Self-focusing in the presence of small time dispersion and nonparaxiality

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We analyze the combined effect of small time dispersion and nonparaxiality on self-focusing and their ability to arrest the blowup of laser pulses by deriving reduced equations which depend only on the propagation distance and time. We calculate the pulse duration for which time dispersion dominates over nonparaxiality, or vice versa. We identify additional terms (shock term, group velocity nonparaxiality, etc) which should be retained when time dispersion and nonparaxiality are of comparable magnitude. These additional terms lead to temporal asymmetry and in the visible spectrum they can dominate over both time dispersion and nonparaxiality.

The simplest model for optical self-focusing is the nonlinear Schrödinger equation (NLS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0 \ , \quad \psi(0,r) = \psi_0(r) \ . \tag{1}$$

Here, $\psi(z,r)$ is the electric field envelope of a laser beam propagating in a medium with Kerr nonlinearity, z is the distance in the direction of the propagation, $r = \sqrt{x^2+y^2}$ is the radial coordinate and $\Delta_\perp = \partial^2/\partial r^2 + (1/r)(\partial/\partial r)$ is the Laplacian in the transverse two-dimensional plane. It is well known that if the initial power is above a critical value (i.e. $\int |\psi_0|^2 r dr \geq N_c \cong 1.86$), solutions of eq. (1) may blow-up in a finite distance z. Since physical quantities do not become infinite, it is clear that the validity of eq. (1) breaks down near the focal point and that additional physical mechanisms, which are initially small, become important there and prevent singularity formation.

In this Letter we focus on the combined effect of two mechanisms which may arrest blowup and which are neglected when approximating Maxwell's equations by NLS: Small time dispersion and beam nonparaxiality. In previous studies it was suggested that small nonparaxiality arrests self-focusing and leads to an oscillatory focusing-defocusing behavior [1,2]. In other studies it was shown that small normal time dispersion delays the onset of self-focusing and causes the temporal splitting of the pulse into two peaks which continue to focus [3,4]. However, it is still unknown at present whether the solution will ultimately blow up or not. Recently, pulse splitting was observed experimentally [5].

An important question which arises when modeling physical self-focusing is whether time dispersion and/or nonparaxiality should be included in the model. In this Letter we answer this question by identifying the regimes where each mechanism dominates. While doing this, additional terms are identified which should be kept in the model when time-dispersion and nonparaxiality are of the same order. In fact, these additional terms can even dominate over both time dispersion and nonparaxiality in the visible spectrum. We then derive reduced equations which describe self-focusing when all of the above mechanisms are present. We use these reduced equations to analyze the combined effect of normal time-dispersion and nonparaxiality - both of which arrest self-focusing, the case of anomalous time-dispersion and nonparaxiality- which have opposite focusing effects, and the influence of the additional terms.

We begin by deriving NLS with nonparaxiality and time dispersion. If we neglect vectorial effects [6], the electric field can be assumed to have the form

$$\vec{E}(x,y,z,t) = \vec{e}A(x,y,z,t)\exp(ik_0 z - i\omega_0 t)$$

where the unit vector $\vec{e}$ is perpendicular to the z axis. The equation for the slowly varying envelope A is [7]:

$$A_{zz} + 2ik_0 A_z + \Delta_\perp A + \frac{2ik_0}{c_g}A_t - \left(\frac{1}{c_g^2} + k_0 k_{\omega\omega}\right)A_{tt}$$
$$= \frac{2n_2 n_0}{c^2}\exp(i\omega_0 t)\Big(|A|^2 A \exp(-i\omega_0 t)\Big)_{tt}$$

where $k = \omega n_0(\omega)/c$, $k_0 = k(\omega_0)$, $c_g^{-1} = (dk/d\omega)_{\omega_0}$, n_0 is the linear index of refraction and n_2 is the Kerr coefficient. We change to a nondimensional moving-frame coordinate system with

$$\tilde{r} = \frac{r}{r_0} \ , \quad \tilde{z} = \frac{z}{2L_{\text{diff}}} \ , \quad \tilde{t} = \frac{t - z/c_g}{T} \ , \quad \psi = r_0 k_0 \sqrt{\frac{2n_2}{n_0}}A \ ,$$

where r_0 is the initial pulse width, $L_{\text{diff}} = r_0^2 k_0$ is the diffraction length and T is the pulse duration. Dropping the tilde and neglecting the $(|A|^2A)_{tt}$ term, which is $O(\epsilon_2^2)$, the equation for the nondimensional envelope ψ is

$$i\psi_z + \triangle_\perp\psi + |\psi|^2\psi + \epsilon_1\psi_{zz} + \epsilon_2\left[2i\frac{n_0 c_g}{c}(|\psi|^2\psi)_t - \psi_{zt}\right] - \epsilon_3\psi_{tt} = 0 \tag{2}$$

where

$$\epsilon_1 = \frac{1}{4r_0^2k_0^2}\ , \quad \epsilon_2 = \frac{1}{c_g k_0 T} = \frac{1}{\omega_0 T}\frac{c}{n_0 c_g}\ , \quad \epsilon_3 = \frac{L_{\rm diff}k_{\omega\omega}}{T^2}\ . \tag{3}$$

The dimensionless parameter $\epsilon_1 \sim$ (wavelength/radial pulse width)^2, $\epsilon_2 \sim$ (period of one oscillation/pulse duration), and ϵ_3 is a dimensionless measure of group velocity dispersion (GVD). Note that

$$\epsilon_2^2 = \epsilon_1\epsilon_3 F\ , \quad F = \frac{4}{c_g^2 k_0 k_{\omega\omega}}\ .$$

The first component of the ϵ_2 term is sometimes called the shock term [8]. The second component can be replaced by:

$$-\epsilon_2\psi_{zt} \sim -i\epsilon_2\left[\Delta_\perp\psi_t + (|\psi|^2\psi)_t\right] \tag{4}$$

and its linear part $(-i\epsilon_2\Delta_\perp\psi_t)$ was interpreted by Rothenberg as the effect of the variation of the group velocity of a tilted ray projected onto the z-axis [8].

Let us define T_b as the pulse duration for which time dispersion and nonparaxiality are of the same magnitude (i.e. $\epsilon_1 = |\epsilon_3|$):

$$T_b = 2L_{\rm diff}\sqrt{|k_0 k_{\omega\omega}|} = \frac{4}{\sqrt{|F|}}\frac{L_{\rm diff}}{c_g}\ .$$

If F is $O(1)$ then, when $T \ll T_b$ time dispersion will initially dominate and $\epsilon_1 \ll \epsilon_2 \ll \epsilon_3$, but as the pulse becomes narrower $\epsilon_1 \sim r^{-2}$ increases while $\epsilon_3 \sim r^2$ decreases. When $T \gg T_b$ nonparaxiality dominates and $\epsilon_1 \gg \epsilon_2 \gg \epsilon_3$. Note that it is not possible to include in the model both the ϵ_1 and ϵ_3 terms without retaining also the ϵ_2 term.

The ϵ_2 term is usually assumed to be small compared with either time dispersion or nonparaxiality. However, we now show that in the visible spectrum it can dominate both. The index of refraction of optical materials such as water [9] or silica [10] in the range of transparency is almost constant and $|\omega n_\omega| \ll 1$ [11]. For example, by using data digitized from [9] it was estimated that for water in the visible spectrum $|\omega n_\omega| \sim 0.03$ [12]. Therefore, $c_g \sim c/n_0$, $\epsilon_2 > 0$ and

$$|F| \sim \frac{2n_0}{|\omega n_\omega|} \gg 1\ ,$$

with $F \sim 100$ for water, for example. This implies that in the visible regime and with $T = O(T_b)$, both ϵ_1 and ϵ_3 are small $(O(1/\sqrt{|F|}))$ compared with ϵ_2. When $T = T_b\sqrt{|F|}$ (or $T = T_b/\sqrt{|F|}$), $\epsilon_1 = \epsilon_2$ (or $\epsilon_3 = \epsilon_2$) and $\epsilon_3/\epsilon_2 = O(1/F)$ (or $\epsilon_1/\epsilon_2 = O(1/F)$). Only when $T \gg T_b\sqrt{|F|}$ (or $T \ll T_b/\sqrt{|F|}$) do we have that $\epsilon_3 \ll \epsilon_2 \ll \epsilon_1$ (or $\epsilon_1 \ll \epsilon_2 \ll \epsilon_3$). Moreover, using (4) and $c_g \sim c/n_0$, eq. (2) reduces to

$$i\psi_z + \triangle_\perp\psi + |\psi|^2\psi + \epsilon_1\psi_{zz} + i\epsilon_2\left[(|\psi|^2\psi)_t - \Delta_\perp\psi_t\right] - \epsilon_3\psi_{tt} = 0 \tag{5}$$

The separate effects of small time dispersion and nonparaxiality were analyzed before [2,4] using a perturbation method that allows the derivation of simplified equations [13]. Briefly, near the focal point the solutions of (2) or (5) have the form

$$\psi(z,t,r) \sim \frac{1}{L(z,t)}R\left(\frac{r}{L}\right)\exp\left(i\zeta(z,t) + i\frac{L_z r^2}{4L}\right)$$

where $R(r) > 0$, the radial profile (Townes soliton), satisfies $\Delta_\perp R - R + R^3 = 0$ and $\int R^2\, r dr = N_c$. By averaging over the transverse coordinates we find that the modulation functions L and ζ must satisfy the reduced equations

$$\zeta_z(z,t) = \frac{1}{L^2}\ , \quad L_{zz}(z,t) = -\frac{\beta(z,t)}{L^3}\ , \tag{6}$$

$$\beta_z(z,t) = -\gamma_1\left(\frac{1}{L^2}\right)_z - \gamma_2\left(\frac{1}{L^2}\right)_t + \gamma_3\zeta_{tt}\ , \tag{7}$$

where $\gamma_1 = 2\epsilon_1 N_c/M$, $\gamma_2 = \epsilon_2(6c_g n_0/c - 2)N_c/M$ for eq. (2) and $\gamma_2 = 4\epsilon_2 N_c/M$ for eq. (5), $\gamma_3 = 2\epsilon_3 N_c/M$ and $M = 1/4 \int R^2 r^3\, rdr \cong 0.55$. The modulation functions that solve the reduced system have the following meaning [13,14]: β is proportional to the excess cross-sectional power above critical, L is the non-dimensional radial pulse width, and is also inversely proportional to the on-axis intensity $|\psi(z,t,r=0)|$ so that blowup occurs when $L=0$, and ζ is the rescaled axial distance. The system (6–7) is much easier both for analysis and simulations, since the radial dependence in eq. (2) has been eliminated.

In the pulse splitting experiment [5] the values of the nondimensional parameters are: $\epsilon_1 = 1.3\times 10^{-6}$, $\epsilon_2 = 5\times 10^{-3}$, $\epsilon_3 = 1.5\times 10^{-1}$. Using these values and the initial conditions $L(0,t)\equiv 1$, $\beta(0,t) = N_c(1.05 exp(-t^2)-1)/M$, we integrated eqs. (6–7). These initial conditions may not be close to those of the experiment in the focusing regime, which are unknown, but they do give an idea of how the pulse evolves. We observe (Fig. 1) pulse splitting (due to normal time dispersion), accompanied by a temporal shift of the focus towards later times and enhanced focusing of the second peak (due to the ϵ_2 term).

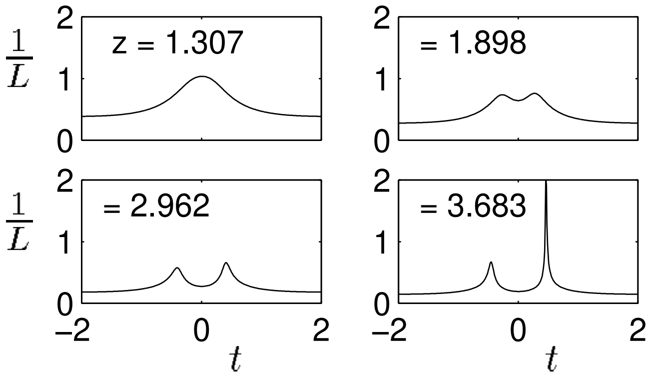

FIG. 1. Evolution of the on-axis intensity $(1/L)$ versus time according to Eqs. (6–7) at the propagation distances indicated.

Following [4], we can analyze the initial effect of the three terms in (2) by looking at special solutions of (6–7). Away from the focal point, the three perturbing terms in (2) are small and each t cross-section of the pulse (i.e. the 2D plane $t=$ const in the (x,y,t) space) focuses independently with

$$L(z,t) = L(Z_c(t)-z)\ ,\quad \beta(z,t) = \beta(Z_c(t)-z)\ ,\quad \zeta(z,t) = \zeta(Z_c(t)-z)\ . \tag{8}$$

Here $Z_c(t)$ is the location of the focus in the (z,t) plane when $\epsilon_1=\epsilon_2=\epsilon_3=0$ [14]. Therefore, eq. (7) becomes

$$\beta_z = -\gamma_1\left(\frac{1}{L^2}\right)_z + \gamma_2\dot{Z}_c\left(\frac{1}{L^2}\right)_z + \gamma_3(-\ddot{Z}_c\zeta_z + \dot{Z_c}^2\zeta_{zz})\ ,\quad \dot{} = \frac{d}{dt}\ . \tag{9}$$

This equation can be transformed into a nonlinear Airy equation [4]

$$g_{ss} = sg + \kappa g^3\ ,\quad \text{with}\quad g = L^{-1} > 0\ . \tag{10}$$

Here

$$s = \left(\beta_0 - \gamma_3\ddot{Z}_c\zeta\right)\left(\gamma_3\ddot{Z}_c\right)^{-2/3}\ ,\quad \beta_0 \sim \beta(0,t)\ ,$$
$$\kappa = -(\gamma_1 - \gamma_2\dot{Z}_c - \gamma_3\dot{Z_c}^2)\left(\gamma_3\ddot{Z}_c\right)^{-2/3}$$

The initial conditions for eq. (10) are given at

$$s_0(t) := s(z=0,t) \sim \beta(0,t)\left(\gamma_3\ddot{Z}_c\right)^{-2/3}\ .$$

At the time t_0 of the initial peak power of the pulse, $Z_c(t)$ attains its minimum, $\dot{Z}_c(t_0)=0$ and the evolution is given by (10) with $\kappa = -\gamma_1\left(\gamma_3\ddot{Z}_c\right)^{-2/3} < 0$. Because $\ddot{Z}_c(t_0)>0$, as $z\to Z_c$ and $\zeta\to+\infty$, $s\to-\infty$ for normal time dispersion ($\epsilon_3>0$), and both time-dispersion and nonparaxiality (first and second terms on the right-hand-side of (10), respectively) contribute to the arrest of the blowup by preventing g from becoming infinite. When time dispersion is anomalous ($\epsilon_3 < 0$), it enhances blowup ($s \to +\infty$) while nonparaxiality opposes it. Eventually, as $s \to +\infty$ nonparaxiality prevails and the solution of (10) will decay (no blowup).

In the case of normal time dispersion and $\epsilon_1 = \epsilon_2 = 0$, blowup is arrested only in an exponentially small neighborhood of t_0 [4], where pulse splitting occurs. In order to assess the added effects of nonparaxiality and the mixed term, we note that the condition for blowup [4] in (10) as $s \to -\infty$ is $\kappa > 2L^2(0,t)Ai^2(s_0)$ or

$$\gamma_3 \dot{Z}_c^{\,2} > \gamma_1 - \gamma \dot{Z}_c + 2L^2(0,t)Ai^2(s_0)\left(\gamma_3 \ddot{Z}_c\right)^{2/3}$$

where $Ai(s)$ is the Airy function. Therefore, if nonparaxiality dominates, arrest of blowup occurs over a much larger region (possibly everywhere). If the ϵ_2 term dominates, blowup will occur when $\epsilon_3 > -\epsilon_2/\dot{Z}_c$, i.e. only for $t > t_0$. Note that as the solution starts to deviate from that of the unperturbed NLS, the 2D self-similar structure (8) will gradually break down. Therefore, for later z this 2D self-similar argument becomes invalid and the full 3D nature of (7) has to be considered.

From eq. (9) we see that the effect of the ϵ_2 term on a self-focusing pulse is a temporal power transfer towards later times (recall that β is proportional to the excess power above critical). This will result in an asymmetric temporal development of the pulse, with a greatly enhanced trailing portion and a suppressed leading part, in agreement with previous results on the effect of the shock term [15] and of the linear component of the ϵ_2 term [8].

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