^\varepsilon(0,\mathbf{x}) = \phi_0(\frac{\mathbf{x}}{\varepsilon},\mathbf{x}). \tag{2.30}$$

To describe the passage from (2.30) to the transport equation in its simplest form we will set $V_0(\mathbf{x}) = 0$ and drop the subscript one from $V_1(\mathbf{x})$. A $V_0(\mathbf{x})$ that is not zero will not change the scattering terms in the radiative transport equation. Now (2.19) for W^ε has the form

$$\frac{\partial W^\varepsilon}{\partial t} + \mathbf{k}\cdot\nabla_{\mathbf{x}} W^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{\frac{\mathbf{x}}{\varepsilon}} W^\varepsilon = 0 \tag{2.31}$$

where the operator $\mathcal{L}_{\frac{\mathbf{x}}{\varepsilon}}$, a rescaled form of (2.20), is given by

$$\mathcal{L}_{\frac{\mathbf{x}}{\varepsilon}} Z(\mathbf{x},\mathbf{k}) = i\int e^{-i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\hat{V}(\mathbf{p})\left(Z(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}) - Z(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\right)d\mathbf{p}. \tag{2.32}$$

The behavior of this operator as $\varepsilon \to 0$ is very different from (2.22) when V is slowly varying. We can find the correct results by a multiscale analysis as follows.

Let $\boldsymbol{\xi} = \mathbf{x}/\varepsilon$ be a fast space variable (on the scale of the wavelength) and introduce an expansion of W^ε of the form

$$W^\varepsilon(t,\mathbf{x},\mathbf{k}) = W^{(0)}(t,\mathbf{x},\mathbf{k}) + \varepsilon^{1/2}W^{(1)}(t,\mathbf{x},\boldsymbol{\xi},\mathbf{k}) + \varepsilon W^{(2)}(t,\mathbf{x},\boldsymbol{\xi},\mathbf{k}) + \ldots. \tag{2.33}$$

We assume that the leading term does not depend on the fast scale and that the initial Wigner distribution $W^\varepsilon(0,\mathbf{x},\mathbf{k})$ tends to a smooth function $W_0(\mathbf{x},\mathbf{k})$ which is decaying fast enough at infinity. Then the average of the Wigner distribution, $< W^\varepsilon >$, is close to $W^{(0)}$ which satisfies the transport equation

$$\frac{\partial W}{\partial t} + \mathbf{k}\cdot\nabla_{\mathbf{x}} W = \overline{\mathcal{L}}W$$
$$W(0,\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k}), \tag{2.34}$$

where we have dropped the superscript zero. The operator $\overline{\mathcal{L}}$ is given by

$$\overline{\mathcal{L}}W(\mathbf{x},\mathbf{k}) = 4\pi\int \hat{R}(\mathbf{p}-\mathbf{k})\delta(\mathbf{k}^2-\mathbf{p}^2)(W(\mathbf{x},\mathbf{p}) - W(\mathbf{x},\mathbf{k}))d\mathbf{p}. \tag{2.35}$$

Equation (2.34) has precisely the form (1.1). From (2.8)

$$\omega = \frac{\mathbf{k}^2}{2},$$

since the background potential V_0 is zero. The differential scattering cross-section $\sigma(\mathbf{k}, \mathbf{k}')$ is given by

$$\sigma(\mathbf{k}, \mathbf{p}) = 4\pi \hat{R}(\mathbf{p} - \mathbf{k})\delta(\mathbf{k}^2 - \mathbf{p}^2) \tag{2.36}$$

and the total scattering cross-section $\Sigma(\mathbf{k})$ is given by

$$\Sigma(\mathbf{k}) = 4\pi \int \hat{R}(\mathbf{k} - \mathbf{p})\delta(\mathbf{k}^2 - \mathbf{p}^2) d\mathbf{p}. \tag{2.37}$$

Note also that the transport equation (2.34) has two important properties. First, the total energy

$$E(t) = \iint W(t, \mathbf{x}, \mathbf{k}) d\mathbf{k} d\mathbf{x} \tag{2.38}$$

is conserved and second, the positivity of the solution $W(t, \mathbf{x}, \mathbf{k})$ is preserved, that is, if the initial Wigner distribution $W_0(\mathbf{x}, \mathbf{k})$ is non-negative then $W(t, \mathbf{x}, \mathbf{k}) \geq 0$ for $t > 0$.

We explain in the Appendix how a formal multiscale expansion like (2.33) gives this transport equation starting from (2.31).

In the rest of this paper we extend the analysis of this section to symmetric hyperbolic systems of partial differential equations. The main steps are (i) developing the high frequency approximation in phase space using the Wigner distribution and (ii) getting the scattering cross-sections from the random inhomogeneities of the medium.

3 High Frequency Approximation for General Wave Equations

3.1 General Symmetric Hyperbolic Systems

We will use the Wigner distribution to get the high frequency approximation of symmetric hyperbolic systems [42] in phase space. As we will see in sections 3.2-3.4, many wave equations arising from physical problems can be written as symmetric hyperbolic systems of the form[1]

$$\begin{aligned} A(\mathbf{x})\frac{\partial \mathbf{u}}{\partial t} + D^i \frac{\partial \mathbf{u}}{\partial x^i} &= 0, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), \end{aligned} \tag{3.1}$$

where $\mathbf{u}$ is a complex valued N-vector and $\mathbf{x} \in R^3$. We assume that the matrix $A(\mathbf{x})$ is symmetric and positive definite and that the matrices D^j are symmetric and independent of $\mathbf{x}$ and t.

[1]We use the summation convention as follows: repeated Latin indices are summed, while repeated Greek indices are not summed.

The energy density $\mathcal{E}(t, \mathbf{x})$ for solutions of (3.1) is given by the inner product

$$\mathcal{E}(t, \mathbf{x}) = \frac{1}{2}(A(\mathbf{x})\mathbf{u}(t, \mathbf{x}), \mathbf{u}(t, \mathbf{x})) = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij}(\mathbf{x})u_i(t, \mathbf{x})\bar{u}_j(t, \mathbf{x}) \tag{3.2}$$

and the flux $\boldsymbol{\mathcal{F}}(\mathbf{x})$ by

$$\boldsymbol{\mathcal{F}}_i(t, \mathbf{x}) = \frac{1}{2}(D^i\mathbf{u}(t, \mathbf{x}), \mathbf{u}(t, \mathbf{x})). \tag{3.3}$$

Taking the inner product of (3.1) with $\mathbf{u}(t, \mathbf{x})$ yields the energy conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = 0. \tag{3.4}$$

Integration of (3.4) shows that the total energy is conserved:

$$\frac{d}{dt}\int \mathcal{E}(t, \mathbf{x})d\mathbf{x} = 0. \tag{3.5}$$

It is convinient to introduce the new inner product

$$< \mathbf{u}, \mathbf{v} >_A = (A\mathbf{u}, \mathbf{v}). \tag{3.6}$$

Then the energy density is $\mathcal{E} = \frac{1}{2} < \mathbf{u}, \mathbf{u} >_A$. This inner product is the natural one for the system (3.1).

For N-vector functions we define the Wigner distribution an $N \times N$ matrix,

$$W(t, \mathbf{x}, \mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{u}(t, \mathbf{x} - \mathbf{y}/2)\mathbf{u}^*(t, \mathbf{x} + \mathbf{y}/2)d\mathbf{y}, \tag{3.7}$$

where $\mathbf{u}^* = \bar{\mathbf{u}}^t$ is the conjugate transpose of $\mathbf{u}$. The matrix $W(t, \mathbf{x}, \mathbf{k})$ is Hermitian but not necessarily positive definite. As in the scalar case, $W(t, \mathbf{x}, \mathbf{k})$ has the properties

$$\int W(t, \mathbf{x}, \mathbf{k})d\mathbf{k} = \mathbf{u}(t, \mathbf{x})\mathbf{u}^*(t, \mathbf{x})$$

and

$$\left(\frac{1}{2\pi}\right)^d \int W(t, \mathbf{x}, \mathbf{k})d\mathbf{x} = \hat{\mathbf{u}}(-\mathbf{k}, t)\widehat{\mathbf{u}^*}(-\mathbf{k}, t).$$

It follows that the energy density is expressible in terms of $W(t, \mathbf{x}, \mathbf{k})$ by

$$\begin{aligned} \mathcal{E}(t, \mathbf{x}) &= \frac{1}{2} < \mathbf{u}(t, \mathbf{x}), \mathbf{u}(t, \mathbf{x}) >_A = \frac{1}{2}A_{ij}(\mathbf{x})\ u_i(t, \mathbf{x})\bar{u}_j(t, \mathbf{x}) \\ &= \frac{1}{2}A_{ij}(\mathbf{x})\int W_{ij}(t, \mathbf{x}, \mathbf{k})d\mathbf{k} = \frac{1}{2}\int \mathrm{Tr}(A(\mathbf{x})W(t, \mathbf{x}, \mathbf{k}))d\mathbf{k}. \end{aligned} \tag{3.8}$$

The flux $\boldsymbol{\mathcal{F}}(t,\mathbf{x})$ can be expressed via $W(t,\mathbf{x},\mathbf{k})$ by

$$\mathcal{F}_i(t,\mathbf{x},\mathbf{k}) = \frac{1}{2}D^i_{nm}u_n(t,\mathbf{x})\bar{u}_m(t,\mathbf{x}) = \frac{1}{2}\int \mathrm{Tr}(D^i W(t,\mathbf{x},\mathbf{k}))d\mathbf{k}. \tag{3.9}$$

To study the high frequency approximation of solutions of (3.1), we assume that the coefficients of the matrix $A(\mathbf{x})$ vary on a scale much longer than the scale on which the initial data vary. Let ε be the ratio of these two scales. We rescale space and time coordinates $(\mathbf{x},t)$ by $\mathbf{x}\to\varepsilon\mathbf{x}$, $t\to\varepsilon t$ as in (2.3). In scaled coordinates (3.1) has the form

$$A(\mathbf{x})\frac{\partial \mathbf{u}_\varepsilon}{\partial t} + D^j\frac{\partial \mathbf{u}_\varepsilon}{\partial x^j} = 0 \tag{3.10}$$

$$\mathbf{u}_\varepsilon(0,\mathbf{x}) = \mathbf{u}_0(\frac{\mathbf{x}}{\varepsilon}) \quad \text{or} \quad \mathbf{u}_0(\frac{\mathbf{x}}{\varepsilon},\mathbf{x}). \tag{3.11}$$

Note that the parameter ε does not appear explicitly in (3.10). It enters through the initial conditions (3.11), which may be of the standard geometrical optics form (2.4). The scaled Wigner distribution matrix W^ε is defined, as in the scalar case, by

$$W^\varepsilon(t,\mathbf{x},\mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{u}_\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y}/2)\mathbf{u}^*_\varepsilon(\mathbf{x}+\varepsilon\mathbf{y}/2)d\mathbf{y}. \tag{3.12}$$

Although W^ε is not positive definite, it becomes so in the high frequency limit $\varepsilon\to 0$.

As in (2.19), W^ε satisfies the evolution equation

$$\frac{\partial W^\varepsilon}{\partial t} + \mathcal{Q}^\varepsilon_1 W^\varepsilon + \frac{1}{\varepsilon}\mathcal{Q}^\varepsilon_2 W^\varepsilon = 0 \tag{3.13}$$
$$W^\varepsilon(0,\mathbf{x},\mathbf{k}) = W^\varepsilon_0(\mathbf{x},\mathbf{k}).$$

Here the operators $\mathcal{Q}^\varepsilon_1$ and $\mathcal{Q}^\varepsilon_2$ are given by

$$\begin{aligned}\mathcal{Q}^\varepsilon_1 W^\varepsilon = \frac{1}{2}\int e^{-i\mathbf{p}\cdot\mathbf{x}}\{\widehat{A^{-1}}(\mathbf{p})D^j\frac{\partial W^\varepsilon(t,\mathbf{x},\mathbf{k}+\varepsilon\mathbf{p}/2)}{\partial x^j} + \frac{\partial W^\varepsilon(t,\mathbf{x},\mathbf{k}-\varepsilon\mathbf{p}/2)}{\partial x^j}D^j\widehat{A^{-1}}(\mathbf{p}) \\ + i\widehat{A^{-1}}(\mathbf{p})p_jD^jW^\varepsilon(t,\mathbf{x},\mathbf{k}+\varepsilon\mathbf{p}/2) + W^\varepsilon(\mathbf{k}-\varepsilon\mathbf{p}/2)ip_jD^j\widehat{A^{-1}}(\mathbf{p})\}d\mathbf{p}\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\mathcal{Q}^\varepsilon_2 W^\varepsilon = \int e^{-i\mathbf{p}\cdot\mathbf{x}}\{\; i\;\widehat{A^{-1}}(\mathbf{p})k_jD^jW^\varepsilon(t,\mathbf{x},\mathbf{k}+\varepsilon\mathbf{p}/2) \\ - iW^\varepsilon(t,\mathbf{x},\mathbf{k}-\varepsilon\mathbf{p}/2)k_jD^j\widehat{A^{-1}}(\mathbf{p})\}d\mathbf{p}.\end{aligned} \tag{3.15}$$

The hat denotes the Fourier transform as in (2.21). The initial condition for (3.13) is obtained by inserting (3.11) into (3.12).

A new feature of (3.13), not found in the scalar case (2.19), is the appearance of the factor $1/\varepsilon$ in front of the term $\mathcal{Q}_2^\varepsilon W^\varepsilon$. There is no other term in the equation to balance it. This means that the limiting Wigner distribution $W(t, \mathbf{x}, \mathbf{k})$ ($W^\varepsilon \to W$ as $\varepsilon \to 0$) must belong to the null space of the limit operator $\mathcal{Q}_2$, where $\mathcal{Q}_2^\varepsilon \to \mathcal{Q}_2$ as $\varepsilon \to 0$. From (3.15) this operator acting on a matrix $Z(\mathbf{x}, \mathbf{k})$ has the form

$$\mathcal{Q}_2 Z(\mathbf{x}, \mathbf{k}) = iA^{-1} k_j D^j Z(\mathbf{x}, \mathbf{k}) - iZ(\mathbf{x}, \mathbf{k}) k_j D^j A^{-1}. \tag{3.16}$$

The next term in the expansion of Q_2^ε in ε, $\mathcal{Q}_2^\varepsilon = \mathcal{Q}_2 + \varepsilon \mathcal{Q}_{21} + \ldots$, is given by

$$\mathcal{Q}_{21} Z(\mathbf{x}, \mathbf{k}) = -\frac{1}{2} \frac{\partial A^{-1}}{\partial x^i} k_j D^j \frac{\partial Z}{\partial k_i} - \frac{1}{2} \frac{\partial Z}{\partial k_i} k_j D^j \frac{\partial A^{-1}}{\partial x^i} \tag{3.17}$$

This introduces the term with the gradient with respect to $\mathbf{k}$ into the transport equation, as we shall see. Similarly, the limit operator $\mathcal{Q}_1$, $\mathcal{Q}_1^\varepsilon \to \mathcal{Q}_1$ as $\varepsilon \to 0$ is given by

$$\mathcal{Q}_1 Z(\mathbf{x}, \mathbf{k}) = \frac{1}{2} A^{-1} D^j \frac{\partial Z}{\partial x^j} + \frac{1}{2} \frac{\partial Z}{\partial x^j} D^j A^{-1} - \frac{1}{2} \frac{\partial A^{-1}}{\partial x^j} D^j Z - \frac{1}{2} Z D^j \frac{\partial A^{-1}}{\partial x^j}. \tag{3.18}$$

This operator introduces the term with the $\mathbf{x}$-gradient. The undifferentiated terms in $\mathcal{Q}_1$ also contribute to the transport equation, as we explain below. With the expansions of the $\mathcal{Q}$'s given by (3.16)-(3.18) equation (3.13) becomes

$$\frac{\partial W^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \mathcal{Q}_2 W^\varepsilon + (\mathcal{Q}_{21} + \mathcal{Q}_1) W^\varepsilon + O(\varepsilon) = 0. \tag{3.19}$$

We analyze (3.19) by expanding W^ε

$$W^\varepsilon(t, \mathbf{x}, \mathbf{k}) = W^{(0)}(t, \mathbf{x}, \mathbf{k}) + \varepsilon W^{(1)}(t, \mathbf{x}, \mathbf{k}) + \ldots$$

This leads to the following equations for $W^{(0)}$ and $W^{(1)}$

$$\mathcal{Q}_2 W^{(0)} = 0 \tag{3.20}$$

and

$$\mathcal{Q}_2 W^{(1)} = -\{\frac{\partial W^{(0)}}{\partial t} + (\mathcal{Q}_{21} + \mathcal{Q}_1) W^{(0)}\}. \tag{3.21}$$

We introduce the *dispersion* matrix $L(\mathbf{x}, \mathbf{k})$, defined by

$$L(\mathbf{x}, \mathbf{k}) = A^{-1}(\mathbf{x}) k_i D^i. \tag{3.22}$$

It is self-adjoint with respect to the inner product $<, >_A$:

$$< L\mathbf{u}, \mathbf{v} >_A = (AL\mathbf{u}, \mathbf{v}) = (k_j D^j \mathbf{u}, \mathbf{v}) = (\mathbf{u}, k_j D^j \mathbf{v}) = (A\mathbf{u}, A^{-1} k_j D^j \mathbf{v}) =< \mathbf{u}, L\mathbf{v} >_A .$$

Therefore, all its eigenvalues ω_τ are real and the corresponding eigenvectors $\mathbf{b}^\tau$ can be chosen to be orthonormal with respect to $<,>_A$:

$$L(\mathbf{x},\mathbf{k})\mathbf{b}^\tau(\mathbf{x},\mathbf{k}) = \omega_\tau(\mathbf{x},\mathbf{k})\mathbf{b}^\tau(\mathbf{x},\mathbf{k}) \ , \quad < \mathbf{b}^\tau, \mathbf{b}^\beta >_A = \delta_{\tau\beta}.$$

We assume throughout that the eigenvalues have constant multiplicity independent of $\mathbf{x}$ and $\mathbf{k}$. This hypothesis is satisfied in the case of acoustic, electromagnetic and elastic waves. In terms of the dispersion matrix L, (3.20) becomes

$$\mathcal{Q}_2 W^{(0)}(t,\mathbf{x},\mathbf{k}) = iL(\mathbf{x},\mathbf{k})W^{(0)}(t,\mathbf{x},\mathbf{k}) - iW^{(0)}(t,\mathbf{x},\mathbf{k})L^*(\mathbf{x},\mathbf{k}) = 0$$

The structure of this null space when all the eigenvalues of $L(\mathbf{x},\mathbf{k})$ are distinct is different from that when there are some multiple eigenvalues.

We assume first that all the eigenvalues $\omega_\tau(\mathbf{x},\mathbf{k})$ are simple. Define the matrices $B^\tau(\mathbf{x},\mathbf{k})$ by

$$B^\tau(\mathbf{x},\mathbf{k}) = \mathbf{b}^\tau(\mathbf{x},\mathbf{k})\mathbf{b}^{\tau *}(\mathbf{x},\mathbf{k}) \tag{3.23}$$

They span the null space of $\mathcal{Q}_2$, so the limit Wigner matrix $W^{(0)}(t,\mathbf{x},\mathbf{k})$ has the form

$$W^{(0)}(t,\mathbf{x},\mathbf{k}) = \sum_{\tau=1}^{N} a^\tau(t,\mathbf{x},\mathbf{k})B^\tau(\mathbf{x},\mathbf{k}). \tag{3.24}$$

The $a^\tau(t,\mathbf{x},\mathbf{k})$ are scalar functions determined by projection

$$a^\tau = \mathrm{Tr}(AW^{(0)*}AB^\tau).$$

We now insert (3.24) into equation (3.21) for $W^{(1)}$, which is an inhomogeneous form of (3.20). The operator $\frac{1}{i}\mathcal{Q}_2$ is self-adjoint with respect to the matrix inner product $<< X, Y >>= \mathrm{Tr}(AX^*AY)$. Since the null space of $\mathcal{Q}_2$ is spanned by the matrices B^τ given by (3.23), the solvability condition for (3.21) is that its right hand side be orthogonal to these matrices, relative to the $<<,>>$ inner product. This leads to the following equations for the functions a^τ:

$$\frac{\partial a^\tau}{\partial t} + \nabla_{\mathbf{k}}\omega_\tau \cdot \nabla_{\mathbf{x}} a^\tau - \nabla_{\mathbf{x}}\omega_\tau \cdot \nabla_{\mathbf{k}} a^\tau = 0. \tag{3.25}$$

These are Liouville equations in phase space.

We see, therefore, that in the absence of polarization (simple eigenvalues of the dispersion matrix) the amplitudes a^τ decouple from each other and each satisfies the Liouville equation with Hamiltonian equal to the corresponding eigenvalue ω_τ. We see also that the Liouville equation is not satisfied by the limiting Wigner distribution but by its projections on the eigenspaces generated by the matrices B^τ given by (3.23). Moreover, we do not have a single Liouville equation but several decoupled ones. When small random perturbations are present the Liouville equations are coupled (section 4.1).

Consider now the case where the dispersion matrix $L(\mathbf{x},\mathbf{k})$ has multiple eigenvalues. Let $\omega_\tau(\mathbf{x},\mathbf{k})$ be an eigenvalue of multiplicity r and let the corresponding eigenvectors $\mathbf{b}^{\tau,i}$, $i=1,\ldots,r$ be orthonormal with respect to $<,>_A$. Given a pair of eigenvectors $\mathbf{b}^{\tau,i}$, $\mathbf{b}^{\tau,j}$ we define the $N\times N$ matrix

$$B^{\tau,ij} = \mathbf{b}^{\tau,i}\mathbf{b}^{\tau,j*}, \tag{3.26}$$

with $i,j=1\ldots r$. These matrices span the null space of the operator $\mathcal{Q}_2$ and so the limiting Wigner matrix $W^{(0)}(t,\mathbf{x},\mathbf{k})$ has the representation

$$W^{(0)}(t,\mathbf{x},\mathbf{k}) = \sum_{\tau,i,j} a^\tau_{ij}(t,\mathbf{x},\mathbf{k})B^{\tau,ij}(\mathbf{x},\mathbf{k}), \tag{3.27}$$

where $a^\tau_{ij}(t,\mathbf{x},\mathbf{k})$ are scalar functions. Define the $r\times r$ *coherence* matrices $W^\tau(t,\mathbf{x},\mathbf{k})$ by

$$W^\tau_{ij}(t,\mathbf{x},\mathbf{k}) = a^\tau_{ij}(t,\mathbf{x},\mathbf{k})\ , \quad i,j=1\ldots r. \tag{3.28}$$

The multiplicity r of the eigenvalue ω_τ depends on τ but we do not indicate this explicitly. The functions a^τ_{ij} are given by

$$a^\tau_{ij}(t,\mathbf{x},\mathbf{k}) =<< W^{(0)}(t,\mathbf{x},\mathbf{k}), B^{\tau,ij}(\mathbf{x},\mathbf{k}) >> .$$

Then, by applying the solvability condition for (3.21) as before, we find that each of the coherence matrices $W^\tau(t,\mathbf{x},\mathbf{k})$ satisfies the transport equation

$$\frac{\partial W^\tau}{\partial t} + \nabla_{\mathbf{k}}\omega_\tau\cdot\nabla_{\mathbf{x}}W^\tau - \nabla_{\mathbf{x}}\omega_\tau\cdot\nabla_{\mathbf{k}}W^\tau + W^\tau N^\tau - N^\tau W^\tau = 0. \tag{3.29}$$

The skew-symmetric *coupling* matrices $N^\tau(\mathbf{x},\mathbf{k})$ are given by

$$N^\tau_{mn}(\mathbf{x},\mathbf{k}) = (\mathbf{b}^{\tau,n}, D^i\frac{\partial\mathbf{b}^{\tau,m}}{\partial x^i}) - \frac{\partial\omega_\tau}{\partial x^i}(A(\mathbf{x})\mathbf{b}^{\tau,n}, \frac{\partial\mathbf{b}^{\tau,m}}{\partial k_i}) - \frac{1}{2}\frac{\partial^2\omega_\tau}{\partial x^i\partial k_i}\delta_{nm}. \tag{3.30}$$

The last term in (3.30) is retained to make the coupling matrices N skew symmetric even though it cancels in the transport equation (3.29).

The coherence matrices $W^\tau(t,\mathbf{x},\mathbf{k})$ are Hermitian and positive definite because they are projections of the limiting Wigner matrix $W^{(0)}(t,\mathbf{x},\mathbf{k})$ which is Hermitian and positive definite. Equations (3.29) preserve both of these properties: if the initial conditions for W^τ are Hermitian and positive definite then the solution is Hermitian and positive definite for all t. The fact that the coupling matrices N are skew-symmetric is important for these properties.

We see that in the case of polarized waves, i.e. waves for which the eigenvalues of the dispersion matrix have multiplicity larger than one, the quantities satisfying the transport equations are not scalars but matrices. Their sizes are equal to the degeneracies of the corresponding wave modes. However, modes corresponding to different eigenvalues still decouple from each other. Random inhomogeneities will couple them in general (section 4.2).

The reason we call the $W^\tau(t,\mathbf{x},\mathbf{k})$ coherence matrices is because their off-diagonal terms capture cross-polarization effects. Their diagonal terms represent the phase space energy density in each state of polarization. That is, since $\mathrm{Tr}(AB^{\tau,ij}) = \delta_{ij}$, the energy density (3.8) is given by

$$\mathcal{E}(t,\mathbf{x}) = \frac{1}{2}\int \mathrm{Tr}(A(\mathbf{x})W(t,\mathbf{x},\mathbf{k}))d\mathbf{k} = \frac{1}{2}\int \sum_\tau \mathrm{Tr}W^\tau(t,\mathbf{x},\mathbf{k})d\mathbf{k} \tag{3.31}$$

and the flux (3.9) is given by

$$\begin{aligned}\mathcal{F}_i(t,\mathbf{x}) &= \frac{1}{2}\mathrm{Tr}\int D^i W(t,\mathbf{x},\mathbf{k})d\mathbf{k} \\ &= \frac{1}{2}\int \sum_\tau \frac{\partial \omega_\tau}{\partial k_i}\mathrm{Tr}W^\tau(t,\mathbf{x},\mathbf{k})d\mathbf{k}\ , \quad i=1,2,3.\end{aligned} \tag{3.32}$$

These relations hold because

$$\begin{aligned}\mathrm{Tr}D^i W(t,\mathbf{x},\mathbf{k}) &= \sum_{\tau,n,m} a^\tau_{nm}(t,\mathbf{x},\mathbf{k})\mathrm{Tr}\{D^i b^{\tau,n}(\mathbf{x},\mathbf{k}) b^{\tau,m*}(\mathbf{x},\mathbf{k})\} \\ &= \sum_{\tau,n,m} a^\tau_{nm}(t,\mathbf{x},\mathbf{k})(D^i b^{\tau,n}(\mathbf{x},\mathbf{k}), b^{\tau,m}(\mathbf{x},\mathbf{k})) \\ &= \sum_{\tau,n,m} a^\tau_{nm}(t,\mathbf{x},\mathbf{k})(\frac{\partial\omega_\tau}{\partial k_i}Ab^{\tau,n} + \omega_\tau A\frac{\partial b^{\tau,n}}{\partial k_i} - k_j D^j\frac{\partial b^{\tau,n}}{\partial k_i}, b^{\tau,m}) \\ &= \sum_{\tau,n,m} a^\tau_{nm}(t,\mathbf{x},\mathbf{k})\frac{\partial\omega_\tau}{\partial k_i}(Ab^{\tau,n}, b^{\tau,m}) = \sum_\tau \frac{\partial\omega_\tau}{\partial k_i}\mathrm{Tr}W^\tau(t,\mathbf{x},\mathbf{k}).\end{aligned}$$

Here we have used the fact that $L\mathbf{b}^\tau = A^{-1}k_iD^i\mathbf{b}^\tau = \omega_\tau\mathbf{b}^\tau$, which implies after differentiation with respect to k_i, that

$$D^i\mathbf{b}^\tau = \frac{\partial\omega_\tau}{\partial k_i}A\mathbf{b}^\tau + \omega_\tau A\frac{\partial\mathbf{b}^\tau}{\partial k_i} - k_jD^j\frac{\partial\mathbf{b}^\tau}{\partial k_i}.$$

The energy equation (3.4) follows from (3.29) when $\mathcal{E}$ and $\mathcal{F}$ are defined by (3.31) and (3.32), respectively. Thus, the total energy

$$\int \mathcal{E}(t,\mathbf{x})d\mathbf{x}$$

is conserved by the transport equations (3.29).

Expressions (3.31) and (3.32) for the energy and flux are similar to (2.12) and (2.13) because $\mathbf{k}W$, which is the flux density for the Schrödinger equation, can be written as $\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})W(t,\mathbf{x},\mathbf{k})$ where $\omega(\mathbf{x},\mathbf{k})$ is the Hamiltonian (2.8).

In the case of multiple eigenvalues, there is a basis of eigenvectors $\mathbf{b}^{\tau,i}(\mathbf{x},\mathbf{k})$ such that the transport equations (3.29) for the coherence matrices have the form (3.25); that is, we can eliminate the matrices N^τ from (3.29) by a rotation of the basis. Small random perturbations couple the components of the coherence matrices, and to keep the coupling explicit we do not use a basis which eliminates the N's.

3.2 High Frequency Approximation for Acoustic Waves

We will now apply the results of the previous section to acoustic waves. We will also review the usual form of the high frequency approximation and make explicit the relation between the phase space form of the high frequency approximation and the usual one.

The acoustic equations for the velocity and pressure disturbances $\mathbf{u}$ and p are

$$\begin{aligned} \rho\frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0 \\ \kappa\frac{\partial p}{\partial t} + \mathrm{div}\,\mathbf{u} = 0. \end{aligned} \tag{3.33}$$

Here $\rho = \rho(\mathbf{x})$ is the density and $\kappa = \kappa(\mathbf{x})$ is the compressibility. Equations (3.33) can be put in the form of a symmetric hyperbolic system

$$A(\mathbf{x})\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} + \sum_{i=1}^{3} D^i \frac{\partial}{\partial x^i}\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} = 0.$$

The matrix $A(\mathbf{x}) = \mathrm{diag}(\rho(\mathbf{x}),\rho(\mathbf{x}),\rho(\mathbf{x}),\kappa(\mathbf{x}))$ and each of the matrices D^i has all zero entries except for D^i_{i4} and D^i_{4i} which are equal to one. Then the dispersion matrix $L(\mathbf{x},\mathbf{k})$, defined by (3.22), is

$$L = \begin{pmatrix} 0 & 0 & 0 & k_1/\rho \\ 0 & 0 & 0 & k_2/\rho \\ 0 & 0 & 0 & k_3/\rho \\ k_1/\kappa & k_2/\kappa & k_3/\kappa & 0 \end{pmatrix}. \tag{3.34}$$

It has one double eigenvalue $\omega_1 = \omega_2 = 0$ and two simple eigenvalues

$$\omega_+ = v(\mathbf{x})|\mathbf{k}|\ ,\quad \omega_- = -v(\mathbf{x})|\mathbf{k}|\ , \tag{3.35}$$

where $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ and v is the sound speed

$$v(\mathbf{x}) = \frac{1}{\sqrt{\kappa(\mathbf{x})\rho(\mathbf{x})}}. \tag{3.36}$$

The corresponding basis of eigenvectors orthonormal with respect to the inner product $<,>_A$ is

$$\begin{aligned} \mathbf{b}^1 &= \frac{1}{\sqrt{\rho}}(\mathbf{z}^{(1)}(\mathbf{k}), 0)^t, \\ \mathbf{b}^2 &= \frac{1}{\sqrt{\rho}}(\mathbf{z}^{(2)}(\mathbf{k}), 0)^t, \\ \mathbf{b}^+ &= (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, \frac{1}{\sqrt{2\kappa}})^t, \\ \mathbf{b}^- &= (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, -\frac{1}{\sqrt{2\kappa}})^t, \end{aligned} \tag{3.37}$$

the vectors $\hat{\mathbf{k}}$, $\mathbf{z}^{(1)}(\mathbf{k})$, $\mathbf{z}^{(2)}(\mathbf{k})$, which form an orthonormal triplet, are

$$\hat{\mathbf{k}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \mathbf{z}^{(1)} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \mathbf{z}^{(2)} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{3.38}$$

The physical meaning of the eigenvectors is as follows. The eigenvectors $\mathbf{b}^1(\mathbf{x},\mathbf{k})$ and $\mathbf{b}^2(\mathbf{x},\mathbf{k})$ correspond to transverse advection modes, orthogonal to the direction of propagation. These modes do not propagate because $\omega_{1,2} = 0$. The eigenvectors $\mathbf{b}^+(\mathbf{x},\mathbf{k})$ and $\mathbf{b}^-(\mathbf{x},\mathbf{k})$ represent acoustic waves, which are longitudinal , and which propagate with the sound speed $\pm v(\mathbf{x})$ given by (3.36).

The energy density (3.2) for acoustic waves is given by

$$\mathcal{E}(t,\mathbf{x}) = \frac{1}{2}\rho(\mathbf{x})|\mathbf{u}(t,\mathbf{x})|^2 + \frac{1}{2}\kappa(\mathbf{x})p^2(t,\mathbf{x}) \tag{3.39}$$

and the flux (3.3) by

$$\boldsymbol{\mathcal{F}}(t,\mathbf{x}) = p(t,\mathbf{x})\mathbf{u}(t,\mathbf{x}). \tag{3.40}$$

We now express the *unscaled* amplitudes $a^j(t,\mathbf{x},\mathbf{k})$, in terms of the acoustic velocity and pressure fields $\underline{\mathbf{u}} = (\mathbf{u}, p)^t$. The amplitudes $a^{\pm}(t,\mathbf{x},\mathbf{k})$ are given by

$$a^{\pm}(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^3}\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}} f_{\pm}(t,\mathbf{x},\mathbf{x}-\mathbf{y}/2,\mathbf{k}) f_{\pm}^*(t,\mathbf{x},\mathbf{x}+\mathbf{y}/2,\mathbf{k}), \tag{3.41}$$

where

$$f_{\pm}(t,\mathbf{x},\mathbf{z},\mathbf{k}) = < \underline{\mathbf{u}}(t,\mathbf{z}), b^{\pm}(\mathbf{x},\mathbf{k}) >_A = \sqrt{\frac{\rho(\mathbf{x})}{2}}(\mathbf{u}(t,\mathbf{z})\cdot\hat{\mathbf{k}}) \pm \sqrt{\frac{\kappa(\mathbf{x})}{2}}p(t,\mathbf{z}). \tag{3.42}$$

This shows that

$$a^+(t,\mathbf{x},\mathbf{k}) = a^-(t,\mathbf{x},-\mathbf{k}) \tag{3.43}$$

and therefore we need only keep track of $a^+(t,\mathbf{x},\mathbf{k})$. The advective mode amplitudes are given by

$$a^0_{ij}(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^3}\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\frac{\rho(\mathbf{x})}{2}(\mathbf{u}(t,\mathbf{x}-\mathbf{y}/2)\cdot\mathbf{z}^{(i)}(\mathbf{k}))\overline{(\mathbf{u}(t,\mathbf{x}+\mathbf{y}/2)\cdot\mathbf{z}^{(j)}(\mathbf{k}))}. \tag{3.44}$$

By direct computation we verify that

$$\begin{aligned}&\int a^+(t,\mathbf{x},\mathbf{k})d\mathbf{k} + \frac{1}{2}\int\{a^0_{11}(t,\mathbf{x},\mathbf{k}) + a^0_{22}(t,\mathbf{x},\mathbf{k})\}d\mathbf{k} \\ &= \frac{1}{2}\rho(\mathbf{x})|\mathbf{u}(t,\mathbf{x})|^2 + \frac{1}{2}\kappa(\mathbf{x})p^2(t,\mathbf{x}) = \mathcal{E}(t,\mathbf{x})\end{aligned} \tag{3.45}$$

and

$$\int \hat{\mathbf{k}} v(\mathbf{x}) a^+(t,\mathbf{x},\mathbf{k})d\mathbf{k} = p(t,\mathbf{x})\mathbf{u}(t,\mathbf{x}) = \boldsymbol{\mathcal{F}}(t,\mathbf{x}). \tag{3.46}$$

The first integral in (3.45) represents the part of the energy density which is propagating with speed v. The second integral gives the energy of the non-propagating waves.

Equation (3.29) for W^0 is of the form $\frac{\partial W^0}{\partial t} = 0$ and so $W^0(t,\mathbf{x},\mathbf{k}) = 0$ if it is zero initially. Then from(3.45)

$$\int a^+(t,\mathbf{x},\mathbf{k})d\mathbf{k} = \frac{1}{2}\rho(\mathbf{x})|\mathbf{u}(t,\mathbf{x})|^2 + \frac{1}{2}\kappa(\mathbf{x})p^2(t,\mathbf{x}). \tag{3.47}$$

This shows that when $W^0 = 0$, the amplitude $a^+(t,\mathbf{x},\mathbf{k})$ is the phase space energy density. In the high frequency limit it satisfies the Liouville equation (3.25)

$$\frac{\partial a^+}{\partial t} + v(\mathbf{x})\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a^+ - |\mathbf{k}|\nabla_{\mathbf{x}}v(\mathbf{x})\cdot\nabla_{\mathbf{k}}a^+ = 0. \tag{3.48}$$

with the initial condition

$$a^+(0,\mathbf{x},\mathbf{k}) = a_0(\mathbf{x},\mathbf{k}). \tag{3.49}$$

Next we establish the connection with the usual high frequency approximation. We consider (3.33) with initial data of the form

$$\underline{\mathbf{u}}(0,\mathbf{x}) = \underline{\mathbf{u}}_0(\mathbf{x})e^{iS_0(\mathbf{x})/\varepsilon}, \tag{3.50}$$

where $\underline{\mathbf{u}} = (\mathbf{u}, p)$ and S_0 is the real valued initial phase function. We look for a solution of (3.33) in the form

$$\underline{\mathbf{u}}(t, \mathbf{x}) = (\underline{A}_0(t, \mathbf{x}) + \varepsilon \underline{A}_1 + \dots)e^{iS(t,\mathbf{x})/\varepsilon}, \tag{3.51}$$

where $\underline{A}_0 = (\mathbf{u}_0, p_0)$. We insert (3.51) into (3.33) to get to leading order in ε

$$\begin{pmatrix} \rho S_t & \nabla S \\ \nabla S \cdot & \kappa S_t \end{pmatrix} \begin{pmatrix} \mathbf{u}_0 \\ p_0 \end{pmatrix} = 0. \tag{3.52}$$

The next term in the expansion yields

$$-i \begin{pmatrix} \rho S_t & \nabla S \\ \nabla S \cdot & \kappa S_t \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} \rho \partial_t & \nabla \\ \nabla \cdot & \kappa \partial_t \end{pmatrix} \begin{pmatrix} \mathbf{u}_0 \\ p_0 \end{pmatrix}. \tag{3.53}$$

Equation (3.52) gives the eiconal equation for the phase S

$$\frac{1}{v^2} S_t^2 - (\nabla S)^2 = 0. \tag{3.54}$$

Then assuming that $S_t = +v|\nabla S|$ we have

$$\begin{pmatrix} \mathbf{u}_0 \\ p_0 \end{pmatrix} = \mathcal{A}(\mathbf{x})\mathbf{b}^+(\mathbf{x}, \nabla S(t, \mathbf{x})), \tag{3.55}$$

where $\mathbf{b}^+$ is given by (3.37).The amplitude $\mathcal{A}(t, \mathbf{x})$ is determined by the solvability condition for (3.53), which gives the transport equation

$$\frac{\partial}{\partial t}|\mathcal{A}|^2 + \nabla \cdot (|\mathcal{A}|^2 v \frac{\nabla S}{|\nabla S|}) = 0. \tag{3.56}$$

The terminology 'transport equation' is standard in high frequency asymptotics for this equation and should not be confused with the radiative transport equations which are defined in phase space. As expected, equation (3.56) is the same as (3.4), to principal order in ε when $\underline{\mathbf{u}}$ is of the form (3.51) and (3.55). It is also the same as the transport equation (2.7) for the Schrödinger equation and both can be written in the form

$$\frac{\partial}{\partial t}|\mathcal{A}|^2 + \nabla \cdot (|\mathcal{A}|^2 \nabla_{\mathbf{k}} \omega(\mathbf{x}, \nabla S)) = 0. \tag{3.57}$$

The Hamiltonian for the acoustic waves is the eigenvalue $\omega(\mathbf{x}, \mathbf{k}) = v(\mathbf{x})|\mathbf{k}|$ and for the Schrödinger equation it is given by (2.8).

The eiconal and transport equations (3.54) and (3.56) can also be derived from (3.48) as follows. In the high frequency limit, initial conditions of the form (3.50) imply that

$$a^+(0, \mathbf{x}, \mathbf{k}) = |\mathcal{A}_0(\mathbf{x})|^2 \delta(\mathbf{k} - \nabla S_0(\mathbf{x})). \tag{3.58}$$

Let the functions $S(t,\mathbf{x})$ and $|\mathcal{A}(t,\mathbf{x})|^2$ be the solutions of the eiconal and transport equations (3.54) and (3.56), respectively, with the initial conditions $S(0,\mathbf{x}) = S_0(\mathbf{x})$ and $|\mathcal{A}(0,\mathbf{x})|^2 = |\mathcal{A}_0(\mathbf{x})|^2$. Then the solution of equation (3.48) is

$$a^+(t,\mathbf{x},\mathbf{k}) = |\mathcal{A}(t,\mathbf{x})|^2\delta(\mathbf{k} - \nabla S(t,\mathbf{x})). \tag{3.59}$$

Conversely, given initial conditions of the form (3.58) for (3.48) and a^+ given by (3.59), then S and $\mathcal{A}$ must satisfy the eiconal and transport equations (3.54) and (3.56), respectively. This is because the eiconal equation follows by integrating (3.48) with respect to $\mathbf{k}$ while the transport equation follows by multiplying it by $\mathbf{k}$ and then integrating with respect to $\mathbf{k}$. This shows that we can recover from the Liouville equation (3.25) the usual high frequency approximation.

3.3 Geometrical Optics for Electromagnetic Waves

Maxwell's equations in an isotropic medium and in suitable units are

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= \frac{1}{\epsilon}\mathrm{curl}\mathbf{H} \\ \frac{\partial \mathbf{H}}{\partial t} &= -\frac{1}{\mu}\mathrm{curl}\mathbf{E} \end{aligned} \tag{3.60}$$

where the dielectric permittivity [2] is $\epsilon(\mathbf{x})$ and the relative magnetic permeability is $\mu(\mathbf{x})$. As a symmetric hyperbolic system they are

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0. \tag{3.61}$$

These equations imply that if at some initial time we have

$$\begin{aligned} \mathrm{div}(\epsilon\mathbf{E}) &= 0 \\ \mathrm{div}(\mu\mathbf{H}) &= 0 \end{aligned} \tag{3.62}$$

then these equations hold for all time. We assume (3.3.3) holds. The 6×6 dispersion matrix L defined by (3.22) is

$$L = -\begin{pmatrix} 0 & 0 & 0 & 0 & -k_3/\epsilon & k_2/\epsilon \\ 0 & 0 & 0 & k_3/\epsilon & 0 & -k_1/\epsilon \\ 0 & 0 & 0 & -k_2/\epsilon & k_1/\epsilon & 0 \\ 0 & k_3/\mu & -k_2/\mu & 0 & 0 & 0 \\ -k_3/\mu & 0 & k_1/\mu & 0 & 0 & 0 \\ k_2/\mu & -k_1/\mu & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.63}$$

[2] Throughout this section and when we consider electromagnetic waves ϵ denotes the dielectric permittivity while the small parameter is denoted by ε.

or in block form

$$L = \begin{pmatrix} 0 & -\frac{1}{\epsilon}P \\ \frac{1}{\mu}P & 0 \end{pmatrix}.$$

The matrix $P(\mathbf{k})\mathbf{p} = \mathbf{k} \times \mathbf{p}$ or

$$P(\mathbf{k}) = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}. \tag{3.64}$$

The dispersion matrix L has three eigenvalues, each with multiplicity two. They are $\omega_0 = 0$, $\omega_+ = v|\mathbf{k}|$, $\omega_- = -v|\mathbf{k}|$ with the speed of propagation v given by

$$v(\mathbf{x}) = \frac{1}{\sqrt{\epsilon(\mathbf{x})\mu(\mathbf{x})}}. \tag{3.65}$$

The basis formed by the corresponding eigenvectors is

$$\begin{aligned} \mathbf{b}^{(01)} &= \frac{1}{\sqrt{\epsilon}}(\hat{\mathbf{k}}, 0),\ \mathbf{b}^{(02)} = \frac{1}{\sqrt{\mu}}(0, \hat{\mathbf{k}}), \\ \mathbf{b}^{(+,1)} &= (\sqrt{\frac{1}{2\epsilon}}\mathbf{z}^{(1)}, \sqrt{\frac{1}{2\mu}}\mathbf{z}^{(2)}),\ \mathbf{b}^{(+,2)} = (\sqrt{\frac{1}{2\epsilon}}\mathbf{z}^{2}, -\sqrt{\frac{1}{2\mu}}\mathbf{z}^{(1)}), \\ \mathbf{b}^{(-,1)} &= (\sqrt{\frac{1}{2\epsilon}}\mathbf{z}^{(1)}, -\sqrt{\frac{1}{2\mu}}\mathbf{z}^{(2)}),\ \mathbf{b}^{(-,2)} = (\sqrt{\frac{1}{2\epsilon}}\mathbf{z}^{(2)}, \sqrt{\frac{1}{2\mu}}\mathbf{z}^{(1)}), \end{aligned} \tag{3.66}$$

where the vectors $\mathbf{z}^{(1)}(\mathbf{k})$ and $\mathbf{z}^{(2)}(\mathbf{k})$ are given by (3.38). The eigenvectors $\mathbf{b}^{(01)}$ and $\mathbf{b}^{(02)}$ represent the non-propagating longitudinal modes and do not satisfy (3.62) so they will be assumed to be absent from the solution. The other eigenvectors correspond to transverse modes propagating with the speed $v(\mathbf{x})$. As in the acoustic case, we need only consider the eigenspace corresponding to ω_+. With this choice for the basis of eigenvectors, the skew symmetric coupling matrix $N(\mathbf{x}, \mathbf{k})$, given by (3.30), is

$$N = \frac{\partial v}{\partial x^i}|\mathbf{k}|\mathbf{z}^{(1)} \cdot \frac{\partial \mathbf{z}^{(2)}}{\partial k_i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.67}$$

Note that the vector $\mathbf{z}^{(2)}(\mathbf{k})$ does not depend on k_3. From (3.67) we conclude that if the medium is layered, so that $v = v(x_3)$, then the coupling matrix N vanishes. This means that in the case of a layered medium there is no coupling between the two polarizations of the electromagnetic field, a well known fact. We note also that there is a choice of the vectors $\mathbf{z}^{(1)}(\mathbf{k})$, $\mathbf{z}^{(2)}(\mathbf{k})$, different from (3.38), which eliminates the coupling terms [44]. As explained earlier, we will use (3.38) because they are convenient for the analysis of random effects.

The transport equation (3.29) for the matrix W^+ is

$$\frac{\partial W^+}{\partial t} + v(\mathbf{x})\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W^+ - |\mathbf{k}| \nabla_{\mathbf{x}} v(\mathbf{x}) \cdot \nabla_{\mathbf{k}} W^+ + W^+ N - N W^+ = 0. \tag{3.68}$$

The energy density (3.2) for the electromagnetic waves is given by

$$\mathcal{E}(t,\mathbf{x}) = \frac{1}{2}\epsilon(\mathbf{x})|\mathbf{E}(t,\mathbf{x})|^2 + \frac{1}{2}\mu(\mathbf{x})|\mathbf{H}(t,\mathbf{x})|^2 \tag{3.69}$$

while the energy flux (3.3) is the Poynting vector

$$\boldsymbol{\mathcal{F}}(t,\mathbf{x}) = \mathbf{E}(t,\mathbf{x}) \times \mathbf{H}(t,\mathbf{x}). \tag{3.70}$$

Let $\underline{\mathbf{u}}(t,\mathbf{x}) = (\mathbf{E},\mathbf{H})$. Then, as in the case of acoustic waves, we will consider the *unscaled* amplitudes $a_{ij}^{\pm}(t,\mathbf{x},\mathbf{k})$

$$a_{ij}^{\pm}(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{y}} f_i^{\pm}(t,\mathbf{x},\mathbf{x}-\mathbf{y}/2,\mathbf{k}) f_j^{\pm *}(t,\mathbf{x},\mathbf{x}+\mathbf{y}/2,\mathbf{k}) d\mathbf{y}, \tag{3.71}$$

where

$$\begin{aligned} f_i(t,\mathbf{x},\mathbf{z},\mathbf{k}) = < \underline{\mathbf{u}}(t,\mathbf{z}), \mathbf{b}^{\pm i}(\mathbf{x},\mathbf{k}) >_A = & \sqrt{\frac{\epsilon(\mathbf{x})}{2}}(\mathbf{E}(t,\mathbf{z}) \cdot \mathbf{z}^{(i)}(\mathbf{k})) \\ \pm & \sqrt{\frac{\mu(\mathbf{x})}{2}}(\mathbf{H}(t,\mathbf{z}) \cdot (\hat{\mathbf{k}} \times \mathbf{z}^{(i)}(\mathbf{k}))). \end{aligned} \tag{3.72}$$

The amplitudes of the longitudinal, nonpropagating modes are

$$\begin{aligned} a_{11}^0(t,\mathbf{x},\mathbf{k})) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{y}} \epsilon(\mathbf{x})(\mathbf{E}(t,\mathbf{x}-\mathbf{y}/2)\cdot\hat{\mathbf{k}})\overline{(\mathbf{E}(t,\mathbf{x}+\mathbf{y}/2)\cdot\hat{\mathbf{k}})} d\mathbf{y} \\ a_{12}^0(t,\mathbf{x},\mathbf{k}) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{y}} \sqrt{\epsilon(\mathbf{x})\mu(\mathbf{x})}(\mathbf{E}(t,\mathbf{x}-\mathbf{y}/2)\cdot\hat{\mathbf{k}})\overline{(\mathbf{H}(t,\mathbf{x}+\mathbf{y}/2)\cdot\hat{\mathbf{k}})} d\mathbf{y} \\ a_{21}^0(t,\mathbf{x},\mathbf{k}) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{y}} \sqrt{\epsilon(\mathbf{x})\mu(\mathbf{x})}(\mathbf{H}(t,\mathbf{x}-\mathbf{y}/2)\cdot\hat{\mathbf{k}})\overline{(\mathbf{E}(t,\mathbf{x}+\mathbf{y}/2)\cdot\hat{\mathbf{k}})} d\mathbf{y} \\ a_{22}^0(t,\mathbf{x},\mathbf{k}) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{y}} \mu(\mathbf{x})(\mathbf{H}(t,\mathbf{x}-\mathbf{y}/2)\cdot\hat{\mathbf{k}})\overline{(\mathbf{H}(t,\mathbf{x}+\mathbf{y}/2)\cdot\hat{\mathbf{k}})} d\mathbf{y}. \end{aligned} \tag{3.73}$$

As in section 3.1, we denote the coherence matrices by $W^{\pm} = (a_{ij}^{\pm})$ and $W^0 = (a_{ij}^0)$. The latter is zero since there are no longitudinal modes. Moreover, as in the acoustic case, we have the symmetry

$$W^-(t,\mathbf{x},-\mathbf{k}) = \begin{pmatrix} W_{11}^+(\mathbf{k}) & -W_{12}^+(\mathbf{k}) \\ -W_{21}^+(\mathbf{k}) & W_{22}^+(\mathbf{k}) \end{pmatrix}. \tag{3.74}$$

Hence, by direct computation, we get the energy relation

$$\int \mathrm{Tr} W^+(t,\mathbf{x},\mathbf{k}) d\mathbf{k} = \frac{1}{2}\epsilon(\mathbf{x})|\mathbf{E}(t,\mathbf{x})|^2 + \frac{1}{2}\mu(\mathbf{x})|\mathbf{H}(t,\mathbf{x})|^2 = \mathcal{E}(t,\mathbf{x}). \tag{3.75}$$

Thus, $\mathrm{Tr}W^+(t,\mathbf{x},\mathbf{k})$ is the phase space energy density. By a similar calculation using (3.71) we find that the Poynting vector (3.70) is

$$\mathcal{F}(t,\mathbf{x}) = \mathbf{E}(t,\mathbf{x}) \times \mathbf{H}(t,\mathbf{x}) = v(\mathbf{x}) \int \hat{\mathbf{k}} \mathrm{Tr} W^+(t,\mathbf{x},\mathbf{k}) d\mathbf{k} \tag{3.76}$$

The coherence matrix $W^+(t,\mathbf{x},\mathbf{k})$ is related to the four Stokes parameters [1,20], which are commonly used for the description of polarized light because they are directly measurable. Let l and r be two directions orthogonal to the direction of propagation and let $I = I_l + I_r$ be the the total intensity of light, with I_l and I_r denoting the intensities in the directions l and r, respectively. Let $Q = I_l - I_r$ be the difference between the two intensities. Also let $U = 2 < E_l E_r \cos\delta >$ and $V = 2 < E_l E_r \sin\delta >$ denote the intensity coherence, with fixed phase shift δ, between the amplitude of light in the directions l and r, respectively. Light is unpolarized if $U = V = Q = 0$. If the directions l and r are chosen to be $\mathbf{z}^{(1)}(\mathbf{k})$ and $\mathbf{z}^{(2)}(\mathbf{k})$, given by (3.38), then the coherence matrix $W^+(t,\mathbf{x},\mathbf{k})$ is related to the Stokes parameters (I,Q,U,V) by

$$W^+(t,\mathbf{x},\mathbf{k}) = \frac{1}{2}\begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}. \tag{3.77}$$

When the light is unpolarized, then the coherence matrix W^+ is proportional to the 2×2 identity matrix I.

3.4 High Frequency Approximation for Elastic Waves

The equations of motion for small displacemets $u_i(t,\mathbf{x}),\ i = 1,2,3$ of an elastic medium are

$$\rho\frac{d^2 u_i}{dt^2} = \frac{\partial \tau_{ij}}{\partial x^j},\ i = 1,2,3. \tag{3.78}$$

Here $\rho(\mathbf{x})$ is the density, $\tau_{ij}(t,\mathbf{x})$ is the stress tensor, which, in an isotropic medium is

$$\tau_{ij} = \lambda(\mathbf{x})\frac{\partial u_k}{\partial x^k}\delta_{ij} + \mu(\mathbf{x})(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}), \tag{3.79}$$

and $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are the Lame parameters. Equation (3.78) is then

$$\rho\frac{d^2 u_i}{dt^2} = \frac{\partial}{\partial x^i}(\lambda \mathrm{div}\mathbf{u}) + \frac{\partial}{\partial x^j}(\mu\frac{\partial u_j}{\partial x^i} + \mu\frac{\partial u_i}{\partial x^j}). \tag{3.80}$$

We now write these equations as a symmetric hyperbolic system (3.1) and apply the high frequency analysis to them.

We introduce new dependent variables by

$$p = \lambda \mathrm{div}\mathbf{u}, \qquad \xi_i = \dot{u}_i, \qquad \varepsilon_{ij} = \mu(\frac{\partial u_i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}), \tag{3.81}$$

where dot stands for derivative with respect to time. Clearly p is similar to pressure, $\boldsymbol{\xi}$ is the velocity of the medium and ε_{ij} is part of the stress tensor. Equations (3.80) are equivalent to

$$\begin{aligned} \rho\dot{\xi}_i &= \frac{\partial p}{\partial x^i} + \frac{\partial \varepsilon_{ij}}{\partial x^j} \\ \dot{\varepsilon_{ij}} &= \mu(\frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i}) \\ \dot{p} &= \lambda \text{div} \boldsymbol{\xi}. \end{aligned} \tag{3.82}$$

Note that if the shear modulus μ is zero in (3.82) then $\varepsilon_{ij} = 0$ and we have the acoustic equations (3.33) for the velocity $\boldsymbol{\xi}$ and pressure p. From these variables we form the 10-vector $\mathbf{w} = (\xi_1, \xi_2, \xi_3, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}, p)$ and rewrite (3.82) as a system

$$A(\mathbf{x})\frac{\partial \mathbf{w}}{\partial t} + D^i \frac{\partial \mathbf{w}}{\partial x^i} = 0, \tag{3.83}$$

with the 10×10 matrix $A(\mathbf{x}) = \text{diag}(\rho, \rho, \rho, 1/2\mu, 1/2\mu, 1/2\mu, 1/\mu, 1/\mu, 1/\mu, 1/\lambda)$. The 10×10 matrices D^i are constant and symmetric and the dispersion matrix $L(\mathbf{x}, \mathbf{k})$ defined by (3.22) is

$$L = -\begin{pmatrix} 0 & 0 & 0 & k_1/\rho & 0 & 0 & 0 & k_3/\rho & k_2/\rho & k_1/\rho \\ 0 & 0 & 0 & 0 & k_2/\rho & 0 & k_3/\rho & 0 & k_1/\rho & k_2/\rho \\ 0 & 0 & 0 & 0 & 0 & k_3/\rho & k_2/\rho & k_1/\rho & 0 & k_3/\rho \\ 2\mu k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu k_3 & \mu k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu k_3 & 0 & \mu k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu k_2 & \mu k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda k_1 & \lambda k_2 & \lambda k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.84}$$

In block form

$$L = -\begin{pmatrix} 0 & K(\mathbf{k})/\rho & M(\mathbf{k})/\rho & \frac{1}{\rho}\mathbf{k} \\ 2\mu K(\mathbf{k}) & 0 & 0 & 0 \\ \mu M(\mathbf{k}) & 0 & 0 & 0 \\ \lambda \mathbf{k}^t & 0 & 0 & 0 \end{pmatrix}, \tag{3.85}$$

where the matrix $K(\mathbf{k}) = \text{diag}(k_1, k_2, k_3)$ and

$$M(\mathbf{k}) = \begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix}. \tag{3.86}$$

The matrix $M(\mathbf{k})$ is a symmetrized version of the matrix $P(\mathbf{k})$ in (3.64) that appears in Maxwell's equations.

The eigenvalues of the dispersion matrix L are

$$\begin{aligned} \omega_0 &= 0 \text{ with multiplicity four,} \\ \omega_\pm^P &= \pm v_P|\mathbf{k}| \text{ each with multiplicity one,} \\ \omega_\pm^S &= \pm v_S|\mathbf{k}| \text{ each with multiplicity two,} \end{aligned} \tag{3.87}$$

with the corresponding compressional and shear speeds given by

$$v_P = \sqrt{(2\mu+\lambda)/\rho}\ , \quad v_S = \sqrt{\mu/\rho}. \tag{3.88}$$

The eigenvectors of the dispersion matrix are orthonormal with respect to the inner product $<,>_A$, defined in (3.6), and are given by

$$\begin{aligned} \mathbf{b}_\pm^P &= (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, \mp\frac{2\mu K(\hat{\mathbf{k}})\hat{\mathbf{k}}}{\sqrt{2(2\mu+\lambda)}}, \mp\frac{\mu M(\hat{\mathbf{k}})\hat{\mathbf{k}}}{\sqrt{2(2\mu+\lambda)}}, \mp\frac{\lambda}{\sqrt{2(2\mu+\lambda)}}) \\ \mathbf{b}_\pm^{Sj} &= (\frac{\mathbf{z}^{(j)}}{\sqrt{2\rho}}, \mp\frac{2\sqrt{\mu}K(\hat{\mathbf{k}})\mathbf{z}^{(j)}}{\sqrt{2}}, \mp\frac{\sqrt{\mu}M(\hat{\mathbf{k}})\mathbf{z}^{(j)}}{\sqrt{2}}, 0), \qquad j=1,2 \\ \mathbf{b}^{0j} &= (0, \sqrt{2\mu}K(\mathbf{z}^{(j)})\mathbf{z}^{(j)}, \sqrt{\frac{\mu}{2}}M(\mathbf{z}^{(j)})\mathbf{z}^{(j)}, 0), \qquad j=1,2 \\ \mathbf{b}^{03} &= (0, 2\sqrt{\mu}K(\mathbf{z}^{(1)})\mathbf{z}^{(2)}, \sqrt{\mu}M(\mathbf{z}^{(1)})\mathbf{z}^{(2)}, 0) \\ \mathbf{b}^{04} &= (0, \frac{2\sqrt{\lambda\mu}K(\hat{\mathbf{k}})\hat{\mathbf{k}}}{\sqrt{2(\lambda+2\mu)}}, \sqrt{\frac{\lambda\mu}{2(\lambda+2\mu)}}M(\hat{\mathbf{k}})\hat{\mathbf{k}}, -\frac{2\sqrt{\lambda\mu}}{\sqrt{2(\lambda+2\mu)}}). \end{aligned} \tag{3.89}$$

The orthonormal triple $\hat{\mathbf{k}}, \mathbf{z}^{(1)}(\mathbf{k}), \mathbf{z}^{(2)}(\mathbf{k})$ is defined by (3.38). The eigenvectors $\mathbf{b}_\pm^P$ represent longitudinal or compressional modes, the P waves. They are similar to the acoustic longitudinal modes and if $\mu = 0$ then $\mathbf{b}_\pm^P$ is equivalent to the vector $\mathbf{b}^\mp$ for acoustics (3.37). The eigenvectors $\mathbf{b}_\pm^{Sj}$ represent transverse or shear waves, the S waves. They are similar to the eigenvectors (3.66) in Maxwell's equations, because they correspond to transverse waves admitting two states of polarization. The eigenvectors $\mathbf{b}^{0j}$, $j = 1, \ldots 4$ correspond to non-propagating modes.

The energy density for elastic waves is given by

$$\mathcal{E}(t,\mathbf{x}) = \frac{1}{2}\rho(\mathbf{x})|\dot{\mathbf{u}}(t,\mathbf{x})|^2 + \frac{1}{2}\lambda(\mathbf{x})(\mathrm{div}\,\mathbf{u}(\mathbf{x}))^2 + \frac{1}{2}\mu(\mathbf{x})\mathrm{Tr}(\nabla\mathbf{u}(t,\mathbf{x}) + \nabla^t\mathbf{u}(t,\mathbf{x}))^2. \tag{3.90}$$

The first term is the kinetic energy and the sum of the last two terms is the strain energy. The energy flux of the elastic waves is

$$\boldsymbol{\mathcal{F}}(t,\mathbf{x}) = \{\lambda\mathrm{div}\,\mathbf{u}(\mathbf{x})) + \mu(\mathbf{x})(\nabla\mathbf{u}(t,\mathbf{x}) + \nabla^t\mathbf{u}(t,\mathbf{x}))\}\dot{\mathbf{u}}(t,\mathbf{x}), \tag{3.91}$$

which in view of (3.79) is also

$$\boldsymbol{\mathcal{F}}(t,\mathbf{x}) = \boldsymbol{\tau}(t,\mathbf{x})\dot{\mathbf{u}}(t,\mathbf{x}).$$

The *unscaled* amplitudes $a^P_\pm(t,\mathbf{x},\mathbf{k})$ are

$$a^P_\pm = \left(\frac{1}{2\pi}\right)^3 \int e^{i\mathbf{k}\cdot\mathbf{y}} f^P_\pm(t,\mathbf{x},\mathbf{x}-\mathbf{y}/2,\mathbf{k})\bar{f}^P_\pm(t,\mathbf{x},\mathbf{x}+\mathbf{y}/2,\mathbf{k})d\mathbf{y}, \tag{3.92}$$

where

$$f^P_\pm(t,\mathbf{x},\mathbf{z},\mathbf{k}) = <\underline{\mathbf{u}}(t,\mathbf{z}),\mathbf{b}^{P\pm}(\mathbf{x},\mathbf{k})>_A = \sqrt{\frac{\rho(\mathbf{x})}{2}}(\hat{\mathbf{k}}\cdot\dot{\mathbf{u}}(t,\mathbf{z})) \mp$$
$$\frac{\mu(\mathbf{x})}{\sqrt{2(2\mu(\mathbf{x})+\lambda(\mathbf{x}))}}(\hat{\mathbf{k}}\cdot(\nabla\mathbf{u}(t,\mathbf{z})+\nabla^t\mathbf{u}(t,\mathbf{z}))\hat{\mathbf{k}}) \mp \frac{\lambda(\mathbf{x})\mathrm{div}\,\mathbf{u}(t,\mathbf{z})}{\sqrt{2(2\mu(\mathbf{x})+\lambda(\mathbf{x}))}}.$$

The 2×2 coherence matrices $W^S_\pm$ for the S waves are

$$W^S_{\pm ij}(t,\mathbf{x},\mathbf{k}) = \left(\frac{1}{2\pi}\right)^3 \int e^{i\mathbf{k}\cdot\mathbf{y}} f^{S\pm}_i(t,\mathbf{x},\mathbf{x}-\mathbf{y}/2,\mathbf{k})\bar{f}^{S\pm}_j(t,\mathbf{x},\mathbf{x}+\mathbf{y}/2,\mathbf{k})d\mathbf{y}, \tag{3.93}$$

where

$$f^{S\pm}_i(t,\mathbf{x},\mathbf{z},\mathbf{k}) = \sqrt{\frac{\rho(\mathbf{x})}{2}}(\mathbf{z}^{(i)}(\mathbf{k})\cdot\dot{\mathbf{u}}(t,\mathbf{z})) \mp \sqrt{\frac{\mu(\mathbf{x})}{2}}(\hat{\mathbf{k}}\cdot(\nabla\mathbf{u}(\mathbf{z})+\nabla^t u(\mathbf{z}))\mathbf{z}^{(i)}(\mathbf{k})).$$

The entries of the 4×4 coherence matrix for the nonpropagating modes are

$$a^0_{ij}(t,\mathbf{x},\mathbf{k}) = \left(\frac{1}{2\pi}\right)^3 \int e^{i\mathbf{k}\cdot\mathbf{y}} f^0_i(t,\mathbf{x},\mathbf{x}-\mathbf{y}/2,\mathbf{k})\bar{f}^0_j(t,\mathbf{x},\mathbf{x}+\mathbf{y}/2,\mathbf{k})d\mathbf{y}, \tag{3.94}$$

where

$$\begin{aligned}
f^0_j(t,\mathbf{x},\mathbf{z},\mathbf{k}) &= \sqrt{\frac{\mu(\mathbf{x})}{2}}(\mathbf{z}^{(j)}(\mathbf{k})\cdot(\nabla\mathbf{u}(t,\mathbf{z})+\nabla^t\mathbf{u}(t,\mathbf{z}))\mathbf{z}^{(j)}(\mathbf{k})), \qquad j=1,2\\
f^0_3(t,\mathbf{x},\mathbf{z},\mathbf{k}) &= \sqrt{\mu(\mathbf{x})}(\mathbf{z}^{(1)}(\mathbf{k})\cdot(\nabla\mathbf{u}(t,\mathbf{z})+\nabla^t\mathbf{u}(t,\mathbf{z}))\mathbf{z}^{(2)}(\mathbf{k}))\\
f^0_4(t,\mathbf{x},\mathbf{z},\mathbf{k}) &= \sqrt{\frac{\lambda(\mathbf{x})\mu(x)}{2(\lambda(\mathbf{x})+2\mu(\mathbf{x}))}}(\hat{\mathbf{k}}\cdot(\nabla\mathbf{u}(t,\mathbf{z})+\nabla^t\mathbf{u}(t,\mathbf{z}))\hat{\mathbf{k}}) - \frac{2\sqrt{\lambda(\mathbf{x})\mu(\mathbf{x})}\mathrm{div}\,\mathbf{u}(t,\mathbf{z})}{\sqrt{2(2\mu(\mathbf{x})+\lambda(\mathbf{x}))}}.
\end{aligned}$$

Note that (3.92) implies that the amplitudes a^P_+ and a^P_- are related by

$$a^P_+(t,\mathbf{x},\mathbf{k}) = a^P_-(t,\mathbf{x},-\mathbf{k}), \tag{3.95}$$

which is analogous to (3.43), while the coherence matrices W^S_+ and W^S_- are related by the analog of (3.74) and

$$\mathrm{Tr}W^S_+(t,\mathbf{x},\mathbf{k}) = \mathrm{Tr}W^S_-(t,\mathbf{x},-\mathbf{k}). \tag{3.96}$$

A direct calculation using (3.92-3.94) shows that the energy density (3.90) is

$$\mathcal{E}(t,\mathbf{x}) = \int (a_+^P + \mathrm{Tr} W_+^S) d\mathbf{k} + \frac{1}{2}\int \sum_{i=1}^{4} a_{ii}^0 d\mathbf{k}. \tag{3.97}$$

The first term is the energy density of the P and S waves while the second is the energy of the zero velocity waves. The flux (3.91) is

$$\boldsymbol{\mathcal{F}}(t,\mathbf{x}) = \int \hat{\mathbf{k}}[v_P a_+^P(t,\mathbf{x},\mathbf{k}) + v_S \mathrm{Tr} W_+^S(t,\mathbf{x},\mathbf{k})] d\mathbf{k}. \tag{3.98}$$

Using the eigenvalues (3.87) and (3.88) in (3.25) and (3.29) we obtain the transport equations for the scalar amplitude a_+^P and the coherence matrix W_+^S:

$$\frac{\partial a_+^P}{\partial t} + v_P(\mathbf{x})\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a_+^P - |\mathbf{k}| \nabla_{\mathbf{x}} v_P(\mathbf{x}) \cdot \nabla_{\mathbf{k}} a_+^P = 0 \tag{3.99}$$

$$\frac{\partial W_+^S}{\partial t} + v_S(\mathbf{x})\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W_+^S - |\mathbf{k}| \nabla_{\mathbf{x}} v_S(\mathbf{x}) \cdot \nabla_{\mathbf{k}} W_+^S + W_+^S N - N W_+^S = 0. \tag{3.100}$$

The coupling matrix $N(\mathbf{x},\mathbf{k})$ is exactly the same as in the case of Maxwell's equations (3.67) with the speed $v = v_S$. In the high frequency limit the longitudinal P waves behave exactly like acoustic waves. This is because in both cases the waves correspond to a simple eigenvalue of the dispersion matrix. The S waves behave exactly like electromagnetic waves. The same results were obtained in [44] by ray methods.

4 Waves in Random Media

4.1 Transport Equations without Polarization

We now consider wave propagation in a slowly varying background with small random perturbations. The symmetric hyperbolic system (3.1) is

$$A(\mathbf{x})\{I + \varepsilon^{1/2} V(\frac{\mathbf{x}}{\varepsilon})\} \frac{\partial \mathbf{u}}{\partial t} + D^j \frac{\partial \mathbf{u}}{\partial x^j} = 0, \tag{4.1}$$

where $V(\mathbf{x})$ is a statistically homogeneous matrix-valued random process with mean zero that models the parameter fluctuations. The scale of variation of the fluctuations is of order ε and therefore comparable to the wave length so that the random inhomogeneities can interact fully with the propagating waves. The magnitude $\sqrt{\varepsilon}$ of the fluctuations is chosen, as in the case of the Schrödinger equation (2.30), so that the effect of scattering by the inhomogeneites be comparable to the effect of the slowly varying background. In order that the system (4.1) remain symmetric hyperbolic the random inhomogeneities must satisfy the condition

$$A(\mathbf{x})V(\mathbf{y}) = V^*(\mathbf{y})A(\mathbf{x}). \tag{4.2}$$

for all $\mathbf{x}$ and $\mathbf{y}$, which implies conservation of energy. The matrices A and D^j are symmetric and A is positive definite. In all three cases considered here – acoustic, electromagnetic and elastic waves – condition (4.2) is satisfied. In this section we will assume that the dispersion matrix (3.22) for the deterministic background has simple eigenvalues. The case of polarization (multiple eigenvalues) is considered in the next section.

The covariance functions $R_{ijkl}(\mathbf{x})$ and the power spectral densities $\hat{R}_{ijkl}(\mathbf{k})$ are defined by

$$R_{ijkl}(\mathbf{x}) = \langle V_{ij}(\mathbf{y})V_{kl}(\mathbf{x}+\mathbf{y})\rangle = \int e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{R}_{ijkl}(\mathbf{p})d\mathbf{p}, \tag{4.3}$$

where $<,>$ denotes statistical average. Spatial homogeneity implies

$$\langle\widehat{V_{ij}}(\mathbf{p})\widehat{V_{kl}}(\mathbf{q})\rangle = \hat{R}_{ijkl}(\mathbf{p})\delta(\mathbf{p}+\mathbf{q}). \tag{4.4}$$

and

$$\hat{R}_{ijkl}(\mathbf{p}) = \hat{R}_{klij}(-\mathbf{p}). \tag{4.5}$$

We assume that the power spectral densities $\hat{R}_{ijkl}(\mathbf{p})$ are real, which is equivalent to

$$\hat{R}_{ijkl}(\mathbf{p}) = \hat{R}_{ijkl}(-\mathbf{p}) \tag{4.6}$$

and holds when the covariance functions $R_{ijkl}(\mathbf{x})$ are even . This is the case when the fluctuations are isotrpopic in space, that is

$$R_{ijkl}(\mathbf{x}) = R_{ijkl}(|\mathbf{x}|). \tag{4.7}$$

The symmetry condition (4.2) implies that the matrix A and the covariance tensor R_{ijkl} satisfy the relations

$$A_{ni}A_{mk}R_{ijkl} = A_{ji}A_{mk}R_{inkl} = A_{ji}A_{lk}R_{inkm}. \tag{4.8}$$

When (4.1) holds, the evolution equation (3.13) for W^ε has the form

$$\frac{\partial W^\varepsilon}{\partial t} + \mathcal{Q}_1^\varepsilon W^\varepsilon + \frac{1}{\varepsilon}\mathcal{Q}_2^\varepsilon W^\varepsilon - \frac{1}{\sqrt{\varepsilon}}\mathcal{P}_2^\varepsilon W^\varepsilon - \sqrt{\varepsilon}\mathcal{P}_1^\varepsilon W^\varepsilon = 0, \tag{4.9}$$

where the operators $\mathcal{Q}_1^\varepsilon$ and $\mathcal{Q}_2^\varepsilon$ are defined by (3.14) and (3.15). The operators $\mathcal{P}_1^\varepsilon$ and $\mathcal{P}_2^\varepsilon$ come from the random inhomogeneites and are given by

$$\mathcal{P}_1^\varepsilon W^\varepsilon = \frac{1}{2}\int\!\!\int \frac{e^{i\mathbf{q}\cdot\mathbf{y}}\,d\mathbf{y}\,d\mathbf{q}}{(2\pi)^d}\left\{ V(\frac{\mathbf{x}}{\varepsilon}+\mathbf{y})A^{-1}(\mathbf{x}+\varepsilon\mathbf{y})D^j\frac{\partial W^\varepsilon(\mathbf{k}+\mathbf{p}/2)}{\partial x^j} + \frac{\partial W^\varepsilon(\mathbf{k}-\mathbf{p}/2)}{\partial x^j}D^jA^{-1}(\mathbf{x}+\varepsilon\mathbf{y})V^*(\frac{\mathbf{x}}{\varepsilon}+\mathbf{y})\right\} \tag{4.10}$$

and

$$\mathcal{P}_2^\varepsilon W^\varepsilon = i\int\!\!\int \frac{e^{i\mathbf{q}\cdot\mathbf{y}}\,d\mathbf{y}\,d\mathbf{q}}{(2\pi)^d}\left\{ (k_j+\frac{q_j}{2})V(\frac{\mathbf{x}}{\varepsilon}+\mathbf{y})A^{-1}(\mathbf{x}+\varepsilon\mathbf{y})W^\varepsilon(\mathbf{k}+\mathbf{q}/2) - W^\varepsilon(\mathbf{k}-\mathbf{q}/2)(k_j-\frac{q_j}{2})D^jA^{-1}(\mathbf{x}+\varepsilon\mathbf{y})V^*(\frac{\mathbf{x}}{\varepsilon}+\mathbf{y})\right\}. \tag{4.11}$$

The double integrals enter in (4.10) and (4.11) because we inserted the Fourier transform $\hat{V}$ into (3.14) and (3.15). The operator $\mathcal{P}_1^\varepsilon$ corresponds to the terms in (3.14) involving the $\mathbf{x}$-gradient of W^ε, while the undifferentiated terms in (3.14) and (3.15) combine to produce the operator $\mathcal{P}_2^\varepsilon$.

We analyze equation (4.9) by a multiple scales expansion, following section 2.3 and Appendix. We introduce the fast space variable $\boldsymbol{\xi} = \mathbf{x}/\varepsilon$ and the expansion

$$W^\varepsilon(t,\mathbf{x},\boldsymbol{\xi},\mathbf{k}) = W^{(0)}(t,\mathbf{x},\mathbf{k}) + \varepsilon^{1/2}W^{(1)}(t,\mathbf{x},\boldsymbol{\xi},\mathbf{k}) + \varepsilon W^{(2)}(t,\mathbf{x},\boldsymbol{\xi},\mathbf{k}) + \ldots \tag{4.12}$$

We replace $\frac{\partial}{\partial x^i}$ by

$$\frac{\partial}{\partial x^i} + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi^i} \tag{4.13}$$

and expand the $\mathcal{Q}$ and $\mathcal{P}$ operators in powers of ε:

$$\begin{aligned}
\mathcal{Q}_1^\varepsilon &= \frac{1}{\varepsilon}\tilde{\mathcal{Q}}_1 + \mathcal{Q}_1 + \tilde{\mathcal{Q}}_{11} + \ldots\\
\mathcal{Q}_2^\varepsilon &= \mathcal{Q}_2 + \varepsilon\mathcal{Q}_{21} + \ldots\\
\mathcal{P}_1^\varepsilon &= \frac{1}{\varepsilon}\mathcal{P}_1(\frac{\partial}{\partial\boldsymbol{\xi}}) + \mathcal{P}_1(\frac{\partial}{\partial\mathbf{x}}) + \ldots\\
\mathcal{P}_2^\varepsilon &= \mathcal{P}_2 + \ldots
\end{aligned}$$

The operator $\tilde{\mathcal{Q}}_1$ is

$$\tilde{\mathcal{Q}}_1 Z = \frac{1}{2}A^{-1}D^j\frac{\partial Z}{\partial \xi^j} + \frac{1}{2}\frac{\partial Z}{\partial \xi^j}D^jA^{-1} \tag{4.14}$$

and the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ are

$$\mathcal{P}_1 Z(\mathbf{x},\boldsymbol{\xi},\mathbf{k}) = \frac{1}{2}\int d\mathbf{q}e^{-i\mathbf{q}\cdot\boldsymbol{\xi}}\left\{ \hat{V}(\mathbf{q})A^{-1}(\mathbf{x})D^j\frac{\partial Z(\mathbf{k}+\mathbf{q}/2)}{\partial x^j} + \frac{\partial Z(\mathbf{k}-\mathbf{q}/2)}{\partial x^j}D^jA^{-1}(\mathbf{x})\widehat{V^*}(\mathbf{q})\right\} \tag{4.15}$$

and

$$\mathcal{P}_2 Z(\mathbf{x},\boldsymbol{\xi},\mathbf{k}) = i\int d\mathbf{q} e^{-i\mathbf{q}\cdot\boldsymbol{\xi}} \left\{ \hat{V}(\mathbf{p})A^{-1}(\mathbf{x})(k_j+q_j/2)D^j Z(\mathbf{k}+\mathbf{q}/2) \right. \\ \left. - Z(\mathbf{k}-\mathbf{q}/2)(k_j-p_j/2)D^j A^{-1}(\mathbf{x})\widehat{V^*}(\mathbf{q})\right\}. \tag{4.16}$$

We do not give an explicit expression for $\tilde{\mathcal{Q}}_{11}$ since we shall not need it. It is the first order term in the expansion in ε of the part involving the $\boldsymbol{\xi}$-gradient of the operator $\mathcal{Q}_1(\frac{\partial}{\partial\boldsymbol{\xi}})$. With these definitions, (4.9) becomes

$$\frac{\partial W^\varepsilon}{\partial t} + \left\{\frac{1}{\varepsilon}\mathcal{Q}_2 + \mathcal{Q}_{21} + \frac{1}{\varepsilon}\tilde{\mathcal{Q}}_1 + \mathcal{Q}_1 + \tilde{\mathcal{Q}}_{11} - \frac{1}{\sqrt{\varepsilon}}\mathcal{P}_2 - \frac{1}{\sqrt{\varepsilon}}\mathcal{P}_1(\frac{\partial}{\partial\boldsymbol{\xi}}) + O(\varepsilon)\right\} W^\varepsilon = 0. \tag{4.17}$$

We assume that the average of the leading term $W^{(0)}$ in the expansion (4.9) depends only on the slow space variable $\mathbf{x}$. This is discussed further in Appendix. To simplify the presentation we will assume that $W^{(0)}$ itself is independent of $\boldsymbol{\xi}$. We insert expansion (4.12) into (4.9) and find that $W^{(0)}$ satisfies

$$Q_2 W^{(0)} = 0 \tag{4.18}$$

as in (3.20). We assume in this section that all the eigenvalues of the dispersion matrix $L(\mathbf{x},\mathbf{k})$ in (3.22) are simple. The case of multiple eigenvalues is considered in section 4.2. Then the Wigner matrix $W^{(0)}$ has the form

$$W^{(0)}(t,\mathbf{x},\mathbf{k}) = \sum_{\tau=1}^{N} a^\tau(t,\mathbf{x},\mathbf{k})B^\tau(\mathbf{x},\mathbf{k}), \tag{4.19}$$

where the martrices $B^\tau(\mathbf{x},\mathbf{k})$ are defined by (3.23), as in (3.24).

The term $W^{(1)}$ satisfies

$$\mathcal{Q}_2 W^{(1)} + \tilde{\mathcal{Q}}_1 W^{(1)} = \mathcal{P}_2 W^{(0)}. \tag{4.20}$$

We insert (4.19) into (4.20) and solve this equation explicitly for $F^{(1)}(t.\mathbf{x},\mathbf{p},\mathbf{k})$, the Fourier transform in $\boldsymbol{\xi}$ of $W^{(1)}$:

$$F^{(1)} = \frac{1}{\omega_j(\mathbf{k}+\frac{\mathbf{p}}{2}) - \omega_i(\mathbf{k}-\frac{\mathbf{p}}{2}) - i\nu}\left\{\omega_i(\mathbf{k}-\frac{\mathbf{p}}{2})a^i(\mathbf{k}-\frac{\mathbf{p}}{2})c_m^j(\mathbf{k}+\frac{\mathbf{p}}{2})\hat{V}_{ml}(\mathbf{p})b_l^i(\mathbf{k}-\frac{\mathbf{p}}{2}) \right. \\ \left. - \omega_j(\mathbf{k}+\frac{\mathbf{p}}{2})a^j(\mathbf{k}+\frac{\mathbf{p}}{2})c_m^i(\mathbf{k}-\frac{\mathbf{p}}{2})\hat{V}_{ml}(\mathbf{p})b_l^j(\mathbf{k}+\frac{\mathbf{p}}{2})\right\}\mathbf{b}^i(\mathbf{k}-\frac{\mathbf{p}}{2})\mathbf{b}^{j*}(\mathbf{k}+\frac{\mathbf{p}}{2}). \tag{4.21}$$

Here the vectors $\mathbf{b}^j(\mathbf{x},\mathbf{k})$ are the right eigenvectors of the dispersion matrix $L(\mathbf{x},\mathbf{k})$, orthonomal with respect to the inner product $<,>_A$, and the vectors $\mathbf{c}^i(\mathbf{x},\mathbf{k})$ are the left eigenvectors of the dispersion matrix, given by

$$\mathbf{c}^i(\mathbf{x},\mathbf{k}) = A(\mathbf{x})\mathbf{b}^i(\mathbf{x},\mathbf{k}). \tag{4.22}$$

The second order term $W^{(2)}$ satisfies the equation

$$\mathcal{Q}_2 W^{(2)} + \tilde{\mathcal{Q}}_1 W^{(2)} = -\frac{\partial W^{(0)}}{\partial t} - \mathcal{Q}_{21} W^{(0)} - \mathcal{Q}_1 W^{(0)} + \mathcal{P}_2 W^{(1)} + \mathcal{P}_1(\frac{\partial}{\partial \boldsymbol{\xi}}) W^{(1)}, \tag{4.23}$$

because $\tilde{\mathcal{Q}}_{11} W^{(0)} = 0$ since $W^{(0)}$ is independent of $\boldsymbol{\xi}$. As discussed in Appendix for the analogous situation for the Schrödinger equation, the average

$$< \tilde{\mathcal{Q}}_1 W^{(2)} >= 0$$

and so the average of the right side of (4.23) is orthogonal to the null space of $\mathcal{Q}_2$. We insert expression (4.21) for $W^{(1)}$ into (4.23), average it and obtain from the orthogonality condition that the amplitudes a^τ satisfy the radiative transport equations

$$\frac{\partial a^\tau}{\partial t} + \nabla_{\mathbf{k}}\omega_\tau \cdot \nabla_{\mathbf{x}} a^\tau - \nabla_{\mathbf{x}}\omega_\tau \cdot \nabla_{\mathbf{k}} a^\tau = \int \sigma_{\tau i}(\mathbf{k},\mathbf{k}') a^i(\mathbf{k}') d\mathbf{k}' - \Sigma_\tau(\mathbf{k}) a^\tau(\mathbf{k}). \tag{4.24}$$

The differential scattering cross-sections $\sigma_{\tau i}(\mathbf{k},\mathbf{k}')$ and the total scattering cross-sections $\Sigma_\tau(\mathbf{k})$ are given by

$$\sigma_{\tau i}(\mathbf{k},\mathbf{k}') = 2\pi\omega_\tau^2(\mathbf{k}) c_s^\tau(\mathbf{k}) c_l^\tau(\mathbf{k}) b_v^i(\mathbf{k}') b_w^i(\mathbf{k}') \hat{R}_{svlw}(\mathbf{k}-\mathbf{k}') \delta(\omega_\tau(\mathbf{k}) - \omega_i(\mathbf{k}')) \tag{4.25}$$

and

$$\Sigma_\tau(\mathbf{k}) = \sum_i \int \sigma_{\tau i}(\mathbf{k},\mathbf{k}') d\mathbf{k}' \tag{4.26}$$

Equation (4.24) has the form (1.1). The scattering cross-sections $\sigma_{\tau i}(\mathbf{k},\mathbf{k}')$ defined by (4.25) are always positive because the power spectral densities $\hat{R}_{ijkl}(\mathbf{k})$ are positive definite matrices with respect to the pairs of indices ik and jl, by Bochner's theorem [45]. Two modes generated by the eigenvalues ω_i and ω_j are coupled only if ω_i and ω_j coincide for some values of the wave vectors $\mathbf{k}$, $\mathbf{k}'$, that is if for a fixed $\mathbf{k}$ there exists a hypersurface of solutions $\mathbf{k}'$ to the equation

$$\omega_\tau(\mathbf{k}) = \omega_i(\mathbf{k}'). \tag{4.27}$$

If there is scattering between two modes then the symmetries (4.5), (4.6) and (4.2), and (4.25) imply that the differential scattering cross-sections of the direct and reverse scattering processes are the same, i.e.,

$$\sigma_{\tau i}(\mathbf{k},\mathbf{k}') = \sigma_{i\tau}(\mathbf{k}',\mathbf{k}). \tag{4.28}$$

This implies that the total energy

$$E(t) = \int\!\!\int \sum_{j=1}^{N} a^j(t,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} \tag{4.29}$$

is conserved.

4.2 Transport Equations with Polarization

When the eigenvalues of the dispersion matrix $L(\mathbf{x},\mathbf{k})$ have multiplicities greater than one the perturbation analysis of the previous section must be modified. Equation (4.18) implies that the Wigner matrix $W^{(0)}$ has the form

$$W^{(0)}(t,\mathbf{x},\mathbf{k}) = \sum_{\tau,i,j} a^{\tau}_{ij}(t,\mathbf{x},\mathbf{k})B^{\tau,ij}(\mathbf{x},\mathbf{k}) \tag{4.30}$$

where the matrices $B^{\tau,ij}$ are defined by (3.26), as in (3.27). We define the coherence matrices $W^{\tau}(t,\mathbf{x},\mathbf{k})$ as in (3.28) by

$$W^{\tau}_{ij} = a^{\tau}_{ij}. \tag{4.31}$$

We express $W^{(1)}$ through the coherence matrix using (4.20) and insert it into (4.23). We average (4.23) and use the orthogonality conditions to obtain the radiative transport equations for the coherence matrices

$$\begin{aligned}&\frac{\partial W^{\tau}}{\partial t} + \nabla_{\mathbf{k}}\omega_{\tau}\cdot\nabla_{\mathbf{x}}W^{\tau} - \nabla_{\mathbf{x}}\omega_{\tau}\cdot\nabla_{\mathbf{k}}W^{\tau} + W^{\tau}N^{\tau} - N^{\tau}W^{\tau} \\ &\quad = \int \sigma^{\tau i}(\mathbf{k},\mathbf{k}')[W^{i}(\mathbf{k}')]\delta(\omega_i(\mathbf{k}')-\omega_{\tau}(\mathbf{k}))d\mathbf{k}' - \Sigma^{\tau}(\mathbf{k})W^{\tau}(\mathbf{k}) - W^{\tau}(\mathbf{k})\Sigma^{\tau *}(\mathbf{k}).\end{aligned} \tag{4.32}$$

The differential scattering cross-section matrix is

$$\left(\sigma^{\tau i}(\mathbf{k},\mathbf{k}')[W^{i}(\mathbf{k}')]\right)_{mj} = 2\pi\omega_{\tau}^2(\mathbf{k})b_v^{i,q}(\mathbf{k}')b_w^{i,r}(\mathbf{k}')c_l^{\tau,j}(\mathbf{k})c_s^{\tau,m}(\mathbf{k})\hat{R}_{svlw}(\mathbf{k}-\mathbf{k}')W^{i}_{qr}(\mathbf{k}') \tag{4.33}$$

and the total scattering cross-section matrix Σ^{τ} is

$$\Sigma^{\tau} = \frac{1}{2}\sum_{j}\int \sigma^{\tau j}(\mathbf{k},\mathbf{k}')[I]\delta(\omega_{\tau}(\mathbf{k})-\omega_i(\mathbf{k}'))d\mathbf{k}' - \frac{i}{2}\int \frac{1}{\omega_{\tau}(\mathbf{k})-\omega_i(\mathbf{k}')}\sigma^{\tau j}(\mathbf{k},\mathbf{k}')[I]d\mathbf{k}'. \tag{4.34}$$

The singular integrals in (4.34) should be interpreted in the principal value sense. The imaginary terms in (4.34) are related to the anisotropy of the random perturbations. We will see in particular examples that they are absent when the random perturbations are isotropic.

The radiative transport equations (4.32) preserve W^j as positive definite Hermitian matrices; that is if all the $W^j(0, \mathbf{x}, \mathbf{k})$ are Hermitian and positive definite then $W^j(t, \mathbf{x}, \mathbf{k})$ is Hermitian and postive definite for $t > 0$ and all j. Another important property of equations (4.32) is that they conserve the total energy

$$E(t) = \sum_j \int\int \mathrm{Tr} W^j(t, \mathbf{x}, \mathbf{k}) d\mathbf{x} d\mathbf{k} = \text{const.} \tag{4.35}$$

4.3 Transport Equations for Acoustic Waves

We will now apply the results of section 4.1 to the acoustic equations (3.33). The symmetric hyperbolic system for acoustic waves has simple structure because all the non-zero speeds of propagation are distinct and there is no scattering between different modes, even in the presence of random inhomogeneities. This is because the frequency (3.35) $\omega_+(\mathbf{k})$ is always positive and the frequency $\omega_-(\mathbf{k})$ is negative for all $\mathbf{k} \neq 0$ and so the radiative transport equations (4.24) for the amplitudes a^+ and a^- are decoupled from each other. Moreover, these amplitudes are related by (3.43) and so we consider only $a^+(t, \mathbf{x}, \mathbf{k})$, which we denote by $a(t, \mathbf{x}, \mathbf{k})$.

The perturbed matrix A of the symmetric hyperbolic system (3.33) is

$$\begin{pmatrix} \rho I & 0 \\ 0 & \kappa \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{\varepsilon} \begin{pmatrix} \tilde{\rho} I & 0 \\ 0 & \tilde{\kappa} \end{pmatrix} \right] \tag{4.36}$$

where I is the 3×3 identity matrix and $\tilde{\rho}$ and $\tilde{\kappa}$ are the fluctuations in the density and compressibility, respectively. Therefore the power spectral densities $\hat{R}_{svlw}(\mathbf{p})$ in (4.3) have therefore the form

$$\begin{aligned} \hat{R}_{svlw}(\mathbf{p}) = {} & \delta_{sv}\delta_{lw}\delta_{s\le 3}\delta_{l\le 3}\hat{R}_{\rho\rho}(\mathbf{p}) + \delta_{sv}\delta_{s\le 3}\delta_{lw}\delta_{l,4}\hat{R}_{\rho\kappa}(\mathbf{p}) \\ & + \delta_{sv}\delta_{s,4}\delta_{lw}\delta_{l,4}\hat{R}_{\kappa\kappa}(\mathbf{p}) + \delta_{sv}\delta_{s,4}\delta_{lw}\delta_{l\le 3}\hat{R}_{\rho\kappa}(\mathbf{p}). \end{aligned} \tag{4.37}$$

Here $\hat{R}_{\rho\rho}$, $\hat{R}_{\rho\kappa}$, $\hat{R}_{\kappa\kappa}$ are the power spectral densities of the fluctuations of the density ρ and compressibility κ. The indices go from 1 to 4 and we use the notation $\delta_{l\le 3}$ which is equal to one if $l \le 3$ and to zero otherwise.

We insert into (4.25) the expression (4.37) for the power spectral densities, the eigenvalues (3.35) and the eigenvectors (3.37) and obtain for the phase space energy density $a(t, \mathbf{x}, \mathbf{k})$ the radiative transport equation (4.24) in the form

$$\frac{\partial a}{\partial t} + v\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} a - |\mathbf{k}|\nabla_{\mathbf{x}} v\cdot\nabla_{\mathbf{k}} a = \frac{\pi v^2|\mathbf{k}|^2}{2}\int \delta(v|\mathbf{k}| - v|\mathbf{k}'|)[a(\mathbf{k}') - a(\mathbf{k})]$$
$$\cdot\left\{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \hat{R}_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\right\} d\mathbf{k}'. \tag{4.38}$$

This is equation (1.1) with the scattering cross-section as in (1.3). It is also similar to the radiative transport equation (2.34) for the Schrödinger equation but the scattering cross-sections differ.

4.4 Transport Equations for Electromagnetic Waves

Electromagnetic waves are polarized so propagation of wave energy is described by the coherence matrices $W^+(t,\mathbf{x},\mathbf{k})$ and $W^-(t,\mathbf{x},\mathbf{k})$ that satisy the relation (3.74). Note that the frequency $\omega_+(\mathbf{x},\mathbf{k}) = v(\mathbf{x})|\mathbf{k}|$, with v given by (3.65), is always positive while the frequency $\omega_-(\mathbf{x},\mathbf{k}) = -v(\mathbf{x})|\mathbf{k}|$ is always negative. According to (4.32) this implies that the radiative transport equations for the coherence matrices W^+ and W^- are not coupled so we consider only the radiative transport equation for W^+ and drop the superscript $+$.

We assume that the random fluctuations of the medium properties are isotropic with perturbed A matrix in (3.61) given by

$$\begin{pmatrix}\epsilon I & 0\\ 0 & \mu I\end{pmatrix}\left[\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} + \sqrt{\varepsilon}\begin{pmatrix}\tilde{\epsilon} I & 0\\ 0 & \tilde{\mu} I\end{pmatrix}\right].$$

Here I is the 3×3 identity matrix and $\tilde{\epsilon}$ and $\tilde{\mu}$ are the fluctuations in the dielectric permittivity and the magnetic permeability, respectively. The power spectral densities of the fluctuations (4.3), $\hat{R}_{svlw}(\mathbf{k})$, have the form

$$\hat{R}_{svlw}(\mathbf{k}) = \delta_{sv}\delta_{lw}\delta_{s\le 3}\delta_{w\le 3}\hat{R}_{\epsilon\epsilon}(|\mathbf{k}|) + \delta_{sv}\delta_{lw}\delta_{s\le 3}\delta_{w\ge 4}\hat{R}_{\epsilon\mu}(|\mathbf{k}|) +$$
$$\delta_{sv}\delta_{lw}\delta_{s\ge 4}\delta_{w\le 3}\hat{R}_{\epsilon\mu}(|\mathbf{k}|) + \delta_{sv}\delta_{lw}\delta_{s\ge 4}\delta_{w\ge 4}\hat{R}_{\mu\mu}(|\mathbf{k}|), \tag{4.39}$$

where $\hat{R}_{ij}(\mathbf{k})$, $i,j=\epsilon,\mu$ are the power spectral densities of the fluctuations of ϵ and μ. In (4.39) the indices run from 1 to 6 and we use the delta notation as in (4.37).

We introduce the 2×2 matrices $T(\mathbf{k},\mathbf{k}')$ and $X(\mathbf{k},\mathbf{k}')$ by

$$T_{ij}(\mathbf{k},\mathbf{p}) = \mathbf{z}^{(i)}(\mathbf{k})\cdot\mathbf{z}^{(j)}(\mathbf{p}) \tag{4.40}$$

and

$$X_{ij} = \tilde{\mathbf{z}}^{(i)}(\mathbf{k})\cdot\tilde{\mathbf{z}}^{(j)}(\mathbf{k}), \tag{4.41}$$

where the vectors $\mathbf{z}^{(i)}(\mathbf{k})$ are given by (3.38), and $\tilde{\mathbf{z}}^{(1)}(\mathbf{k}) = -\mathbf{z}^{(2)}(\mathbf{k})$ and $\tilde{\mathbf{z}}^{(2)}(\mathbf{k}) = \mathbf{z}^{(1)}(\mathbf{k})$. These matrices are related by

$$T(\mathbf{k},\mathbf{p})X^*(\mathbf{k},\mathbf{p}) = (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})I \tag{4.42}$$

where I denotes 2×2 matrix. Moreover

$$\begin{aligned} T^*(\mathbf{k},\mathbf{p}) &= T(\mathbf{p},\mathbf{k}) \\ X^*(\mathbf{k},\mathbf{p}) &= X(\mathbf{p},\mathbf{k}). \end{aligned} \tag{4.43}$$

We now calculate the scattering cross-sections in terms of the matrices T and X and the power spectral densities by using in the general formulas (4.33) and (4.34), the eigenvalues and eigenvectors (3.66) and the power spectral densities (4.39). The power spectral density tensor (4.39) has four terms and each one generates a term in the differential scattering cross-section. The one with $\hat{R}_{\epsilon\epsilon}$ is

$$\begin{aligned} \sigma_1(\mathbf{k},\mathbf{k}')[W(\mathbf{k}')]_{mj} &= 2\pi v^2|\mathbf{k}|^2\sqrt{\frac{1}{2\epsilon}}z_v^{(q)}(\mathbf{k}')\sqrt{\frac{1}{2\epsilon}}z_w^{(r)}(\mathbf{k}')\sqrt{\frac{\epsilon}{2}}z_w^{(j)}(\mathbf{k})\sqrt{\frac{\epsilon}{2}}z_v^{(m)}(\mathbf{k})W_{qr}(\mathbf{k}') \\ &\quad\cdot \hat{R}_{\epsilon\epsilon}(\mathbf{k}-\mathbf{k}') \\ &= \frac{\pi v^2|\mathbf{k}|^2}{2}\hat{R}_{\epsilon\epsilon}(\mathbf{k}-\mathbf{k}')T_{mq}(\mathbf{k},\mathbf{k}')W_{qr}(\mathbf{k}')T_{rj}(\mathbf{k}',\mathbf{k}) \end{aligned} \tag{4.44}$$

The other terms in the scattering cross-section are calculated in the same way and they yield

$$\begin{aligned} \sigma[W](\mathbf{k},\mathbf{k}') = \frac{\pi v^2|\mathbf{k}|^2}{2}\Big\{ &\hat{R}_{\epsilon\epsilon}(|\mathbf{k}-\mathbf{k}'|)T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k}) \\ &+\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)X(\mathbf{k},\mathbf{k}')W(\mathbf{k}')X(\mathbf{k}',\mathbf{k}) \\ &+\hat{R}_{\epsilon\mu}(|\mathbf{k}-\mathbf{k}'|)[T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')X(\mathbf{k}',\mathbf{k}) + X(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k})]\Big\}. \end{aligned} \tag{4.45}$$

This differential scattering cross-section has the correct structure so that the radiative transport equation (4.47) below conserves the Hermitian and positive definite properties of the coherence matrix W.

By direct calculation we find that $\int \sigma(\mathbf{k},\mathbf{k}')[I]d\Omega(\hat{\mathbf{p}})$ is proportional to the identity matrix and the imaginary terms in (4.34) vanish. The total scattering cross-section $\Sigma(\mathbf{k})$ is therefore

$$\Sigma(|\mathbf{k}|) = \frac{\pi^2|\mathbf{k}|^4}{2\sqrt{\epsilon\mu}}\int_{-1}^{1}[(\hat{R}_{\epsilon\epsilon}(|\mathbf{k}|\sqrt{2-2\eta}) + \hat{R}_{\mu\mu}(|\mathbf{k}|\sqrt{2-2\eta}))(1+\eta^2) + 4\eta\hat{R}_{\epsilon\mu}(|\mathbf{k}|\sqrt{2-2\eta})]d\eta. \tag{4.46}$$

Thus the radiative transport equation (4.32) for the coherence matrix W is

$$\frac{\partial W}{\partial t} + v\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}W - |\mathbf{k}|\nabla_{\mathbf{x}}v\cdot\nabla_{\mathbf{k}}W + WN - NW$$

$$= \frac{\pi|\mathbf{k}|^4}{2\sqrt{\epsilon\mu}} \int\limits_{|\mathbf{k}'|=|\mathbf{k}|} [\hat{R}_{\epsilon\epsilon}(|\mathbf{k}-\mathbf{k}'|)T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k}) \tag{4.47}$$

$$+ \hat{R}_{\epsilon\mu}(|\mathbf{k}-\mathbf{k}'|)(T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')X(\mathbf{k}',\mathbf{k}) + X(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k}))$$

$$+ \hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)X(\mathbf{k},\mathbf{p})W(\mathbf{p})X(\mathbf{p},\mathbf{k})]d\Omega(\hat{\mathbf{p}}) - \Sigma(|\mathbf{k}|)W(\mathbf{k}).$$

The coupling matrix N is given by (3.67).

When the power spectral denisties of the fluctuations $\hat{R}_{ij}$ are constants, the scattering cross-sections are proportional to $|\mathbf{k}|^4$, which corresponds to Rayleigh scattering. If, in addition, the magnetic permittivity has no fluctuations then the radiative transport equation (4.47) in a uniform background medium coincides, up to a normalization constant, with Chandrasekhar's equation of radiative transfer (equation (212) in [1]).

In the transport equations corresponding to Maxwell's equations, there is scattering only between modes propagating with the same speed. This is not true in general, as we saw in section 4.2.

4.5 Transport Equations for Elastic Waves

The elastic wave equations in a random medium are given by the symmetric hyperbolic system (3.83) with the perturbed A matrix

$$\begin{pmatrix} \rho I & 0 & 0 & 0 \\ 0 & \frac{1}{2\mu}I & 0 & 0 \\ 0 & 0 & \frac{1}{\mu}I & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix} \left[\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sqrt{\varepsilon} \begin{pmatrix} \tilde{\rho} I & 0 & 0 & 0 \\ 0 & \tilde{\theta} I & 0 & 0 \\ 0 & 0 & \tilde{\theta} I & 0 \\ 0 & 0 & 0 & \tilde{\psi} \end{pmatrix} \right]. \tag{4.48}$$

Here I is the 3×3 identity matrix and $\tilde{\theta}$ and $\tilde{\psi}$ are the fluctuations of $\frac{1}{\mu}$ and $\frac{1}{\lambda}$, respectively. The power spectral densities of the fluctuations $\hat{R}_{svlw}(\mathbf{k})$ have the form

$$\hat{R}_{svlw}(\mathbf{k}) = \delta_{sv}\delta_{lw}\{\delta_{s\le 3}\delta_{l\le 3}\hat{R}_{\rho\rho}(|\mathbf{k}|) + \delta_{4\le s\le 6}\delta_{l\le 3}\hat{R}_{\mu\rho}(|\mathbf{k}|)$$

$$+ \delta_{s\le 3}\delta_{4\le l\le 6}\hat{R}_{\mu\rho}(|\mathbf{k}|) + \delta_{7\le s\le 9}\delta_{l\le 3}\hat{R}_{\mu\rho}(|\mathbf{k}|) + \delta_{s\le 3}\delta_{7\le l\le 9}\hat{R}_{\mu\rho}(|\mathbf{k}|)$$

$$+ \delta_{s,10}\delta_{l\le 3}\hat{R}_{\rho\lambda}(|\mathbf{k}|) + \delta_{s\le 3}\delta_{l,10}\hat{R}_{\rho\lambda}(|\mathbf{k}|) + \delta_{4\le s\le 6}\delta_{4\le l\le 6}\hat{R}_{\mu\mu}(|\mathbf{k}|)$$

$$+ \delta_{4\le s\le 6}\delta_{7\le l\le 9}\hat{R}_{\mu\mu}(|\mathbf{k}|) + \delta_{7\le s\le 9}\delta_{4\le s\le 6}\hat{R}_{\mu\mu}(|\mathbf{k}|) + \delta_{4\le s\le 6}\delta_{l,10}\hat{R}_{\mu\lambda}(|\mathbf{k}|)$$

$$+ \delta_{s,10}\delta_{4\le l\le 6}\hat{R}_{\mu\lambda}(|\mathbf{k}|) + \delta_{7\le s\le 9}\delta_{7\le l\le 9}\hat{R}_{\mu\mu}(|\mathbf{k}|) + \delta_{7\le s\le 9}\delta_{l,10}\hat{R}_{\mu\lambda}(|\mathbf{k}|)$$

$$+ \delta_{s,10}\delta_{7\le l\le 9}\hat{R}_{\mu\lambda}(|\mathbf{k}|) + \delta_{s,10}\delta_{l,10}\hat{R}_{\lambda\lambda}(|\mathbf{k}|)\}. \tag{4.49}$$

The subscripts μ and λ refer to the fluctuations of $1/\mu$ and $1/\lambda$ and the subscript ρ corresponds to the fluctuations of the density ρ. The indices in (4.49) run from 1 to 10 and the delta's are as in (4.37).

The P to S wave resonance condition (4.27) is

$$\omega_+^S(\mathbf{k}) = \omega_+^P(\mathbf{k}')$$

with the P and S wave frequencies given by (3.87). For a fixed S wave vector $\mathbf{k}$ there is a sphere of resonant P wave vectors $|\mathbf{k}'| = \sqrt{\mu/(2\mu+\lambda)}|\mathbf{k}|$, so the transport equation (4.32) for the P wave energy density $a_+^P(t, \mathbf{x}, \mathbf{k}))$ and the transport equation for the S wave coherence matrix $W_+^S(t, \mathbf{x}, \mathbf{k})$ are coupled. Moreover, as in the electromagnetic case, there is no coupling to backward travelling waves so it is enough to consider the two forward modes and to omit the subscript $+$. As we noted earlier, the P wave energy transport is similar to that of acoustic waves, and the S wave energy transport is similar to that of electromagnetic waves. Therefore the system of transport equations for elastic waves will have the form (4.38) for the P waves coupled to a system of the form (4.47) for the S waves. They are given by (1.13) and (1.14).

We now outline the calculation of the scattering cross-sections. We present two calculations: the part of σ^{SS} in (1.16) that involves $\hat{R}_{\rho\rho}$ and the part containing $\hat{R}_{\mu\mu}$. Using the eigenvalues (3.87) and eigenvectors (3.89) of the dispersion matrix (3.84) and the power spectral densities (4.49) in (4.33) we have

$$\begin{aligned} & 2\ \pi v_S^2|\mathbf{k}|^2\sqrt{\frac{1}{2\rho}}z_s^{(q)}(\mathbf{k}')\sqrt{\frac{1}{2\rho}}z_n^{(r)}(\mathbf{k}')\sqrt{\frac{\rho}{2}}z_n^{(j)}(\mathbf{k})\sqrt{\frac{\rho}{2}}z_s^{(m)}(\mathbf{k})W_{qr}^S(\mathbf{k}') \\ & \cdot \hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|) \qquad (4.50) \\ & = \frac{\pi v_S^2|\mathbf{k}|^2}{2}\hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|)\{T(\mathbf{k},\mathbf{k}')W^S(\mathbf{k}')T(\mathbf{k}',\mathbf{k})\}_{mj}. \qquad (4.51) \end{aligned}$$

We show next that the differential scattering cross-section for the S-to-S scattering (1.16) differs slightly from the differential scattering cross-section for electromagnetic waves (4.45). The part of the differential scattering cross-section σ^{SS} involving the power spectral density $\hat{R}_{\mu\mu}$ is given by

$$\begin{aligned} & \frac{\pi\mu\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)}{2\rho|\mathbf{k}|^2}(2K(\mathbf{k})K(\mathbf{k}')\mathbf{z}^{(r)}(\mathbf{k}') + M(\mathbf{k})M(\mathbf{k}')\mathbf{z}^{(r)}(\mathbf{k}'), \mathbf{z}^{(j)}(\mathbf{k})) \\ & \cdot(2K(\mathbf{k})K(\mathbf{k}')\mathbf{z}^{(q)}(\mathbf{k}') + M(\mathbf{k})M(\mathbf{k}')\mathbf{z}^{(q)}(\mathbf{k}'), \mathbf{z}^{(m)}(\mathbf{k}))W_{qr}^S(\mathbf{k}') \\ & = \frac{\pi\mu\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)|\mathbf{k}'|^2}{2\rho},\ _{mq}(\mathbf{k},\mathbf{k}')W_{qr}^S(\mathbf{k}'),\ _{rj}(\mathbf{k}',\mathbf{k}) \\ & = \frac{\pi v_S^2|\mathbf{k}'|^2\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)}{2}\{,\ (\mathbf{k},\mathbf{k}')W^S(\mathbf{k}'),\ (\mathbf{k}',\mathbf{k})\}_{mj}. \qquad (4.52) \end{aligned}$$

The matrix , is given by (1.17) or equivalently by

$$,_{mq}(\mathbf{k},\mathbf{k}') = (2K(\hat{\mathbf{k}})K(\hat{\mathbf{k}}')\mathbf{z}^{(q)}(\mathbf{k}') + M(\hat{\mathbf{k}})M(\hat{\mathbf{k}}')\mathbf{z}^{(q)}(\mathbf{k}'), \mathbf{z}^{(m)}(\mathbf{k})). \tag{4.53}$$

with M defined by (3.86) and $K = \operatorname{diag}(k_1, k_2, k_3)$. The differential scattering cross-section has the form (1.16) with

$$\begin{aligned}
\sigma_{ss}^{TT} &= \frac{\pi v_S^2 |\mathbf{k}|^2}{2} \hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|) \\
\sigma_{ss}^{\Gamma\Gamma} &= \frac{\pi v_S^2 |\mathbf{k}|^2}{2} \hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|) \\
\sigma_{ss}^{\Gamma T} &= \frac{\pi v_S^2 |\mathbf{k}|^2}{2} \hat{R}_{\rho\mu}(|\mathbf{k}-\mathbf{k}'|).
\end{aligned} \tag{4.54}$$

This is the same as (4.45) in the electromagnetic case with the matrix X replaced by , . A direct calculation shows that the imaginary terms in (4.34) vanish in this case. The rest of the calculations are similar and we omit them.

The transport equations for the P wave amplitude a^P and the S wave coherence matrix W^S have the form (1.13) and (1.14) with the differential scattering cross-section σ^{SS} given by (4.54) and the functions σ_{pp} and σ_{ps} given by

$$\begin{aligned}
\sigma_{pp}(\mathbf{k},\mathbf{k}') = \frac{\pi|\mathbf{k}|^2(2\mu+\lambda)}{2\rho} \Bigg\{ & \frac{\lambda^2}{(2\mu+\lambda)^2}\hat{R}_{\lambda\lambda}(|\mathbf{k}-\mathbf{k}'|) + \frac{4\lambda\mu}{(2\mu+\lambda)^2}(\hat{\mathbf{k}},\hat{\mathbf{k}}')^2\hat{R}_{\lambda\mu}(|\mathbf{k}-\mathbf{k}'|) \\
& + \frac{4\mu^2}{(2\mu+\lambda)^2}(\hat{\mathbf{k}},\hat{\mathbf{k}}')^4\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|) + (\hat{\mathbf{k}},\hat{\mathbf{k}}')^2\hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|) \\
& + \frac{2\lambda}{2\mu+\lambda}(\hat{\mathbf{k}},\hat{\mathbf{k}}')\hat{R}_{\lambda\rho}(|\mathbf{k}-\mathbf{k}'|) + \frac{4\mu}{2\mu+\lambda}(\hat{\mathbf{k}},\hat{\mathbf{k}}')^3\hat{R}_{\rho\mu}(|\mathbf{k}-\mathbf{k}'|) \Bigg\}
\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}
\sigma_{ps}(\mathbf{k},\mathbf{k}') = \frac{\pi\mu}{2\rho}\{ & |\mathbf{k}'|^2\hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|) + 4|\mathbf{k}|^2(\hat{\mathbf{k}},\hat{\mathbf{k}}')^2\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|) \\
& + 4|\mathbf{k}||\mathbf{k}'|(\hat{\mathbf{k}},\hat{\mathbf{k}}')\hat{R}_{\mu\rho}(|\mathbf{k}-\mathbf{k}'|)\}
\end{aligned} \tag{4.56}$$

The P-to-P part of (1.13) coincides with the transport equation for the acoustic waves when $\mu = 0$. The S-to-S part coincides with the electromagnetic case with the replacement of , by X. The scattering operator on the right side of the transport equations (1.13) and (1.14) is symmetric in a^P and W^S. This is an important property that is used in the analysis of the transport equations in the diffusion regime (section 5.3).

5 The Diffusion Approximation

5.1 Diffusion Approximation for Acoustic Waves

The diffusion approximation for transport equations like (1.1) is valid at propagation distances much longer than the transport mean free path $|\nabla_{\mathbf{k}}\omega|/\Sigma$ [17]. We show in sections 5.2 and 5.3 that solutions of transport equations for polarized waves also exhibit diffusive behaviour and that the waves become approximately depolarized in this regime. For simplicity we will consider only the case when the background is homogeneous and isotropic, in which case the eigenvalues of the dispersion matrix (3.22) are given by $\omega_i(\mathbf{k}) = v_i|\mathbf{k}|$ with the speeds v_i independent of $\mathbf{x}$. We shall consider only conservative transport equations so that (1.2) or (1.9) holds. The results, however, can be generalized to variable backgrounds and to weakly dissipative scattering provided that the background variations and the dissipation are on the scale of the propagation distance.

To derive the diffusion approximation we introduce a dimensionless small parameter ε, not related to the small parameter used in the previous sections. It is ratio of the mean free path to the propagation distance. Then, by rescaling time and space variables by $t \to \varepsilon^2 t$ and $\mathbf{x} \to \varepsilon\mathbf{x}$, we can write (1.1) as

$$\begin{aligned} \varepsilon^2\frac{\partial a}{\partial t} + \varepsilon v\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a &= \int_{|\mathbf{k}|=|\mathbf{k}'|}\sigma(|\mathbf{k}|,\hat{\mathbf{k}},\hat{\mathbf{k}}')a(\mathbf{k}')d\Omega(\hat{\mathbf{k}}') - \Sigma(|\mathbf{k}|)a(|\mathbf{k}|) \\ a(0,\mathbf{x},\mathbf{k}) &= a_0(\mathbf{x},\mathbf{k}). \end{aligned} \tag{5.1}$$

The total scattering cross-section Σ is

$$\Sigma(|\mathbf{k}|) = \int_{|\mathbf{k}|=|\mathbf{k}'|}\sigma(|\mathbf{k}|,\hat{\mathbf{k}},\hat{\mathbf{k}}')d\Omega(\hat{\mathbf{k}}') \tag{5.2}$$

and $d\Omega$ denotes the surface element on the unit sphere. We shall consider only rotationally invariant scattering so that the differential scattering cross-section $\sigma(\mathbf{k},\mathbf{k}')$ is a non-negative function that depends only on $|\mathbf{k}|$ and $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'$.

We expand the solution of (5.1) in powers of ε

$$a(t,\mathbf{x},\mathbf{k}) = a^{(0)}(t,\mathbf{x},\mathbf{k}) + \varepsilon a^{(1)}(t,\mathbf{x},\mathbf{k}) + \varepsilon^2 a^{(2)}(t,\mathbf{x},\mathbf{k}) + \dots \tag{5.3}$$

and insert this expansion into (5.1). We find that the leading term $a^{(0)}(t,\mathbf{x},\mathbf{k})$ satisfies

$$\int_{|\mathbf{k}|=|\mathbf{k}'|}\sigma(|\mathbf{k}|,\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')a^{(0)}(t,\mathbf{x},\mathbf{k}')d\Omega(\hat{\mathbf{k}}') = \Sigma(|\mathbf{k}|)a^{(0)}(t,\mathbf{x},\mathbf{k}). \tag{5.4}$$

This is an eigenfunction equation for $a^{(0)}$ involving the integral operator $\mathcal{A}$, defined by the left side. The kernel of $\mathcal{A}$ is the scattering cross-section and it is positive. From the general theory of such operators it follows that they have the following properties [43]:

(i) the eigenvalue with the largest absolute value is simple,

(ii) the eigenfunction corresponding to this eigenvalue is non-negative,

(iii) this eigenfunction is the only non-negative eigenfunction of this operator.

From (5.2) we see that if $a^{(0)}$ is independent of the direction $\hat{\mathbf{k}}$ it is a solution of (5.4). This fact and properties (i-iii) show that

$$a^{(0)}(t, \mathbf{x}, \mathbf{k}) = a^{(0)}(t, \mathbf{x}, |\mathbf{k}|). \tag{5.5}$$

This means that $a(t, \mathbf{x}, \mathbf{k})$ is approximately independent of the direction $\hat{\mathbf{k}}$ of the wave vector $\mathbf{k}$.

The first order term $a^{(1)}$ satisfies the equation

$$v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)} = \int_{|\mathbf{k}|=|\mathbf{k}'|} \sigma(|\mathbf{k}|, \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') a^{(1)}(\mathbf{k}') d\Omega(\hat{\mathbf{k}}') - \Sigma(|\mathbf{k}|) a^{(1)}(\mathbf{k}). \tag{5.6}$$

To solve (5.6) we note that the function $u(\mathbf{x}, \mathbf{k}) = \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)}(t, \mathbf{x}, |\mathbf{k}|)$ is an eigenfunction of the operator $\mathcal{A}$ corresponding to the eigenvalue

$$\lambda = 2\pi \int_{-1}^{1} \sigma(|\mathbf{k}|, \mu) \mu d\mu,$$

where $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'$. To show this we let Q be an orthogonal transformation such that $Q\hat{\mathbf{k}} = (0, 0, 1)^t$. Then

$$\begin{aligned} (\mathcal{A}u)(\hat{\mathbf{k}}) &= \int_{|\mathbf{k}|=|\mathbf{k}'|} \sigma(|\mathbf{k}|, \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')(\hat{\mathbf{k}}' \cdot \nabla_{\mathbf{x}} a^{(0)}) d\Omega(\hat{\mathbf{k}}') \\ &= \int_{|\mathbf{k}|=|\mathbf{k}'|} \sigma(|\mathbf{k}|, \hat{k}_3')(\hat{\mathbf{k}}' \cdot Q\nabla_{\mathbf{x}} a^{(0)}) d\Omega(\hat{\mathbf{k}}') \\ &= 2\pi \int_{-1}^{1} \sigma(|\mathbf{k}|, \mu)\mu d\mu (Q\nabla_{\mathbf{x}} a^{(0)})_3 = 2\pi \int_{-1}^{1} \sigma(|\mathbf{k}|, \mu) \mu d\mu (\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)}) = \lambda u(\hat{\mathbf{k}}). \end{aligned} \tag{5.7}$$

Now we write $a^{(1)} = C(|\mathbf{k}|)u$, substitute into (5.6) and use (5.7). Then we can solve for C and u and obtain

$$a^{(1)}(t, \mathbf{x}, \mathbf{k}) = -\frac{v}{\Sigma(|\mathbf{k}|) - \lambda(|\mathbf{k}|)} \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)}(t, \mathbf{x}, |\mathbf{k}|). \tag{5.8}$$

The equation for $a^{(2)}$ is

$$\frac{\partial a^{(0)}}{\partial t} - v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \left(\frac{v}{\Sigma(|\mathbf{k}|) - \lambda(|\mathbf{k}|)} \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)} \right) = \mathcal{A} a^{(2)} - \Sigma(|\mathbf{k}|) a^{(2)}. \tag{5.9}$$

We integrate (5.9) with respect to direction $\hat{\mathbf{k}}$. The integral of the right side vanishes and we get the solvability condition

$$\int_{|\mathbf{k}|=|\mathbf{k}'|} \left(\frac{\partial a^{(0)}}{\partial t} - v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \left(\frac{v}{\Sigma(|\mathbf{k}|) - \lambda(|\mathbf{k}|)} \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)} \right) \right) d\Omega(\hat{\mathbf{k}}) = 0. \tag{5.10}$$

After performing the integration over $\hat{\mathbf{k}}$ in (5.10) we obtain the diffusion equation

$$\frac{\partial a^{(0)}(t,\mathbf{x},|\mathbf{k}|)}{\partial t} = \nabla_{\mathbf{x}} \cdot [D(|\mathbf{k}|)\nabla_{\mathbf{x}} a^{(0)}(t,\mathbf{x},|\mathbf{k}|)]. \tag{5.11}$$

This equation determines the principal term $a^{(0)}$ in the expansion (5.3). We find that the diffusion coeficient $D(|\mathbf{k}|)$ in (5.11) is given by the well known formula [46].

$$D(|\mathbf{k}|) = \frac{v^2}{3(\Sigma(|\mathbf{k}|) - \lambda(|\mathbf{k}|))}. \tag{5.12}$$

Note that $D > 0$ because $\Sigma(|\mathbf{k}|)$ is the largest eigenvalue of $\mathcal{A}$ so it is larger than $\lambda(|\mathbf{k}|)$, which is another eigenvalue. The diffusion coefficient can also be written in the form

$$D(\mathbf{k}) = \frac{vl^*(|\mathbf{k}|)}{3}, \tag{5.13}$$

where the diffusion mean free path $l^*(|\mathbf{k}|)$ is given by

$$l^*(|\mathbf{k}|) = v\left(2\pi\int_{-1}^{1}\sigma(|\mathbf{k}|,\mu)(1-\mu)d\mu\right)^{-1}. \tag{5.14}$$

The diffusion equation (5.11) cannot accomodate the initial condition $a(0,\mathbf{x},\mathbf{k}) = a_0(\mathbf{x},\mathbf{k})$ unless the function $a_0(\mathbf{x},\mathbf{k})$ is independent of the angular direction $\hat{\mathbf{k}}$. To obtain the correct initial conditions for the diffusion equation (5.11) we must consider the initial layer problem as in [18]. We write a in the form

$$a = a^i + a^{il}, \tag{5.15}$$

where a^i is the solution given by the asymptotic expansion (5.3) and a^{il} is the initial layer solution which decays exponentially in time. The initial layer solution a^{il} depends on the fast time $\tau = t/\varepsilon^2$ and satisfies the equation

$$\frac{\partial a^{il}}{\partial \tau} = \int_{|\mathbf{k}|=|\mathbf{k}'|}\sigma(|\mathbf{k}|,\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')a^{il}(\mathbf{k}')d\hat{\mathbf{k}}' - \Sigma(|\mathbf{k}|)a^{il}(\mathbf{k}). \tag{5.16}$$

The solution a^{il} decays exponentially in time if we take as an initial condition for (5.16)

$$a^{il}(0,\mathbf{x},\mathbf{k}) = a_0(\mathbf{x},\mathbf{k}) - \frac{1}{4\pi}\int a_0(\mathbf{x},\mathbf{k}')d\Omega(\hat{\mathbf{k}}'). \tag{5.17}$$

This implies that the initial condition for the diffusion equation (5.11) is the average of a_0,

$$a^{(0)}(0,\mathbf{x},|\mathbf{k}|) = \frac{1}{4\pi}\int a_0(\mathbf{x},\mathbf{k})d\Omega(\hat{\mathbf{k}}), \tag{5.18}$$

as might have been expected from physical considerations.

5.2 Diffusion Approximation for Electromagnetic Waves

We now apply the analysis of the previous section to the transport equation (4.47) for electromagnetic waves. We rewrite this equation in the form

$$\frac{\partial W}{\partial t} + v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W = \mathcal{A}W - \Sigma(|\mathbf{k}|)W. \tag{5.19}$$

Here the integral operator $\mathcal{A}$ acts on matrix valued functions, and is defined by

$$\begin{aligned}\mathcal{A}f(\mathbf{k}) = \frac{\pi v|\mathbf{k}|^4}{2}\int_{|\mathbf{k}|=|\mathbf{k}'|} &\{\hat{R}_{\epsilon\epsilon}(|\mathbf{k}-\mathbf{k}'|)T(\mathbf{k},\mathbf{k}')f(\mathbf{k}')T(\mathbf{k}',\mathbf{k}) \\ &+\hat{R}_{\epsilon\mu}(|\mathbf{k}-\mathbf{k}'|)(T(\mathbf{k},\mathbf{k}')f(\mathbf{k}')X(\mathbf{k}',\mathbf{k}) + X(\mathbf{k},\mathbf{k}')f(\mathbf{k}')T(\mathbf{k}',\mathbf{k})) \\ &+\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)X(\mathbf{k},\mathbf{k}')f(\mathbf{k}')X(\mathbf{k}',\mathbf{k})\}d\Omega(\hat{\mathbf{k}}'),\end{aligned} \tag{5.20}$$

where $v = 1/\sqrt{\epsilon\mu}$. We assume that the transport mean free path is small compared to the propagation distance and we scale space and time variables $(\mathbf{x}, t)$ by $t \to \varepsilon^2 t$, $\mathbf{x} \to \varepsilon\mathbf{x}$ as in section 5.1. The scaled transport equation (5.19) is

$$\varepsilon^2\frac{\partial W}{\partial t} + \varepsilon v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W = \mathcal{A}W - \Sigma(|\mathbf{k}|)W. \tag{5.21}$$

We expand the solution of (5.21) in powers of the small parameter ε

$$W = W^{(0)} + \varepsilon W^{(1)} + \varepsilon^2 W^{(2)} + \ldots \tag{5.22}$$

Inserting this into (5.21), we find that the leading term $W^{(0)}$ satisfies the eigenfunction equation

$$\mathcal{A}W^{(0)}(t,\mathbf{x},\mathbf{k}) = \Sigma(|\mathbf{k}|)W^{(0)}(t,\mathbf{x},\mathbf{k}), \tag{5.23}$$

which is analogous to (5.4). The general theory of positive operators [43] applies to $\mathcal{A}$ and hence $W^{(0)}(t,\mathbf{x},\mathbf{k})$ has the form

$$W^{(0)}(t,\mathbf{x},\mathbf{k}) = \phi(t,\mathbf{x},|\mathbf{k}|)I, \tag{5.24}$$

where $\phi(t,\mathbf{x},|\mathbf{k}|)$ is an unknown scalar function to be determined. Thus, the leading approximation for the coherence matrix is a scalar multiple of the identitity and is independent of the direction $\hat{\mathbf{k}}$. This shows that electromagnetic waves are depolarized in the diffusion approximation.

The first order term $W^{(1)}$ satisfies the equation

$$v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}}\phi I = \mathcal{A}W^{(1)} - \Sigma(|\mathbf{k}|)W^{(1)}. \tag{5.25}$$

The matrix function $u(\hat{\mathbf{k}}) = \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \phi I$ is an eigenfunction of $\mathcal{A}$, defined by (5.20), corresponding to the eigenvalue

$$\lambda(|\mathbf{k}|) = \frac{\pi v |\mathbf{k}|^4}{2} \int_{-1}^{1} \{\pi(\hat{R}_{\epsilon\epsilon}(|\mathbf{k}|\sqrt{2-2\eta}) + \hat{R}_{\mu\mu}(|\mathbf{k}|\sqrt{2-2\eta}))(\eta + \eta^3) + 4\pi \hat{R}_{\epsilon\mu}(|\mathbf{k}|\sqrt{2-2\eta})\eta^2\} d\eta. \tag{5.26}$$

Hence

$$W^{(1)} = \frac{v}{\lambda(|\mathbf{k}|) - \Sigma(|\mathbf{k}|)} (\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \phi) I. \tag{5.27}$$

The second order term $W^{(2)}$ satisfies the equation

$$\frac{\partial W^{(0)}}{\partial t} + v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W^{(1)} = \mathcal{A} W^{(2)} - \Sigma(|\mathbf{k}|) W^{(2)} \tag{5.28}$$

which is solvable only if the left side of (5.28) is orthogonal to functions of the form (5.24). Integrating (5.28) with respect to $\hat{\mathbf{k}}$ and taking the trace we find that ϕ satisfies the diffusion equation

$$\frac{\partial \phi}{\partial t} = \nabla_{\mathbf{x}} \cdot [D^{em}(|\mathbf{k}|) \nabla_{\mathbf{x}} \phi]. \tag{5.29}$$

The diffusion coefficient is

$$D^{em} = \frac{v l^*_{em}}{3},$$

where the diffusion mean free path l^*_{em} is defined by

$$l^*_{em} = \frac{2}{\pi^2 |\mathbf{k}|^4} \left(\int_{-1}^{1} [(\hat{R}_{\epsilon\epsilon}(|\mathbf{k}|\sqrt{2-2\eta}) + \hat{R}_{\mu\mu}(|\mathbf{k}|\sqrt{2-2\eta}))(1 + \eta^2 - \eta - \eta^3) + 4\hat{R}_{\epsilon\mu}(|\mathbf{k}|\sqrt{2-2\eta})(\eta - \eta^2)] d\eta \right)^{-1}. \tag{5.30}$$

The initial condition for the diffusion equation (5.29) is determined as in the scalar case. The initial condition for the initial layer solution must be

$$W^{il}(0, \mathbf{x}, \mathbf{k}) = W_0(\mathbf{x}, \mathbf{k}) - \frac{1}{8\pi} \{ \int \mathrm{Tr} W_0(\mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}) \} I, \tag{5.31}$$

so as to make it decay exponentially in time. Then the initial condition for (5.29) is

$$\phi(0, \mathbf{x}, |\mathbf{k}|) = \frac{1}{8\pi} \int \mathrm{Tr} W_0(\mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}). \tag{5.32}$$

5.3 Diffusion Approximation for Elastic waves

We shall now determine the diffusion approximation for the elastic transport equations (1.13) and (1.14). We shall show that in the diffusion regime the S waves are depolarized and energy is "equipartitioned" between S and P waves (equation (1.24) or (5.37)).

We rescale space and time variables $(t, \mathbf{x})$ by $t \to \varepsilon^2 t$, $\mathbf{x} \to \varepsilon \mathbf{x}$ and rewrite the transport equations (1.13) and (1.14) for elastic waves in the scaled form

$$\begin{aligned} \varepsilon^2 \frac{\partial a^P}{\partial t} + \varepsilon v_P \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^P &= \mathcal{A}_{PP}[a^P] + \mathcal{A}_{PS}[W^S] - (\Sigma^{PP} + \Sigma^{PS}) a^P \\ \varepsilon^2 \frac{\partial W^S}{\partial t} + \varepsilon v_S \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W^S &= \mathcal{A}_{SS}[W^S] + \mathcal{A}_{SP}[a^P] - (\Sigma^{SS} + \Sigma^{SP}) W^S. \end{aligned} \tag{5.33}$$

The integral operators $\mathcal{A}_{ij}$ are defined by comparing (5.33) to (1.13) and (1.14). We expand the solution of (5.33) as

$$\begin{aligned} a^P &= a^{(0)} + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \ldots \\ W^S &= W^{(0)} + \varepsilon W^{(1)} + \varepsilon^2 W^{(2)} + \ldots. \end{aligned} \tag{5.34}$$

By using (5.34) in (5.33) we find that the principal terms $a^{(0)}$ and $W^{(0)}$ must satisfy the equations

$$\begin{aligned} \mathcal{A}_{PP}[a^{(0)}] + \mathcal{A}_{PS}[W^{(0)}] &= (\Sigma^{PP} + \Sigma^{PS}) a^{(0)} \\ \mathcal{A}_{SS}[W^{(0)}] + \mathcal{A}_{SP}[a^{(0)}] &= (\Sigma^{SS} + \Sigma^{SP}) W^{(0)}. \end{aligned} \tag{5.35}$$

This is a pair of coupled equations of the form (5.4) and (5.23). The general theory of positive operators is applicable again and implies that the solutions of (5.35) are of the form

$$a^{(0)}(t, \mathbf{x}, \mathbf{k}) = \phi(t, \mathbf{x}, |\mathbf{k}|) \tag{5.36}$$

$$W^{(0)}(t, \mathbf{x}, \mathbf{k}) = \phi(t, \mathbf{x}, \frac{v_S}{v_P}|\mathbf{k}|) I,$$

where $\phi(t, \mathbf{x}, |\mathbf{k}|)$ is a scalar function to be determined. It follows that

$$a^{(0)}(t, \mathbf{x}, \mathbf{k}) I = W^{(0)}(t, \mathbf{x}, \frac{v_P}{v_S}\mathbf{k}). \tag{5.37}$$

Equation (5.36) implies that in the diffusion regime the S wave is completely depolarized. Equation (5.37) shows that the energy in the wave number shell of interaction in phase space is partitioned between the P waves and each polarization of the S waves.

In physical space the local energy densities

$$\mathcal{E}_P(t, \mathbf{x}) = \int_{R^3} a^P(t, \mathbf{x}, \mathbf{k}) d\mathbf{k} \tag{5.38}$$

and

$$\mathcal{E}_S(t,\mathbf{x}) = \int_{R^3} \mathrm{Tr} W^S(t,\mathbf{x},\mathbf{k}) d\mathbf{k} \tag{5.39}$$

are related by

$$\mathcal{E}_P(t,\mathbf{x}) = \frac{v_S^3}{2v_P^3} \mathcal{E}_S(t,\mathbf{x}). \tag{5.40}$$

This provides an effective cirterion for determining the range of validity of the diffusion regime in the analysis of seismic data. Unless the energy densities of the P and S waves, which can be obtained from measurements, satisfy relation (5.40) the diffusion approximation is not valid. This formula shows that in the diffusion regime most of the energy is in the S waves, no matter how it was distributed initially.

The first order terms satisfy the system of equations

$$\begin{aligned} v_P \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(0)} &= \mathcal{A}_{PP}[a^{(1)}] + \mathcal{A}_{PS}[W^{(1)}] - (\Sigma^{PP} + \Sigma^{PS}) a^{(1)} \\ v_S \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W^{(0)} &= \mathcal{A}_{SS}[W^{(1)}] + \mathcal{A}_{SP}[a^{(1)}] - (\Sigma^{SS} + \Sigma^{SP}) W^{(1)}. \end{aligned} \tag{5.41}$$

As in sections 5.1 and 5.2, the function $u = \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \phi$ is an eigenfunction of all the operators $\mathcal{A}_{PP}$, $\mathcal{A}_{PS}$, $\mathcal{A}_{SP}$ and $\mathcal{A}_{SS}$. Let the corresponding eigenvalues be λ_{pp}, λ_{ps}, λ_{sp} and λ_{ss}, respectively. This implies that if $W^{(1)} = -l_s \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \phi I$ and $a^{(1)} = -l_p \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \phi$, then (5.41) is satisfied provided the constants l_p and l_s solve the system of two linear equations

$$\begin{aligned} -v_P &= \lambda_{pp} l_p + \lambda_{ps} l_s - (\Sigma_{pp} + \Sigma_{ps}) l_p \\ -v_S &= \lambda_{ss} l_s + \lambda_{sp} l_p - (\Sigma_{ss} + \Sigma_{sp}) l_s. \end{aligned} \tag{5.42}$$

Both constants l_s and l_p have the dimension of length and can be considered as diffusion mean free paths for S and P waves, respectively.

The second-order terms in ε satisfy the system of equations

$$\frac{\partial a^{(0)}}{\partial t} + v_P \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a^{(1)} = \mathcal{A}_{PP}[a^{(2)}] + \mathcal{A}_{PS}[W^{(2)}] - (\Sigma^{PP} + \Sigma^{PS}) a^{(2)} \tag{5.43}$$

$$\frac{\partial W^{(0)}}{\partial t} + v_S \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W^{(1)} = \mathcal{A}_{SS}[W^{(2)}] + \mathcal{A}_{SP}[a^{(2)}] - (\Sigma^{SS} + \Sigma^{SP}) W^{(2)}. \tag{5.44}$$

This system has a solution when the sum of the integrals with respect to $\hat{\mathbf{k}}$ of the left side of (5.43) multiplied by v_S^3/v_P^3 and of the trace of the left side of (5.44) vanishes. This implies that the function ϕ must satisfy the diffusion equation

$$\frac{\partial \phi}{\partial t} = \nabla_{\mathbf{x}} \cdot [D^{el}(|\mathbf{k}|) \nabla_{\mathbf{x}} \phi] \tag{5.45}$$

with the diffusion coefficient

$$D^{el} = \frac{1}{\frac{2}{v_S^3} + \frac{1}{v_P^3}} \left(\frac{l_p^* v_P}{3 v_P^3} + \frac{2 l_s^* v_S}{3 v_S^3} \right). \tag{5.46}$$

Thus D^{el} is the weighted mean of "partial diffusion coefficients" for P waves and for each polarization of S waves, where l_s and l_p satisfy (5.42).

In the special case when the power spectral densities are flat (constant) over the wave numbers of interest and there are no density fluctuations, the mean free paths l_p and l_s that satisfy (5.42) are

$$l_p(|\mathbf{k}|) = \frac{(2\mu + \lambda)^2}{\pi^2 |\mathbf{k}|^4} \frac{1}{2\lambda^2 \hat{R}_{\lambda\lambda} + \frac{8}{3}\lambda\mu \hat{R}_{\lambda\mu} + \frac{8}{5}\mu^2 \hat{R}_{\mu\mu} + \frac{4 v_P^5}{15 v_S^5}\mu^2 \hat{R}_{\mu\mu}} \tag{5.47}$$

and

$$l_s(|\mathbf{k}|) = \frac{15 \rho v_S}{\pi^2 |\mathbf{k}|^4 \mu^2 \hat{R}_{\mu\mu}} \frac{1}{\frac{8}{v_P(2\mu+\lambda)} + \frac{26 v_P^2}{v_S^3 \mu}}, \tag{5.48}$$

with all spectral densities $\hat{R}_{ij}$ constant.

The initial condition for the diffusion equation (5.45) is obtained as in the acoustic and electromagnetic cases, and is

$$\phi_0(\mathbf{x}, |\mathbf{k}|) = \frac{1}{12\pi} \int \mathrm{Tr} W_0^S(\mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}) + \frac{1}{12\pi} \int a_0^P(\mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}). \tag{5.49}$$

Here W_0^S and a_0^P are the initial values for W^S and a^P.

6 Conclusions

We have shown that transport equations for the propagation of energy in phase space can be derived for general waves and for acoustic, electromagnetic and elastic waves, in particular. The transport equations have a universal character that depends on the structure of the dispersion relation (matrix) and not on the details of the wave motion. The effect of random inhomogeneities is to introduce scattering of the energy and mode coupling.

Transport equations are a good way to describe the propagation of wave energy when (i) typical wavelengths are short compared to macroscopic features of the medium (high frequency approximation), (ii) correlation lengths of the inhomogeneities are comparable to wavelengths and (iii) the fluctuations of the inhomogeneities are weak. As mentioned in the introduction, condition (ii) is important because it allows for strong interaction between the waves and the inhomogeneities. As a result the influence of the slow background variations is comparable to that of the scattering by the random inhomogeneities. Transport theory is not valid when the inhomogeneities are either very anisotropic or very strong. The role of anisotropy is not appparent in the present formalism. One has to look closely at the details of the analysis to see the breakdown of the transport approximation and the onset of wave localization.

Polarization alters the transport equations substantially and this is important both for electromagnetic and for elastic wave propagation. The transport equations still have a universal character that depends on the structure of the dispersion matrix, and not on details of the wave motion. Thus the transport equations for electromagnetic and elastic shear or S waves have the same form.

We have also shown how to get diffusion approximations for the transport equations, especially for elastic waves. The diffusion regime is important because multiple scattering effects have a simple and universal form there, independent of the details of the scattering and of the excitation. Many applications of transport theory and, in fact, most of the applications in seismology, have been carried out in the diffusion regime. As mentioned in the Introduction (section 1.3), the energy equipartition law (1.24), or (1.25), implies that in the diffusion regime the P to S energy ratio stabilizes independently of the details of the multiple scattering and of the nature of the source. This is similar to the empirical observation of Hansel, Ringdal and Richards [39] regarding the stabilization of the P to Lg energy ratio.

We have not discussed the influence of boundaries and interfaces on the form of the transport equations and the associated boundary or interface conditions that must be satisfied. This is important in many applications, especially in seismology, and needs to be analyzed in detail.

After this work was completed we became aware of the papers of R. Weaver [49,50] in which transport equations for elastic waves are derived by a different method and the equipartition law (5.40) is obtained.

7 Appendix: Multiscale expansion for the Transport Approximation

A multiscale analysis of (2.31) provides a quick formal way to get the transport equations. Detailed analysis is given in [48]. We expand the solution $W^\varepsilon(t, \mathbf{x}, \mathbf{k})$ of (2.31) in powers of ε

$$W^\varepsilon(t, \mathbf{x}, \mathbf{k}) = W^{(0)}(t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) + \sqrt{\varepsilon} W^{(1)}(t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) + \varepsilon W^{(2)}(t, \mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) + \dots \tag{7.1}$$

and assume that the leading term $W^{(0)}$ does not depend on the fast scale $\boldsymbol{\xi} = \mathbf{x}/\varepsilon$ and is deterministic. We replace

$$\nabla_{\mathbf{x}} \to \frac{1}{\varepsilon}\nabla_{\boldsymbol{\xi}} + \nabla_{\mathbf{x}}$$

in (2.31) and insert expansion (7.1) into (2.31). The term $W^{(1)}$ satisfies the equation

$$\mathbf{k}\cdot\nabla_{\boldsymbol{\xi}} W^{(1)} + \theta W^{(1)} = i\int e^{-i\mathbf{p}\cdot\boldsymbol{\xi}}\hat{V}(\mathbf{p})\{W^{(0)}(\mathbf{k}-\frac{\mathbf{p}}{2}) - W^{(0)}(\mathbf{k}+\frac{\mathbf{p}}{2})\}d\mathbf{p}, \tag{7.2}$$

where θ is a regularization parameter which will be set to zero later. This equation can be solved explicitly and the Fourier transform in $\boldsymbol{\xi}$ of $W^{(1)}$ is given by

$$\frac{\hat{V}(\mathbf{p})[W^{(0)}(\mathbf{k}+\frac{\mathbf{p}}{2}) - W^{(0)}(\mathbf{k}-\frac{\mathbf{p}}{2})]}{\mathbf{k}\cdot\mathbf{p}+i\theta}. \tag{7.3}$$

The next term $W^{(2)}$ satisfies the equation

$$\frac{\partial W^{(0)}}{\partial t} + \mathbf{k}\cdot\nabla_{\mathbf{x}}W^{(0)} + \mathbf{k}\cdot\nabla_{\boldsymbol{\xi}}W^{(2)} + i\int e^{-i\mathbf{p}\cdot\boldsymbol{\xi}}\hat{V}(\mathbf{p})[W^{(1)}(\mathbf{k}+\frac{\mathbf{p}}{2}) - W^{(1)}(\mathbf{k}-\frac{\mathbf{p}}{2})]d\mathbf{p} = 0. \tag{7.4}$$

Note that

$$< \frac{\partial W^{(2)}}{\partial \boldsymbol{\xi}} >= 0 \tag{7.5}$$

and so after averaging (7.4) has the form

$$\frac{\partial W^{(0)}}{\partial t} + \mathbf{k}\cdot\nabla_{\mathbf{x}}W^{(0)} + < i\int e^{-i\mathbf{p}\cdot\boldsymbol{\xi}}\hat{V}(\mathbf{p})[W^{(1)}(\mathbf{k}+\frac{\mathbf{p}}{2}) - W^{(1)}(\mathbf{k}-\frac{\mathbf{k}}{2})]d\mathbf{p} >= 0. \tag{7.6}$$

We insert the Fourier transform (7.3) in (7.6) and use (2.29) to obtain as $\theta \to 0$

$$\begin{aligned} &< i\int e^{-i\mathbf{p}\cdot\boldsymbol{\xi}}\hat{V}(\mathbf{p})[W^{(1)}(\mathbf{k}+\frac{\mathbf{p}}{2}) - W^{(1)}(\mathbf{k}-\frac{\mathbf{p}}{2})]d\mathbf{p} > \\ &= \int \hat{R}(\mathbf{p}-\mathbf{k})[W^{(0)}(\mathbf{k}) - W^{(0)}(\mathbf{p})]\frac{2\theta}{\frac{1}{4}(\mathbf{k}^2-\mathbf{p}^2)^2+\theta^2}d\mathbf{p} \\ &\to 4\pi\int \hat{R}(\mathbf{p}-\mathbf{k})[W^{(0)}(\mathbf{k}) - W^{(0)}(\mathbf{p})]\delta(\mathbf{k}^2-\mathbf{p}^2)d\mathbf{p}. \end{aligned} \tag{7.7}$$

This holds because

$$\frac{\theta}{x^2+\theta^2} \to \pi\delta(x)$$

as $\theta \to 0$. We insert (7.7) in (7.6) and find that $W^{(0)}$ satisfies the transport equation (2.34).

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