Traveling waves in a mean field learning model

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Abstract

Lucas and Moll have proposed in [22] a system of forward-backward partial differential equations that model knowledge diffusion and economic growth. It arises from a microscopic model of learning for a mean-field type interacting system of individual agents. In this paper, we prove existence of traveling wave solutions to this system. They correspond to what is known in economics as balanced growth path solutions. We also study the dependence of the solutions and their propagation speed on various economic parameters of the system.

1 Introduction

In this paper, we study an extension of a mean field game model for propagation of knowledge and economic growth, proposed by Lucas and Moll [22]. Mean field models are used in optimal decision making based on stochastic games with a very large population of agents that are statistically identical [7, 10, 11, 13, 17, 21]. In such models, the overall effect of the other agents on a single one can be replaced by an averaged effect, and the optimal behavior of a single agent can be determined as a solution to an optimal control problem that depends on the distribution of the other agents and not on their precise cofiguration. The model consists of two partial differential equations – a forward Kolmogorov equation that keeps track of the distribution of agents, and a backward Hamilton-Jacobi-Bellman (HJB) equation for the value function of the optimal stochastic control problem for each agent. An equilibrium solution is a solution to the coupled system of the two equations valid for all time.

In the model proposed in [22], the propagation of knowledge is a Markovian process that involves jumps, but no diffusion and the economic growth is modeled by a production function, that depends on the knowledge of each agent in the economy. The mean field model is then a coupled system of a forward Kolmogorov equation for the distribution of knowledge, of the linear kinetic type, and a backward Hamilton Jacobi Bellman equation for the value function of an individual in the economy [1, 22]. The two equations are coupled through a search function,

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representing the tradeoff between the time spent learning, with no production, and the time spent producing. We should mention that [22] also contains an excellent survey of the other literature related to models of knowledge diffusion and growth: without any attempt at completeness we mention here [2, 3, 12, 14, 18–20, 23, 24, 26, 27] but an interested reader should consult [22] for an illuminating discussion of various other existing models and further references.

Balanced Growth Paths (BGP) are special solutions to that system, valid for all time, corresponding to a constant growth rate for the economy in equilibrium. In their paper, Lucas and Moll not only propose this interesting model for propagation of knowledge and economic growth, but also study numerically the BGP solutions of the system [22]. Such solutions are shown to exist in [8] and [9] using a fixed point method in certain function spaces.

In this paper, we study a model, similar to that introduced in [22], but with diffusion added to the process modeling the propagation of knowledge (in a logarithmic variable), introducing an additional level of uncertainty. The model is governed by the following system of partial differential equations:

$$\frac{\partial\psi(t,x)}{\partial t} = \kappa\frac{\partial^2\psi}{\partial x^2} + \psi(t,x)\int_{-\infty}^{x}\alpha(s^*(t,y))\psi(t,y)dy - \alpha(s^*(t,x))\psi(t,x)\int_{x}^{\infty}\psi(t,y)dy \tag{1.1}$$

and

$$\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \kappa\frac{\partial^2 V(t,x)}{\partial x^2} + \max_{s\in[0,1]}\Big[(1-s)e^x + \alpha(s)\int_x^\infty [V(t,y)-V(t,x)]\psi(t,y)dy\Big]. \tag{1.2}$$

Here, $\psi(t,x)$ is the density of the distribution of the agents, $V(t,x)$ is the value function obtained from the optimal stochastic control problem for each agent, and $s(t,x)$ is the optimal control for each agent. The function $\alpha(s)$, known as the search function, represents the chance of success if an agent searches for a fraction $s \in [0,1]$ of time. The maximizer in (1.2) is unique if the function $\alpha(s)$ is concave. This assumption is also natural from the economics point of view – if $\alpha(s)$ were convex, the preference to search and not to produce would be overwhelming. The derivation of this model is recalled in Section 2.

We should mention that diffusion was also considered in [9] in the original variables, where BGP solutions were constructed numerically. We choose to add diffusion after the change of variables, because the logarithmic variables are natural from the economics point of view. The economic role of the diffusive term is to incorporate the change of knowledge not due to learning via meeting an outside agent but due to innovation and experimentation, as discussed in [24, 27]. Such experimentation may occasionally lead to a small boost in productivity, and sometimes to a small loss in productivity, and the diffusion term reflects this.

Construction of the BGP solutions in [8] and [9] relied on viewing (1.1), in the original variables, as a linear kinetic equation. Here, we take a different point of view. As has already been pointed out in [1, 22], in the very special case when $\alpha(s)$ is independent of s, the cumulative distribution function of the agents is decoupled from (1.2) and satisfies the Fisher-KPP (Kolmogorov-Petrovski-Piskunov) equation. This assumption means that the success of the search does not depend on the fraction of the time spent searching and is not realistic. However, the structure of the full coupled problem without this assumption, for a general search function $\alpha(s)$, still inherits some Fisher-KPP features that allow us to use a strategy originating in the construction of traveling waves for reaction-diffusion equations in [4–6], albeit with non-trivial modifications coming from the required estimates for the HJB equation.

As we explain below, a Balanced Growth Path (BGP) solution of the original system corresponds to a traveling wave solution of (1.1)-(1.2). These are solutions of the form

$$\Psi(t,x) = F(x-ct), \quad V(t,x) = e^{ct}Q(x-ct), s^*(t,x) = s^*(x-ct), \tag{1.3}$$

where $1-\Psi(t,x)$ is the cumulative distribution function so that $\Psi_x(t,x) = -\psi(t,x)$. As the BGP solutions, traveling waves are an important class of solutions to the infinite time horizon problem. The traveling wave equations satisfied by F and Q in the case $\alpha(s) = \alpha\sqrt{s}$ are written explicitly in (2.30)-(2.31) below.

The main result of this paper is a proof of existence of traveling waves for a specific choice of a search function $\alpha(s) = \alpha\sqrt{s}$.

Theorem 1.1. *There exist ρ_0 that depends on κ and α, and α_0 that depends on κ, so that if $\rho > \rho_0$ and $\alpha > \alpha_0$, then there exists c such that $0 < c < 2\sqrt{\kappa\alpha}$ so that the system (1.1)-(1.2) has a solution of the form (1.3), such that $F(x)$ is monotonically decreasing, $Q(x)$ is monotonically increasing, and*

$$\begin{aligned} &\lim_{x\to-\infty} F(x) = 1, \lim_{x\to+\infty} F(x) = 0,\\ &\lim_{x\to-\infty} Q(x) \quad \text{exists and is positive,}\\ &\lim_{x\to+\infty} (Q(x)e^{-x}) = \frac{1}{\rho-\kappa}. \end{aligned} \tag{1.4}$$

In addition, there exist constants $A_{1,2} > 0$, $B > 0$ such that

$$A_1 e^x \le Q(x) \le A_2 e^x + B, \tag{1.5}$$

and $x_0 \in \mathbb{R}$ so that $s^*(x) = 1$ for all $x < x_0$ and $s^*(x) < 1$ for all $x > x_0$, and $F(x)$ satisfies

$$\int_{-\infty}^{\infty} |F_x|^2 dx < +\infty, \tag{1.6}$$

The point x_0 is the transition point between the agents for $x < x_0$ that do not produce at all, but rather spend all their time acquiring new knowledge, so that $s^*(x) = 1$, and agents for $x > x_0$ that spend a fraction $s^*(x) \in (0,1)$ of the time acquiring new knowledge and a non-trivial fraction $(1-s^*(x))$ of their time producing. Note that $s^*(x) > 0$ for all $x > x_0$. This means that all agents, no matter how advanced, spend a positive fraction of their time learning and not just producing. This is a consequence of the assumption that $\alpha(s) = \alpha\sqrt{s}$, more specifically, of the fact that $\alpha'(0) = +\infty$. Otherwise, if $\alpha'(0) < +\infty$, there would be another transition point x_1 so that $s^*(x) = 0$ for all $x > x_1$ – the very advanced agents would not search at all and will increase their knowledge only by a random experimentation via diffusion. On the other hand, if $\alpha'(0) = +\infty$ then searching even for a small fraction of the time gives a "disproportionally large" chance of success, so that even advanced agents perform a search. This is discussed in more detail in Section 2.3.

The assumption that $\alpha(s) = \alpha\sqrt{s}$ is convenient to simplify some considerations but our result can be generalized to concave functions $\alpha(s)$ such that $\alpha'(0) = +\infty$ in a straightforward manner.

The case of a concave $\alpha(s)$ such that $\alpha'(0) < +\infty$ can also be studied with a similar approach, except for the existence of the second transition point x_1 mentioned above. We choose to work with $\alpha(s) = \alpha\sqrt{s}$ to keep the presentation as simple as possible while still interesting from the economics point of view.

The assumption that the discount rate ρ is sufficiently large in Theorem 1.1 is natural from the economic intuition. If the discount rate is too small, there is not a sufficient incentive to produce today, so that the agents would spend all their time just learning and we expect that the balanced growth paths do not exist. In particular, we expect that the transition point x_0 moves to $+\infty$ as the discount rate approaches the critical value $\rho_0 > 0$ from above, with the parameters α and κ fixed. This is further illustrated numerically in Section 4.4.

As we have mentioned, when $\alpha(s) = \alpha$ is constant, the equation for $\Psi(t, x)$ reduces to the classical Fisher-KPP equation

$$\Psi_t = \kappa\Psi_{xx} + \alpha\Psi(1 - \Psi). \tag{1.7}$$

In that case, traveling waves exist for all $c \geq c_{FKPP} = 2\sqrt{\kappa\alpha}$. One may wonder if for the full system (1.1)-(1.2), there may also exist traveling waves for all speeds c larger than some minimal speed c_*. While we only prove here existence of the wave for a single speed, corresponding to the minimal speed, we can argue that traveling wave does not exist for large speeds. The equation for F, corresponding to the traveling wave profile of Ψ is

$$-cF_x - \kappa F_{xx} = \alpha F \int_{-\infty}^{x} s^*(y)(-F_y)dy, \quad F(-\infty) = 1, \quad F(+\infty) = 0. \tag{1.8}$$

As in the classical Fisher-KPP case, as $x \to +\infty$, the solution to (1.8) has the asymptotics

$$F(x) \sim e^{-\lambda x}, \text{ as } x \to +\infty, \tag{1.9}$$

with the exponential decay rate λ related to the propagation speed c by

$$c\lambda - \kappa\lambda^2 = \alpha\gamma, \quad \lambda = \frac{c - \sqrt{c^2 - 4\alpha\kappa\gamma}}{2\kappa}, \tag{1.10}$$

with

$$\gamma = \int_{-\infty}^{\infty} s^*(y)(-F_y)dy \leq \int_{-\infty}^{\infty} (-F_y)dy = 1. \tag{1.11}$$

The difference with the standard Fisher-KPP situation is that γ is not explicit but the decay rate λ and the traveling wave speed c are still related by (1.10). It follows, in particular, that $\lambda < 1$ if $c > \alpha + \kappa$. However, the value function $Q(x)$ has the asymptotics $Q(x) \sim e^x$ as $x \to +\infty$, that comes both from (1.2) and its economic interpretation. In addition, for the traveling wave solutions to make sense, the expected benefit of the search, given by the integral

$$\int_x^{\infty} [Q(y) - Q(x)](-F_y(y))dy = \int_x^{\infty} Q'(y)F(y)dy, \tag{1.12}$$

that appears in the right side of (1.2), must be finite. This in incompatible with (1.9) if $\lambda < 1$. It follows that traveling waves with speeds $c > \kappa + \alpha$ can not exist. We expect that there exists an interval of speeds $[c_{min}, c_{max})$ so that (1.1)-(1.2) has traveling wave solutions for all $c \in [c_{min}, c_{max})$.

This gives a limit on how fast economy may grow along a balanced growth path, within the parameters of this model.

The upper bound $c < c_{FKPP} = 2\sqrt{\kappa\alpha}$ in Theorem 1.1 is an immediate consequence of the following relation between the speed c and the wave profile constructed in Theorem 1.1, that was conjectured in [1], as a direct analog of the minimal front speed formula $c_{FKPP} = 2\sqrt{\kappa\alpha}$ for (1.7).

Proposition 1.2. *The speed c, the search function $s^*(x)$ and $F(x)$ constructed in Theorem 1.1 are related by*

$$c = 2\sqrt{\kappa\alpha\gamma}, \tag{1.13}$$

with $\gamma < 1$ as in (1.11).

The assumption of Theorem 1.1, that the search effectiveness parameter α is large is also natural from the economic intuition. The proof of Proposition 1.2 shows not only that (1.13) holds but also that the traveling wave $F(x)$ constructed in Theorem 1.1 satisfies (1.9) with

$$\lambda = \frac{c}{2\kappa} = \sqrt{\frac{\alpha\gamma}{\kappa}} < \sqrt{\frac{\alpha}{\kappa}}, \tag{1.14}$$

in agreement with (1.10), due to (1.13). Therefore, if $\alpha \leq \kappa$ is too small, then $\lambda \leq 1$ and the integral in (1.12) would, once again, blow up. As we show below, the transition point x_0 and the integral in (1.12) are related by

$$e^{x_0} = \int_{x_0}^{\infty} Q'(y)F(y)dy. \tag{1.15}$$

As α approaches a critical value α_0 from above, we expect that the integral in (1.12) blows up for any x fixed. It follows then from (1.15) that we must have $x_0 \to +\infty$. In this sense, the effect of small alpha is similar to that of a small discount rate ρ: all agents search rather than produce, though for a different reason. Now, as α approaches α_0 from above, the chance of a successful search is small (even though it does not vanish as $\alpha \downarrow \alpha_0$), and more and more skilled agents have to search, to keep the economy growing along a balanced path. This is also illustrated numerically in Section 4.4.

Another consequence of (1.14) and the requirement that $\lambda > 1$ is that the traveling waves constructed in Theorem 1.1 satisfy

$$c \geq 2\kappa. \tag{1.16}$$

This condition holds also for any traveling wave, not just those we construct in that theorem. Indeed, any traveling wave that moves with a speed c satisfies the decay estimate in (1.9) with λ related to c via (1.10)-(1.11). The requirement that $\lambda > 1$ then implies the lower bound on the speed in (1.16).

The limitation in Theorem 1.1 that ρ is sufficiently large is also a limit on how large the diffusivity κ can be for a given value of the discount rate ρ. One can already see that from the behavior of $Q(x)$ as $x \to +\infty$ in (1.4). Mathematically, this comes from the requirement that ρ is larger than the principal eigenvalue of a certain linear operator with a diffusion term $\kappa\partial_x^2$. A toy model for this phenomenon is that the solution to

$$\frac{\partial\phi}{\partial t} + \rho\phi = \kappa\Delta\phi + e^x \tag{1.17}$$

with, say, zero initial condition, is given by

$$\phi(t,x) = \frac{1}{\kappa} e^x (e^{(\kappa-\rho)t} - e^{-\rho t}).$$

Therefore, for a balanced growth path to exist, the discount rate ρ has to be larger than κ – otherwise, $\phi(t,x)$ blows up as $t \to +\infty$. From the economics point of view, this means that, since the knowledge gained by diffusion already leads to an exponential growth in time, the discount rates need to be sufficiently high for a balanced growth path to exist, so the total production would not blow up.

As we have mentioned, when $\kappa = 0$ the balanced growth paths in the original non-logarithmic variables, which correspond exactly to the traveling wave solutions for (1.1)-(1.2), have been constructed in [8,9] using completely different techniques. These are, however, slightly different objects from the traveling wave we construct in Theorem 1.1 for $\kappa > 0$, as the case $\kappa = 0$ is special even for the Fisher-KPP equation (1.7), in the following sense. Generally, for $\kappa > 0$, traveling waves for (1.7) exist for all speeds $c \geq c_* = 2\sqrt{\kappa\alpha}$. The minimal speed is special in that solutions to the Cauchy problem for (1.7) with all sufficiently rapidly decaying initial conditions converge to a translate of the wave moving with the minimal speed, while traveling waves for $c > c_*$ represent the long time behavior of the solutions to (1.7) that have exactly the same exponential decay at $t = 0$ as the corresponding traveling wave. The economic interpretation of the former case is that the initial distribution of the logarithm of knowledge $\psi(0,x)$ has a right tail that decays quickly and may, in fact, have bounded support. The interpretation of the latter case is that this initial distribution not only has an unbounded support but, in fact, has a precise exponential right tail meaning that the initial distribution of the level of knowledge e^x has a fat right tail (it follows a power law). The role of positive diffusion in overcoming the need for heavy tails of the initial distributions is discussed in detail in [24]. On the other hand, when $\kappa = 0$, so that $c_* = 0$, there is no traveling wave solution for (1.7) moving with the minimal speed but traveling waves do exist for all $c > 0$. The balanced growth paths constructed in [8, 9] for $\kappa = 0$ are the analogs of these "super-critical" Fisher-KPP waves for $\kappa = 0$. On the other hand, the traveling wave constructed in Theorem 1.1 is the analog of the minimal speed wave for the Fisher-KPP equation and thus does not exist for $\kappa = 0$ but only for $\kappa > 0$. In that sense, Theorem 1.1 is a complementary result to [8, 9].

The methods of the present paper also allow to study the long time existence of solutions to the time-dependent coupled forward-backward problem (1.1)-(1.2). This will be discussed elsewhere [25].

Organization of the paper. In Section 2, we review the mean field learning model, presented in [22], and formulate the mean filed system with diffusion added after the logarithmic change of variable. We also discuss the formulation for the specific choice of a search function $\alpha(s) = \alpha\sqrt{s}$.

In Section 3, we prove Theorem 1.1. As we have mentioned, the proof uses a general strategy for the construction of traveling waves originating in [4–6] and is in two steps: first, we consider a suitable approximate problem on a finite interval $[-a, a]$, for a sufficiently large a. The key step is to show that a solution (F^a, Q^a, c^a) to the approximate problem exists. This is done by obtaining a priori bounds on the solutions and a degree argument. The a priori bounds for the coupled system is the main nontrivial difficulty in the present problem compared to the standard reaction-diffusion scalar equations. Next, using the a priori bounds on the solutions to the approximate problem on finite intervals, we pass to the limit along a subsequence $a_n \to +\infty$ and show that $(F^{a_n}, Q^{a_n}, c^{a_n})$ converge uniformly on compact sets to a solution (F, Q, c) to the traveling wave system (2.30), and that the boundary conditions (2.31) are also satisfied by the functions F and Q.

In Section 4, we describe an iterative finite difference numerical algorithm solving the problem on a finite interval $[-a, a]$ and discuss the properties of the numerical solutions. The simulations show clearly the validity of the result and clarify the dependence of the solutions on various parameters that enter the problem. We should mention that different iterative numerical algorithms for BGP solutions of the mean field model are given in [8], [9] and [22]. One difference with the present paper is in the procedure that finds numerically the wave speed.

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2 The mean field learning model

In this section, we recall the basics of the mean field learning model introduced in [22]. In addition, we reformulate the model in the logarithmic variables and add diffusion in the knowledge space. We also define the notion of traveling wave solutions correspond to the balanced growth paths in the original variables.

2.1 The non-diffusive model

Consider a population of agents, such that each agent has a certain knowledge $z \geq 0$ at a given time $t \geq 0$. An agent can either produce or learn at each moment of time, and we denote by $s(t, z) \in [0, 1]$ the fraction of time an agent with knowledge $z \geq 0$ spends learning on a time interval $[t, t + \Delta t]$, so that $(1 - s(t, z))$ is the fraction of time he spends producing on this time interval. His total production between the times t and $t + \Delta t$ is then

$$[1 - s(t, z)]z\Delta t. \tag{2.1}$$

Agents in the economy learn by meeting other agents, with a higher production knowledge. In order to describe the meetings, let $\Phi(t, z)$ be the fraction of the agents with knowledge less or equal to z at time t, and let $\phi(t, z) = \Phi_z(t, z)$ be the corresponding density. The probability that the search by an agent A with knowledge $z \geq 0$ is successful on a time interval $(t, t + \Delta t)$ is $\alpha(s(t, z))\Delta t$, where $\alpha(s) : [0, 1] \to \mathbb{R}^+$ is a given concave function. Given that the search is successful, the probability that A encounters an agent B with knowledge in the interval $(z', z' + \Delta z')$ is proportional to $\phi(t, z')\Delta z'$ – this is the mean field nature of the model. If the production knowledge of agent A is lower than the production knowledge of agent B, then agent A updates his production knowledge to that of agent B. The overall balance leads to the following nonlinear kinetic equation for the density $\phi(t, z)$:

$$\frac{\partial \phi(t, z)}{\partial t} = -\alpha(s(t, z))\phi(t, z) \int_z^\infty \phi(t, y)dy + \phi(t, z) \int_0^z \alpha(s(t, y))\phi(t, y)dy. \tag{2.2}$$

An agent with knowledge z at time t chooses the search time $s(t,z)$, so as to maximize the expected total production (value function) $V(t,z)$, discounted in time:

$$V(t,z) = \max_{s\in\mathcal{A}} \mathbb{E}\Big\{ \int_t^T e^{-\rho(\tau-t)} z(\tau)[1-s(\tau,z(\tau))]d\tau + V_T(z(T))\big|z(t)=z\Big\}. \tag{2.3}$$

Here $\rho > 0$ is a discount parameter and $\mathcal{A}$ is the set of admissible control functions, $T > 0$ is a given terminal time, and $V_T(z)$ is a prescribed terminal value. The value function $V(t,z)$ satisfies a Hamilton-Jacobi-Bellman equation:

$$\rho V(t,z) = \frac{\partial V(t,z)}{\partial t} + \sup_{s\in[0,1]} \Big\{(1-s)z + \alpha(s)\int_z^\infty [V(t,y)-V(t,z)]\phi(t,y)dy\Big\}, \tag{2.4}$$

supplemented by the terminal condition $V(T,z) = V_T(z)$ We will denote by $s^*(t,x)$ the optimal control in (2.4). An informal derivation of (2.2) and (2.4) is given in [22].

In the economics context, especially since we are soon going to introduce the diffusion of knowledge, it is natural to consider an exponential change of variables $\phi(t,z) = \psi(t,\log z)/z$, so that the function $\psi(t,x)$ is also a density but in the logarithmic variables

$$1 = \int_0^\infty \phi(t,z)dz = \int_0^\infty \psi(t,\log z)\frac{dz}{z} = \int_{-\infty}^\infty \psi(t,x)dx. \tag{2.5}$$

This change of variables transforms (2.2), (2.4) into the following system:

$$\frac{\partial\psi(t,x)}{\partial t} = \psi(t,x)\int_{-\infty}^x \alpha(s^*(t,y))\psi(t,y)dy - \alpha(s^*(t,x))\psi(t,x)\int_x^\infty \psi(t,y)dy. \tag{2.6}$$

$$\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \max_{s\in[0,1]} \Big[(1-s)e^x + \alpha(s)\int_x^\infty [V(t,y)-V(t,x)]\psi(t,y)dy\Big], \tag{2.7}$$

with the initial condition $\psi(0,x) = \phi(0,e^x)e^x$, and the terminal condition $V(T,x) = V_T(e^x)$.

2.2 The diffusive model

Equations (2.6)-(2.7) assume that the only changes in the productivity of the agents come from their interactions. It is reasonable from the economics point of view to assume that even in the absence of such interactions the productivity of each agent undergoes some diffusion, so that the agents learn not only from each other but also through experimenting, and it is natural to do that in the logarithmic variables, as in (2.6)-(2.7). Adding diffusion to both equations transforms the system to

$$\frac{\partial\psi(t,x)}{\partial t} = \kappa\frac{\partial^2\psi}{\partial x^2} + \psi(t,x)\int_{-\infty}^x \alpha(s^*(t,y))\psi(t,y)dy - \alpha(s^*(t,x))\psi(t,x)\int_x^\infty \psi(t,y)dy \tag{2.8}$$

and

$$\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \kappa\frac{\partial^2 V(t,x)}{\partial x^2} + \max_{s\in[0,1]} \Big[(1-s)e^x + \alpha(s)\int_x^\infty [V(t,y)-V(t,x)]\psi(t,y)dy\Big]. \tag{2.9}$$

This is the system (1.1)-(1.2).

We will also make use of the cumulative distribution function

$$\Psi(t,x)=\int_x^\infty \psi(t,y)dy,\quad \Psi(-\infty)=1,\ \Psi(+\infty)=0. \tag{2.10}$$

A straightforward computation shows that $\Psi(t,x)$ satisfies the following integro-differential equation:

$$\frac{\partial\Psi}{\partial t}-\kappa\frac{\partial^2\Psi}{\partial x^2}=-\Psi(t,x)\int_{-\infty}^x \alpha(s^*(t,y))\Psi_y(t,y)dy. \tag{2.11}$$

The equation for the value function $V(t,x)$ in terms of $\Psi(t,x)$ is

$$\rho V(t,x)=\frac{\partial V(t,x)}{\partial t}+\kappa\frac{\partial^2 V(t,x)}{\partial x^2}+\max_{s\in[0,1]}\Big[(1-s)e^x+\alpha(s)\int_x^\infty [V(t,y)-V(t,x)](-\Psi_y(t,y))dy\Big]. \tag{2.12}$$

Equation (2.11) should be supplemented by an initial condition for $\Psi(0,x)$ and (2.12) should come with a terminal condition for $V(T,x)=V_T(x)$ at some $T>0$. Existence of the solutions of the resulting forward-backward in time problem will be discussed elsewhere [25]. One natural terminal condition is $V_T(x)=0$, as there is no time left to produce at the end. This, however, is not the only possibility as one could also try to choose $V_T(x)$ so as to approximate the solution to the infinite time horizon problem with $T=+\infty$, so that $V(t,x)$ in (2.3) is re-defined as

$$V(t,z)=\max_{s\in\mathcal{A}}\mathbb{E}\Big\{\int_t^\infty e^{-\rho(\tau-t)}z(\tau)[1-(\tau,z(\tau))]\big|z(t)=z\Big\}. \tag{2.13}$$

A very interesting question, to be addressed in [25], is if the pair of solutions $\Psi_T(t,x)$, $V_T(t,x)$ defined on the time interval $0\le t\le T$, with some prescribed terminal conditions, have a well-defined limit $\Psi(t,x)$, $V(t,x)$ as $T\to+\infty$. This would be a natural candidate for a "correct" solution to the infinite horizon problem, without an explicit terminal condition for $V(t,x)$.

As we have mentioned, in the special case when $\alpha(s)=\alpha$ is a constant, the system (2.11)-(2.12) decouples, and (2.11) becomes the classical Fisher-KPP equation (1.7). Its solutions in the long time limit converge to traveling waves moving with the speed $c_*=2\sqrt{\kappa\alpha}$. This direct analogy to the Fisher-KPP type problems works only in the special case when $\alpha(s)$ is constant. However, in general, one still expects that, as in the FKPP case, the long time behavior of the solutions to (2.11) is governed to the leading order by the linearization as $x\to+\infty$:

$$\frac{\partial\tilde\Psi}{\partial t}-\kappa\frac{\partial^2\tilde\Psi}{\partial x^2}=R(t)\tilde\Psi(t,x),\quad R(t)=\int_{-\infty}^\infty \alpha(s^*(t,y))\Psi_y(t,y)dy, \tag{2.14}$$

Note that, unlike in the true FKPP case, the linearized equation (2.14) is not closed in general as the rate $R(t)$ depends on the function $V(t,y)$ as well. Nevertheless, it is natural to conjecture that solutions to the full problem still belong to the so called class of pulled fronts [15], and significant intuition can be gained from the Fisher-KPP analogy.

2.3 The choice of the search function

The maximization problem in (2.12) is of the form

$$\max_{s\in[0,1]}\Big[(1-s)+B\alpha(s)\Big], \tag{2.15}$$

with

$$B=e^{-x}\int_x^\infty[V(t,y)-V(t,x)](-\Psi_y(t,y))dy, \tag{2.16}$$

so that the optimal s is given by

$$s^*=s^*(B)=\begin{cases}0, \quad B\le \dfrac{1}{\alpha'(0)},\\ \beta(\dfrac{1}{B}), \quad \dfrac{1}{\alpha'(0)}<B\le\dfrac{1}{\alpha'(1)},\\ 1, \quad B>\dfrac{1}{\alpha'(1)},\end{cases} \tag{2.17}$$

where $\beta=(\alpha')^{-1}$. In order to avoid the situation where agents of sufficiently advanced knowledge do not search at all, it is natural to assume that $\alpha'(0)=+\infty$. To simplify some computations, we will make an assumption that $\alpha(s)=\alpha\sqrt{s}$ with some $\alpha>0$. Generalizations of our results to a general concave function $\alpha(s):[0,1]\to[0,1]$ with $\alpha'(0)=+\infty$ are quite straightforward. Now, equations (2.12) and (2.11) become

$$\rho V(t,x)=\frac{\partial V(t,x)}{\partial t}+\kappa\frac{\partial^2 V(t,x)}{\partial x^2}+e^x\max_{s\in[0,1]}\Big[(1-s^2)+\alpha s e^{-x}\int_x^\infty[V(t,y)-V(t,x)](-\Psi_y(t,y))dy\Big], \tag{2.18}$$

and

$$\frac{\partial\Psi}{\partial t}-\kappa\frac{\partial^2\Psi}{\partial x^2}=-\alpha\Psi(t,x)\int_{-\infty}^x s^*(t,y)\Psi_y(t,y)dy, \tag{2.19}$$

To simplify (2.18), we introduce the auxiliary functions

$$r(t,x)=\frac{\alpha}{2}e^{-x}\int_x^\infty[V(t,y)-V(t,x)](-\Psi_y(t,y))dy, \tag{2.20}$$

$$\begin{aligned}H(r(t,x))&=\max_{s\in[0,1]}\Big[(1-s^2)+\alpha s e^{-x}\int_x^\infty[V(t,y)-V(t,x)]\psi(t,y)dy\Big] \\ &=\max_{s\in[0,1]}\Big[(1-s^2)+2sr(t,x)\Big],\end{aligned} \tag{2.21}$$

so that $H(r)$ and the maximizer $S^*(r)$ are given by

$$H(r)=\begin{cases}2r, \ r>1,\\ 1+r^2, \ 0<r<1,\\ 1, \ r<0.\end{cases} \qquad S^*(r)=\begin{cases}1, \ r>1,\\ r, \ 0<r<1,\\ 0, \ r<0.\end{cases} \tag{2.22}$$

Now, we can write (2.18) - (2.19) as

$$\frac{\partial \Psi}{\partial t} - \kappa \frac{\partial^2 \Psi}{\partial x^2} = \alpha \Psi(t,x) \int_{-\infty}^{x} S^*(r(t,y))(-\Psi_y(t,y))dy, \tag{2.23}$$

and

$$\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \kappa \frac{\partial^2 V(t,x)}{\partial x^2} + e^x H(r(t,x)). \tag{2.24}$$

Thus, the new formulation of the problem are equations (2.23)-(2.24) for $\Psi(t,x)$ and $V(t,x)$, with the function $r(t,x)$ defined by (2.20), and $H(r)$ and $S^*(r)$ given by (2.22).

2.4 The traveling wave solutions

The infinite time horizon problem has special solutions that in the original variables are known as the balanced growths path (BGP). These are solutions to (2.2), (2.4) of the form

$$\phi(t,z) = e^{-\gamma t} f(ze^{-\gamma t}), \quad V(t,z) = e^{\gamma t} v(ze^{-\gamma t}), \quad s(t,z) = \sigma(ze^{-\gamma t}), \tag{2.25}$$

with some $\gamma > 0$, and $f(x), v(x) \in C^1(\mathbb{R})$ and $\sigma(x) \in C(\mathbb{R})$. The BGP solutions are interesting from the economics point of view since they give a constant growth rate for the economy, but they also give a well-defined solution to the infinite time horizon problem, and it is natural to conjecture, from the numerical evidence, that they should be the long time limit of the finite horizon problems on a time interval $[0,T]$ as $T \to +\infty$, with a proper terminal condition $V_T(x)$. This is similar to the stability of the Fisher-KPP traveling waves.

It has been shown in [8] that there exists $\mu_0 > 0$ so that the BGP solutions with the asymptotics

$$\phi(z) \sim z^{-\mu}, \text{ as } z \to +\infty, \tag{2.26}$$

exist for all $0 < \mu < \mu_0$, with a corresponding growth rate $\gamma(\mu) \in (0,\rho)$. After the exponential change of variables, a BGP solution defined for $z \geq 0$ transforms to a traveling wave solution for $x \in \mathbb{R}$ that moves with a constant speed equal to the growth rate γ:

$$\psi(t,x) = e^x \phi(t,e^x) = e^{x-\gamma t} f(e^{x-\gamma t}) = \Psi(x-\gamma t). \tag{2.27}$$

Traveling waves are solutions to the system (2.23)-(2.24) of the form

$$\Psi(t,x) = F(x-ct), \quad V(t,x) = e^{ct} Q(x-ct), \quad r(t,x) = R(x-ct). \tag{2.28}$$

They correspond to the balanced growth paths before the logarithmic change of variables. Note that if $F(x)$, $Q(x)$ and $R(x)$ form a traveling wave, with the corresponding search function $s^*(x)$, then for any fixed shift $y \in \mathbb{R}$, the functions

$$F_y(x) := F(x-y), \ s_y^*(x) := s^*(x-y), \ Q_y(x) := e^y Q(x-y), \ R_y(x) := R(x-y) \tag{2.29}$$

also form a traveling wave solution, so that traveling waves form a one parameter family, which is a typical situation in the theory of traveling waves. The only difference is that the value function $Q(y)$ is transformed slightly different under a shift by y.

A traveling wave satisfies the following system:

$$\begin{aligned}
&-cF_x-\kappa F_{xx}=\alpha F(x)\int_{-\infty}^{x}s^*(y)(-F_y(y))dy,\\
&\rho Q(x)=cQ(x)-c\frac{\partial Q(x)}{\partial x}+\kappa\frac{\partial^2 Q(x)}{\partial x^2}+e^xH(R(x)),\\
&R(x)=\frac{\alpha}{2}e^{-x}\int_x^{\infty}[Q(y)-Q(x)](-F_y(y))dy.
\end{aligned}\tag{2.30}$$

with $s^*(x)=\min[1,R(x)]$, and with boundary conditions

$$\begin{aligned}
&\lim_{x\to-\infty}F(x)=1,\ \lim_{x\to\infty}F(x)=0,\\
&\lim_{x\to-\infty}Q(x)\quad\text{exists and is positive},\\
&\lim_{x\to+\infty}(Q(x)e^{-x})=\frac{1}{\rho-\kappa}.
\end{aligned}\tag{2.31}$$

Theorem 1.1 is the existence result for this system.

The proof of Theorem 1.1 is presented in Section 3. As we have mentioned in the introduction, it proceeds in two steps: first, we consider an approximate problem on a finite interval $[-a,a]$, with $a\gg 1$, and an additional normalization $F^a(0)=1/2$ needed to fix the speed c^a. We obtain a priori bounds on the solutions and use a degree argument to show that there exists a solution (F^a,Q^a,c^a) to the approximate problem. In the second step, using the a priori bounds on the finite intervals, we pass to the limit along a subsequence $a_n\to+\infty$ and show that $(F^{a_n},Q^{a_n},c^{a_n})$ converge uniformly on compact sets to a solution (F,Q,c) to (2.30), and that the boundary conditions (2.31) are also satisfied by the functions F and Q. Proposition 1.2 is proved at the end of Section 3.

3 Existence of a traveling wave solution

In this section, we prove Theorem 1.1.

3.1 The finite interval problem

In the first step, we restrict the system (2.30) to a finite interval $[-a,a]$, with $a>0$ and consider the following approximate problem for the functions $F^a(x)$, $Q^a(x)$ and $R^a(x)$, and a speed c^a:

$$-c^aF^a_x-\kappa F^a_{xx}=\alpha F^a(x)\int_{-a}^{x}s^*_a(y)(-F^a_y(y))dy,\tag{3.1}$$

$$\rho Q^a(x)=c^aQ^a(x)-c^a\frac{\partial Q^a(x)}{\partial x}+\kappa\frac{\partial^2 Q^a(x)}{\partial x^2}+e^xH(R^a(x)),\tag{3.2}$$

$$R_a(x)=\frac{\alpha}{2}e^{-x}\int_x^{a}[Q^a(y)-Q^a(x)](-F^a_y(y))dy,\tag{3.3}$$

with $s_a^*(x) = \min[1, R^a(x)]$ and with the boundary conditions

$$F^a(-a) = 1, \quad F^a(a) = 0, \tag{3.4}$$
$$Q_x^a(-a) = 0, \quad Q_x^a(a) = Q^a(a). \tag{3.5}$$

In addition, we impose a normalization for F^a:

$$F^a(0) = 1/2, \tag{3.6}$$

that is needed to obtain uniform bounds on the speed c^a that at the moment is assumed to be unknown. Let us define x_0^a as

$$x_0^a = \sup\{x : \ s_a^*(x) = 1\}. \tag{3.7}$$

The main result of this step is the following proposition:

Proposition 3.1. *There exists $a_0 > 0$ so that for all $a > a_0$ there exists a constant $c_a \in \mathbb{R}$ for which the system (3.1)-(3.3) has a solution such that $F^a(x)$ and $R^a(x)$ are monotonically decreasing, $Q^a(x)$ is increasing, and the boundary conditions (3.4)-(3.5), as well as the normalization (3.6), hold. Moreover, there exists a constant C independent of a, and $a_0 > 0$ such that for all $a > a_0$ we have*

$$|c^a| + \int_{-a}^{a} |F_x^a|^2 dx \le C. \tag{3.8}$$

There also exist constants A_1, A_2, B, x_0^- and x_0^+ that do not depend on a, such that for all $a > a_0$ we have

$$A_1 e^x \le Q(x) \le A_2 e^x + B, \tag{3.9}$$

and

$$x_0^- < x_0^a < x_0^+. \tag{3.10}$$

The proof of this proposition relies on a Leray-Schauder degree argument: we consider a family of systems of equations

$$-c_a^\tau \frac{\partial F_a^\tau}{\partial x} = \kappa \frac{\partial^2 F_a^\tau}{\partial x^2} + \alpha F_a^\tau(x) \int_{-a}^{x} [(1-\tau) + \tau s_{a,\tau}^*(y)]\big(-\frac{\partial F_a^\tau(y)}{\partial y}\big) dy \tag{3.11}$$

$$(\rho - c_a^\tau) Q_a^\tau + c_a^\tau \frac{\partial Q_a^\tau}{\partial x} - \kappa \frac{\partial^2 Q_a^\tau}{\partial x^2} = \tau e^x H(R_a^\tau(x)), \tag{3.12}$$

with

$$R_a^\tau(x) = (1-\tau) + \tau \frac{\alpha}{2} e^{-x} \int_x^a [Q_a^\tau(y) - Q_a^\tau(x)]\big(-\frac{\partial F_a^\tau(y)}{\partial y}\big) dy, \quad s_{a,\tau}^*(x) = \min(1, R_a^\tau(x)), \tag{3.13}$$

and with the boundary conditions

$$F_a^\tau(-a) = 1, \ F_a^\tau(a) = 0, \tag{3.14}$$
$$\frac{\partial Q_a^\tau(-a)}{\partial x} = 0, \quad \frac{\partial Q_a^\tau(a)}{\partial x} = Q_a^\tau(a), \tag{3.15}$$

together with the normalization

$$F_a^\tau(0) = \frac{1}{2}. \tag{3.16}$$

This family is parametrized by $\tau \in [0,1]$, so that at $\tau = 0$ it reduces to the classical Fisher-KPP equation

$$-c_a^0 \frac{\partial F_a^0}{\partial x} = \kappa \frac{\partial^2 F_a^0}{\partial x^2} + F_a^0(1 - F_a^0),$$

and $Q_a^0(x) = 0$, $R_a^0(x) = s_{a,0}^*(x) = 1$ for all $x \in [-a, a]$, while at $\tau = 1$ the system (3.11)-(3.16) is exactly the problem (3.1)-(3.6) that we are interested in. We will show that the above system has a solution for all $\tau \in [0,1]$, and, in particular, for $\tau = 1$. The main difficulty in the proof of Proposition 3.1 is to obtain the uniform a priori bounds on the solutions to (3.11)-(3.16) that do not depend on a.

3.1.1 A priori bounds on a finite interval

We now prove the required a priori bounds for the solutions to (3.11)-(3.16) that are uniform in the parameter τ and do not depend on a for $a > a_0$.

The monotonicity of F_a^τ

We start by establishing monotonicity of F_a^τ for all $\tau \in [0,1]$.

Lemma 3.2. *The function $F_a^\tau(x)$, satisfying (3.11) together with the boundary conditions (3.14) and normalization (3.16) is positive on $(-a, a)$ and decreasing in x for all $\tau \in [0,1]$.*

Proof. It is helpful to write

$$\begin{aligned} -c_a^\tau \frac{\partial F_a^\tau}{\partial x} - \kappa \frac{\partial^2 F_a^\tau}{\partial x^2} &= \alpha F_a^\tau(x) \int_{-a}^{x} [(1-\tau) + \tau s_{a,\tau}^*(y)](-\frac{\partial F_a^\tau(y)}{\partial y}) dy \\ &= \alpha(1-\tau) F_a^\tau(x)(1 - F_a^\tau(x)) + \alpha\tau F_a^\tau(x) \int_{-a}^{x} s_{a,\tau}^*(y)(-\frac{\partial F_a^\tau(y)}{\partial y}) dy. \end{aligned} \tag{3.17}$$

Note that for $\tau = 0$ this is the Fisher-KPP equation

$$-c_a^0 \frac{\partial F_a^0}{\partial x} - \kappa \frac{\partial^2 F_a^0}{\partial x^2} = \alpha F_a^0(x) \int_{-a}^{x} (-\frac{\partial F_a^0(y)}{\partial y}) dy = \alpha F_a^0(x)(1 - F_a^0(x)), \tag{3.18}$$

with the boundary conditions (3.14), for which we know that the solution $F_a^0(x)$ is positive on $(-a, a)$ and is strictly decreasing in x, so that $\partial_x F_a^0(x) < 0$ for all $x \in [-a, a]$. By continuity, we have $0 < F_a^\tau(x) < 1$ for all $x \in (-a, a)$ and $\partial_x F_a^\tau(x) < 0$ for all $x \in [-a, a]$ for $\tau > 0$ sufficiently small. Furthermore, note that if x_0 is the local minimum or maximum of $F_a^\tau(x)$ that is closest to $(-a)$, then $\partial_x F_a^\tau(x)$ does not change sign on $(-a, x_0)$, so that the integral term in the right side of (3.17) is either strictly positive if x_0 is a minimum, or strictly negative if x_0 is a maximum, which immediately gives a contradiction unless $F_a^\tau(x_0) < 0$. To rule out this possibility, let $\tau_1 > 0$ be the smallest $\tau \in [0,1]$ such that either there exists $x' \in (-a, a)$ such that $F_a^{\tau_1}(x') = 0$ or $\partial_x F_a^{\tau_1}(a) = 0$. In the latter case, we have $F_a^{\tau_1}(x) \geq 0$ for all $x \in [-a, a]$, hence $\partial_x F_a^{\tau_1}(a) = 0$

would contradict the Hopf lemma. On the other hand, the former situation would imply that the closest minimum of $F_a^{\tau_1}(x)$ to $(-a)$ is non-negative, which is also a contradiction. Thus, such τ_1 can not exist, which means that $F_a^\tau(x) > 0$ for all $x \in (-a, a)$ and $\partial_x F_a^\tau(a) < 0$ for all $\tau \in [0, 1]$. As a consequence, by the same token, $F_a^\tau(x)$ can not attain a minimum on $[-a, a]$. The only possibility to rule out then is that $F_a^\tau(x)$ would attain a single local maximum on $[-a, a]$. On the other hand, that maximum would have to be larger than 1, and, as we have explained, this is impossible. Now, the conclusion of Lemma 3.2 follows. □

An a priori bound on the speed

Now, we obtain a uniform bound on the speed c_a^τ.

Lemma 3.3. *For any $\varepsilon > 0$ there exists $a_0 > 0$ such that*

$$-\varepsilon < c_a^\tau < 2\sqrt{\kappa\alpha} + \varepsilon \text{ for all } a > a_0 \quad \text{and for all } \tau \in [0, 1]. \tag{3.19}$$

Proof. As $0 \le s^*_{a,\tau}(y) \le 1$ for all y, and $F_a^\tau(y)$ is monotonically decreasing, the function $F_a^\tau(y)$ satisfies

$$-c_a^\tau \frac{\partial F_a^\tau}{\partial x} - \kappa \frac{\partial^2 F_a^\tau}{\partial x^2} \le \alpha F_a^\tau(x)(1 - F_a^\tau(x)) \le \alpha F_a^\tau(x), \tag{3.20}$$

for all $\tau \in [0, 1]$. On the other hand, the function $\psi^A(x) = Ae^{-\beta(x+a)}$ satisfies

$$-c_a^\tau \psi_x^A - \kappa \psi_{xx}^A \ge \alpha \psi^A, \tag{3.21}$$

as long as

$$c_a^\tau \beta \ge \kappa\beta^2 + \alpha. \tag{3.22}$$

Note that if $\beta > 0$ and A is sufficiently large, then $F_a^\tau(x) < \psi_A(x)$ for all $x \in [-a, a]$. As we decrease A, we see from (3.20) and (3.21) that $F_a^\tau(x)$ and $\psi^A(x)$ can not touch except at the boundary. Since $F_a^\tau(a) = 0$, this can only happen at $x = -a$, which means that $A = 1$. It follows that

$$F_a^\tau(x) \le e^{-\beta(x+a)} \text{ for all } -a \le x \le a,$$

and, in particular, we have $F_a^\tau(0) \le e^{-\beta a}$. This is a contradiction to (3.16) if $\beta > \log 2/a$, and the upper bound for c_a^τ in (3.19) follows.

For the lower bound we proceed in a similar way. Once again, monotonicity of $F_a^\tau(x)$ implies that

$$-c_a^\tau \frac{\partial F_a^\tau}{\partial x} - \kappa \frac{\partial^2 F_a^\tau}{\partial x^2} \ge 0. \tag{3.23}$$

However, the function $\psi(x) = 1 - Be^{\beta(x-a)}$ satisfies

$$-c_a^\tau \psi_x(x) - \kappa \psi_{xx} \le 0, \tag{3.24}$$

provided that

$$c_a^\tau \beta + \kappa\beta^2 \le 0. \tag{3.25}$$

Hence, if $c_a^\tau < 0$, we can find $\beta > 0$ such that (3.24) holds. As before, if $B > 0$ is sufficiently large, we automatically have $F_a^\tau(x) > \psi(x)$. Decreasing B, we see that (3.23) and (3.24) do not allow $F_a^\tau(x)$ and $\psi(x)$ to touch inside $[-a,a]$, and they can not intersect at $x=-a$ either. Thus, they touch at $x=a$ for the first time, with $B=1$. It follows that

$$F_a^\tau(x) \geq 1 - e^{\beta(x-a)} \text{ for all } x \in [-a,a],$$

and, in particular, we have

$$\frac{1}{2} = F_a^\tau(0) > 1 - e^{-\beta a},$$

which is a contradiction if $\beta > \log 2/a$, and the lower bound on c_a^τ in (3.19) follows. □

A lower bound for Q_a^τ

We now obtain a series of bounds for the function $Q_a^\tau(x)$. First, we establish a lower bound on $Q_a^\tau(x)$ and, in particular, show that it is positive. To this end, we need the following auxiliary lemma. Consider the eigenvalue problem

$$c\psi'(x) - \kappa\psi''(x) = \mu_a(c)\psi, \quad \psi(x) > 0 \text{ for all } -a < x < a, \tag{3.26}$$

with the boundary conditions

$$\psi'(-a) = 0, \quad \psi'(a) = \psi(a). \tag{3.27}$$

Existence of such principal eigenfunction and eigenvalue follows from the standard Sturm-Liouville theory – see, for instance, Theorem 4.1 in [16]. The next lemma gives a uniform bound on $\mu_a(c)$ as $a \to +\infty$.

Lemma 3.4. *For any $K > 0$ there exists C_K so that $|\mu_a(c)| \leq C_K$ for all $|c| < K$.*

Proof. Writing

$$\psi(x) = \phi(x) \exp\left(\frac{c}{2\kappa}x\right)$$

turns (3.26)-(3.27) into

$$-\phi''(x) = -\gamma_a\phi, \quad \phi(x) > 0 \text{ for all } -a < x < a, \tag{3.28}$$

with

$$\gamma_a = -\frac{1}{\kappa}\left(\mu_a(c) - \frac{c^2}{4\kappa}\right) \tag{3.29}$$

and with the boundary conditions

$$\phi'(-a) = -\frac{c}{2\kappa}\phi(-a), \quad \phi'(a) = \left(1 - \frac{c}{2\kappa}\right)\phi(a). \tag{3.30}$$

Note that if $\gamma_a < 0$ then the eigenfunction is of the form

$$\phi(x) = \cos(\sqrt{(-\gamma_a)}(x - z_a)),$$

with some $z_a \in \mathbb{R}$. As $\phi(x) > 0$ for all $x \in (-a,a)$, it follows that $\sqrt{(-\gamma_a)} \leq \pi/(2a)$ in this case. As $|c| \leq K$, we conclude that there exists $a_0 > 0$ so that for all $a > a_0$ if $\gamma_a \leq 0$, then

$$|\mu_a(c)| \leq \frac{K^2}{4\kappa} + 1. \tag{3.31}$$

Let us now assume that $\gamma_a > 0$ and set

$$r_1 = -\frac{c}{2\kappa}, \quad r_2 = 1 + r_1. \tag{3.32}$$

If $\gamma_a > 0$, then the positive eigenfunction has the form

$$\eta(x) = \exp(\sqrt{\gamma_a} x) + \beta \exp(-\sqrt{\gamma_a} x).$$

As we are only interested in bounds on γ_a, we may assume without loss of generality that

$$|\sqrt{\gamma_a} + r_1| > 1, \tag{3.33}$$

for otherwise γ_a is automatically bounded, and thus so is $\mu_a(c)$. The boundary condition at $x = -a$

$$\sqrt{\gamma_a} \exp(-\sqrt{\gamma_a} a) - \beta\sqrt{\gamma_a} \exp(\sqrt{\gamma_a} a) = r_1 \exp(-\sqrt{\gamma_a} a) + r_1 \beta \exp(\sqrt{\gamma_a} a), \tag{3.34}$$

implies that

$$\beta = \frac{\sqrt{\gamma_a} - r_1}{\sqrt{\gamma_a} + r_1} \exp(-2\sqrt{\gamma_a} a). \tag{3.35}$$

Using this in the boundary condition at $x = a$

$$\sqrt{\gamma_a} \exp(\sqrt{\gamma_a} a) - \beta\sqrt{\gamma_a} \exp(-\sqrt{\gamma_a} a) = r_2 \exp(\sqrt{\gamma_a} a) + r_2 \beta \exp(-\sqrt{\gamma_a} a) \tag{3.36}$$

gives

$$\sqrt{\gamma_a}\Big(1 - \frac{\sqrt{\gamma_a} - r_1}{\sqrt{\gamma_a} + r_1} \exp(-4\sqrt{\gamma_a} a)\Big) = r_2\Big(1 + \frac{\sqrt{\gamma_a} - r_1}{\sqrt{\gamma_a} + r_1} \exp(-4\sqrt{\gamma_a} a)\Big), \tag{3.37}$$

so that

$$\frac{r_2}{\sqrt{\gamma_a}} = \frac{\sqrt{\gamma_a} + r_1 - (\sqrt{\gamma_a} - r_1)\exp(-4\sqrt{\gamma_a} a)}{\sqrt{\gamma_a} + r_1 + (\sqrt{\gamma_a} - r_1)\exp(-4\sqrt{\gamma_a} a)} = 1 - \frac{2(\sqrt{\gamma_a} - r_1)\exp(-4\sqrt{\gamma_a} a)}{\sqrt{\gamma_a} + r_1 + (\sqrt{\gamma_a} - r_1)\exp(-4\sqrt{\gamma_a} a)}. \tag{3.38}$$

Let us assume that there exists a sequence $a_k \to +\infty$ such that

$$\sqrt{\gamma_{a_k}} \geq \frac{1}{\sqrt{a_k}}. \tag{3.39}$$

Then we have

$$|\sqrt{\gamma_{a_k}} - r_1| \exp(-4\sqrt{\gamma_{a_k}} a_k) \leq (\sqrt{\gamma_{a_k}} + |r_1|) \exp(-4\sqrt{\gamma_{a_k}} a_k) \leq \frac{C}{a_k} + |r_1| \exp(-4\sqrt{a_k}). \tag{3.40}$$

Passing to the limit $a_k \to +\infty$ in (3.38) using (3.33) and (3.40) gives in that case

$$\gamma_{a_k} \to r_2^2 \text{ as } k \to +\infty. \tag{3.41}$$

On the other hand, for any sequence $a_k \to +\infty$ for which (3.39) does not hold, we automatically have (3.31). This finishes the proof. □

Now, we can prove the following lower bound on $Q_a^\tau(x)$.

Lemma 3.5. *Let $\rho > C_K$, with $K = 2\sqrt{\kappa\alpha}$, and $g(x)$ be the solution to*

$$(\rho - c_a^\tau)g(x) + c_a^\tau g'(x) - \kappa g''(x) = \tau e^x, \tag{3.42}$$

with the boundary conditions

$$g'(-a) = 0, \quad g'(a) = g(a), \tag{3.43}$$

then $Q_a^\tau(x) \geq g(x)$ for all $x \in [-a, a]$.

Proof. Recall that $Q_a^\tau(x)$ satisfies

$$(\rho - c^a)Q_a^\tau + c^a \frac{\partial Q_a^\tau}{\partial x} - \kappa \frac{\partial^2 Q_a^\tau}{\partial x^2} = \tau e^x H(R_a^\tau) \geq \tau e^x. \tag{3.44}$$

Hence, the difference $f(x) = Q(x) - g(x)$ satisfies

$$(\rho - c^a)f(x) + c^a f'(x) - \kappa f''(x) \geq 0, \tag{3.45}$$

with the boundary conditions $g'(-a) = 0$, $g'(a) = g(a)$. Lemmas 3.3 and 3.4 imply that under the assumptions of the current lemma on the parameter ρ, the principal eigenvalue of the operator in the left side, with the boundary conditions (3.43), is positive, so that the comparison principe applies, thus $f(x) \geq 0$ for all $x \in [-a, a]$. □

As a consequence of Lemma 3.5, we have the following more explicit lower bound.

Lemma 3.6. *There exist $\rho_0 > 0$ and $a_0 > 0$ so that for $\rho > \rho_0$ and $a > a_0$ the function $Q_a^\tau(x)$ satisfies $Q_a^\tau(x) \geq \tau A e^x$ for all $x \in [-a, a]$, and $\tau \in [0, 1]$ and $A < 1/(\rho - \kappa)$.*

Proof. An explicit solution to (3.42)-(3.43) is

$$g(x) = \tau z_1 e^{\lambda_1 x} + \tau z_2 e^{-\lambda_2 x} + \frac{\tau}{\rho - \kappa} e^x, \tag{3.46}$$

where

$$\lambda_1 = \frac{c + \sqrt{c^2 + 4\kappa(\rho - c)}}{2\kappa} > 0, \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4\kappa(\rho - c)}}{2\kappa} > 0, \tag{3.47}$$

and the constants z_1 and z_2 are given by

$$z_1 = \frac{e^{-a}}{\rho - \kappa} \Big(\frac{\lambda_2(\lambda_1 - 1)}{\lambda_2 + 1} e^{(\lambda_1 + 2\lambda_2)a} - \lambda_1 e^{-\lambda_1 a} \Big)^{-1}, \; z_2 = \frac{e^{-a}}{\rho - \kappa} \Big(\lambda_2 e^{\lambda_2 a} - \frac{\lambda_1(\lambda_2 + 1)}{\lambda_1 - 1} e^{-(\lambda_2 + 2\lambda_1)a} \Big)^{-1}. \tag{3.48}$$

Note that $\lambda_1 > 0$ and $\lambda_2 > 0$ for ρ sufficiently large, and for $a > a_0$ sufficiently large we also have that both $z_1 > 0$ and $z_2 > 0$, and the conclusion of the present Lemma follows from Lemma 3.5.

□

The monotonicity of Q_a^τ and R_a^τ

Next, the uniform bound on the speed c^a and positivity of $Q_a^\tau(x)$ allow us to show monotonicity of $Q_a^\tau(x)$ and $R_a^\tau(x)$.

Lemma 3.7. *There exists $a_0 > 0$ so that function $Q_a^\tau(x)$ is increasing in x and the functions $R_a^\tau(x)$ and $s_{a,\tau}^*(x)$ are decreasing in x for all $a > a_0$.*

Proof. Note that if $Q_a^\tau(x)$ is increasing in x, then, as

$$R_a^\tau(x) = 1 - \tau + \frac{\alpha\tau}{2} e^{-x} \int_x^a [Q_a^\tau(y) - Q_a^\tau(x)](-\frac{\partial F_a^\tau(y)}{\partial y})dy = 1 - \tau + \frac{\alpha\tau}{2} e^{-x} \int_x^a \frac{\partial Q_a^\tau(y)}{\partial y} F_a^\tau(y)dy, \tag{3.49}$$

Lemma 3.2 implies that $R_a^\tau(x)$ is decreasing in x. In addition, monotonicity of $R_a^\tau(x)$ implies monotonicity of $s_{a,\tau}^*(x)$, hence we only need to study monotonicity of $Q_a^\tau(x)$. Differentiating (3.12) shows that

$$(\rho - c_a^\tau)Q' + c_a^\tau Q_x' - \kappa Q_{xx}' = \tau e^x H(R_a^\tau) + \tau e^x H'(R_a^\tau)\frac{\partial R_a^\tau}{\partial x}, \tag{3.50}$$

with $Q'(x) = \partial_x Q_a^\tau(x)$, and from (3.49) we see that

$$\frac{\partial R_a^\tau(x)}{\partial x} = -R_a^\tau(x) + 1 - \tau - \frac{\tau\alpha}{2} e^{-x} \frac{\partial Q_a^\tau(x)}{\partial x} F_a^\tau(x). \tag{3.51}$$

Recalling that $H(R)$ is given explicitly by (2.22), we now write (3.50) as

$$(\rho - c_a^\tau)Q' + c_a^\tau Q_x' - \kappa Q_{xx}' = \tau e^x \begin{cases} 2 - 2\tau - \tau\alpha e^{-x} Q'(x) F_a^\tau(x), & \text{if } R_a^\tau > 1, \\ 1 - (R_a^\tau)^2 + 2R_a^\tau(1-\tau) - \tau R_a^\tau \alpha e^{-x} Q'(x) F_a^\tau(x), & \text{if } 0 \le R_a^\tau \le 1, \\ 1, & \text{if } R_a^\tau < 0. \end{cases} \tag{3.52}$$

It follows that

$$-\kappa Q_{xx}' + c_a^\tau Q_x' + (\rho - c_a^\tau)Q'(x) + \tau\alpha F_a^\tau(x) Q'(x) S^*(R_a^\tau(x)) \ge 0.$$

Assumption $\rho > \rho_0$ in Theorem 1.1 together with Lemma 3.3 implies that $c_a^\tau < \rho$ for $a > a_0$ if ρ_0 is sufficiently large. It follows that $Q'(x)$ can not attain an interior negative minimum. We also have $Q'(-a) = 0$ and $Q'(a) = Q_a^\tau(a) > 0$, thus a negative minimum of $Q'(x)$ can not be attained at $x = \pm a$ either. Therefore, we have $Q'(x) \ge 0$ for all $x \in [-a, a]$ and $Q_a^\tau(x)$ is increasing in x. □

An upper bound for $Q_a^\tau(x)$

In this section we show that $Q_a^\tau(x)$ is bounded from above.

Lemma 3.8. *There exist $a_0 > 0$, $\rho_0 > 0$ and $C > 0$ so that if $\rho > \rho_0$ then*

$$Q_a^\tau(x) \le \frac{C}{\rho}(1 + e^x), \text{ for all } a > a_0,\ -a < x < a \text{ and } \tau \in [0,1]. \tag{3.53}$$

Proof. First, note that $Q_a^0 \equiv 0$ trivially satisfies (3.53). Our goal will be to show that if we choose K sufficiently large, then (3.53) with $C = K$ can not be violated for any $\tau \in [0,1]$. To this end, assume that $\tau_1 > 0$ is the smallest $\tau > 0$ such that there exists $x_1 \in [-a,a]$ such that

$$Q_a^\tau(x_1) = K(1 + e^{x_1}). \tag{3.54}$$

Then, we still have

$$Q_a^{\tau_1}(x) \le K(1 + e^x), \text{ for all } -a < x < a. \tag{3.55}$$

As $H(r) \le 1 + 2r$, it follows from (3.44) that

$$\begin{aligned}(\rho - c_a^{\tau_1})Q_a^{\tau_1} + c_a^{\tau_1}\frac{\partial Q_a^{\tau_1}}{\partial x} - \kappa\frac{\partial^2 Q_a^{\tau_1}}{\partial x^2} &\le \tau_1 e^x(1 + 2R_a^{\tau_1}(x)) \\ \le \tau_1 e^x(1 + 2(1 - \tau_1)) + \tau_1^2\alpha\int_x^a \frac{\partial Q_a^{\tau_1}(y)}{\partial y}F_a^{\tau_1}(y)dy.\end{aligned} \tag{3.56}$$

To estimate the integral in the right side of (3.56), we will use the following lemma.

Lemma 3.9. *There exist constants B and a_0, such that*

$$\int_{-a}^{a}(Q_a^\tau)'(x)F_a^\tau(x)dx \le B \tag{3.57}$$

for all $a > 0$ and τ.

Proof. We first note that

$$\begin{aligned}\int_{-a}^{0}\frac{\partial Q_a^{\tau_1}(y)}{\partial y}F_a^{\tau_1}(y)dy &\le \frac{1}{2}Q_a^{\tau_1}(0) + \int_{-a}^{0}Q_a^{\tau_1}(y)\big(-\frac{\partial F_a^{\tau_1}(y)}{\partial y}\big)dy \\ &\le K + 2K\int_{-a}^{0}\big(-\frac{\partial F_a^{\tau_1}(y)}{\partial y}\big)dy = 2K,\end{aligned} \tag{3.58}$$

because of (3.55) and normalization (3.16). To estimate the integral from 0 to a, recall that x_0, as defined in (3.7) is $x_0 = \sup\{x : R_a^{\tau_1}(x) > 1\}$. If $x_0 \le 0$ we have

$$\int_0^a \frac{\partial Q_a^{\tau_1}(x)}{\partial x}F_a^{\tau_1}(x)dx \le \int_{x_0}^a \frac{\partial Q_a^{\tau_1}(x)}{\partial x}F_a^{\tau_1}(x)dx = \frac{2}{\alpha}e^{x_0}R(x_0) \le \frac{2}{\alpha}. \tag{3.59}$$

The case $x_0 \ge 0$ is handled by the following upper bound on F_a^τ.

Lemma 3.10. *There exists $\alpha_0 > 0$ and a constant C, independent of a and τ, such that for all $\alpha > \alpha_0$ if $x_0 > 0$ then*

$$F_a^\tau(x) \le Ce^{-2x}. \tag{3.60}$$

Proof. Note that (3.60) holds automatically for $x < 0$ if $C > 1$, since $F_a^\tau(x) < 1$, thus we may assume without loss of generality that $x > 0$. Since $x_0 > 0$ we have $s_{a,\tau}^*(x) = 1$ for $x < 0$. Therefore, we have for all $x > 0$

$$\int_{-a}^{x}s_{a,\tau}^*(y)(-\frac{\partial F_a^\tau(y)}{\partial y})dy \ge \int_{-a}^{0}s_{a,\tau}^*(y)(-\frac{\partial F_a^\tau(y)}{\partial y})dy = \frac{1}{2}, \tag{3.61}$$

due to (3.16). It follows that for $x > 0$ we have

$$\int_{-a}^{x}[(1-\tau)+\tau s^*_{a,\tau}(y)](-\frac{\partial F^\tau_a(y)}{\partial y})dy = (1-\tau)(1-F^\tau_a(x)) + \tau\int_{-a}^{x} s^*_{a,\tau}(y))(-\frac{\partial F^\tau_a(y)}{\partial y})dy \\ \geq \frac{(1-\tau)}{2} + \frac{\tau}{2} = \frac{1}{2}. \tag{3.62}$$

Therefore, for $x > 0$ we have

$$-c^\tau_a\frac{\partial F^\tau_a}{\partial x} \geq \kappa\frac{\partial^2 F^\tau_a}{\partial x^2} + \frac{\alpha}{2}F^\tau_a. \tag{3.63}$$

We integrate (3.63) from x to y, with $0 < x < y$, to get

$$c^\tau_a F^\tau_a(x) - c^\tau_a F^\tau_a(y) \geq \kappa\frac{\partial F^\tau_a(y)}{\partial x} - \kappa\frac{\partial F^\tau_a(x)}{\partial x} + \frac{\alpha}{2}\int_x^y F^\tau_a(\xi)d\xi. \tag{3.64}$$

Next, integrate (3.64) in y from z to $z+1$, with $z > x$, to get

$$c^\tau_a F^\tau_a(x) - c^\tau_a\int_z^{z+1} F^\tau_a(y)dy \geq \kappa F^\tau_a(z+1) - \kappa F^\tau_a(z) - \kappa\frac{\partial F^\tau_a(x)}{\partial x} + \frac{\alpha}{2}\int_z^{z+1}\int_x^y F^\tau_a(\xi)d\xi dy. \tag{3.65}$$

We can estimate the left side of (3.65) simply as

$$c^\tau_a F^\tau_a(x) - c^\tau_a\int_z^{z+1} F^\tau_a(y)dy \leq c^\tau_a F^\tau_a(x). \tag{3.66}$$

For the right side of (3.65) we have, as F^τ_a is decreasing:

$$\begin{aligned} &\kappa F^\tau_a(z+1) - \kappa F^\tau_a(z) - \kappa\frac{\partial F^\tau_a(x)}{\partial x} + \frac{\alpha}{2}\int_z^{z+1}\int_x^y F^\tau_a(\xi)d\xi dy \\ &\geq \kappa F^\tau_a(z+1) - \kappa F^\tau_a(z) - \kappa\frac{\partial F^\tau_a(x)}{\partial x} + \frac{\alpha}{2}\int_z^{z+1}(y-x)F^\tau_a(y)dy \\ &\geq \kappa F^\tau_a(z+1) - \kappa F^\tau_a(z) - \kappa\frac{\partial F^\tau_a(x)}{\partial x} + \frac{\alpha}{2}F^\tau_a(z+1)\int_z^{z+1}(y-x)dy \\ &\geq \kappa F^\tau_a(z+1) - \kappa F^\tau_a(z) + \frac{\alpha}{2}(z-x)F^\tau_a(z+1). \end{aligned} \tag{3.67}$$

Putting (3.65), (3.66) and (3.67) together, we get, for all $z > x$:

$$c^\tau_a F^\tau_a(x) \geq \kappa F^\tau_a(z+1) - \kappa F^\tau_a(z) + \frac{\alpha}{2}(z-x)F^\tau_a(z+1). \tag{3.68}$$

Adding $\kappa F^\tau_a(x)$ to both sides gives, as $x < z$:

$$(c^\tau_a+\kappa)F^\tau_a(x) \geq \kappa F^\tau_a(z+1) + \kappa F^\tau_a(x) - \kappa F^\tau_a(z) + \frac{\alpha}{2}(z-x)F^\tau_a(z+1) \\ \geq \kappa F^\tau_a(z+1) + \frac{\alpha}{2}(z-x)F^\tau_a(z+1) \tag{3.69}$$

Taking $z = x + 1$ leads to

$$(c_a^\tau + \kappa) F_a^\tau(x) \geq \Big(\kappa + \frac{\alpha}{2}\Big) F_a^\tau(x+2), \tag{3.70}$$

thus

$$F_a^\tau(x+2) \leq \frac{c_a^\tau + \kappa}{\kappa + \alpha/2} F_a^\tau(x). \tag{3.71}$$

Now, (3.60) follows for $x > 0$ if we take α sufficiently large, as $|c_a^\tau| \leq 2\sqrt{\kappa\alpha}$ by Lemma 3.3. □

We now go back to the proof of Lemma 3.9. It follows from Lemma 3.10 that if $x_0 \geq 0$ then

$$\int_0^a \frac{\partial Q_a^{\tau_1}(x)}{\partial x} F_a^{\tau_1}(x) dx \leq C \int_0^a \frac{\partial Q_a^{\tau_1}(x)}{\partial x} e^{-2x} dx \leq C Q_a^{\tau_1}(a) e^{-2a} + 2C \int_0^a Q_a^{\tau_1}(x) e^{-2x} dx \leq CK, \tag{3.72}$$

because of (3.55). Together with (3.58), this gives (3.57). □

We continue the the proof of Lemma 3.8. Using (3.58), (3.59) and (3.72) in (3.56) gives

$$(\rho - c_a^{\tau_1}) Q_a^{\tau_1} + c_a^{\tau_1} \frac{\partial Q_a^{\tau_1}}{\partial x} - \kappa \frac{\partial^2 Q_a^{\tau_1}}{\partial x^2} \leq \tau_1 e^x (1 + 2(1 - \tau_1)) + C\tau_1 K, \tag{3.73}$$

with the boundary conditions

$$\frac{\partial Q_a^{\tau_1}}{\partial x}(-a) = 0, \quad \frac{\partial Q_a^{\tau_1}}{\partial x}(-a) = Q_a^{\tau_1}(a). \tag{3.74}$$

The comparison principle implies that

$$Q_a^\tau(a) \leq (1 + 2(1 - \tau_1)) g(x) + \frac{C\tau_1 K}{\rho - c_a^{\tau_1}} (1 + q(x)), \tag{3.75}$$

where $g(x)$ is the explicit solution to (3.42)-(3.43) given by (3.46), and $q(x)$ is the solution to

$$(\rho - c_a^{\tau_1}) q + c_a^{\tau_1} \frac{\partial q}{\partial x} - \kappa \frac{\partial^2 q}{\partial x^2} = 0, \tag{3.76}$$

with the boundary conditions

$$\frac{\partial q}{\partial x}(-a) = 0, \quad \frac{\partial q}{\partial x}(a) = q(a) - 1. \tag{3.77}$$

The function $q(x)$ has the form

$$q(x) = \mu_1 e^{\lambda_1(x-a)} + \mu_2 e^{-\lambda_2(x+a)}, \tag{3.78}$$

with $\lambda_{1,2} > 0$ given by (3.47), and the coefficients $\mu_{1,2}$ given by

$$\begin{aligned} \mu_1 &= \frac{\lambda_2}{\lambda_1} e^{2\lambda_1 a} \mu_2, \\ \mu_2 &= \Big[\lambda_2 \Big(\frac{1}{\lambda_1} - 1\Big) e^{2\lambda_1 a} - (\lambda_2 - 1) e^{-2\lambda_2 a}\Big]^{-1}, \end{aligned} \tag{3.79}$$

so that for a large we have

$$\mu_1 \approx -\frac{1}{\lambda_1 - 1}, \quad \mu_2 \approx -\frac{\lambda_1}{\lambda_2(\lambda_1 - 1)} e^{-2\lambda_1 a}. \tag{3.80}$$

It follows from (3.47) that $\lambda_1 > 1$ for $\rho > \rho_0$, due to the bounds on $c_a\tau$ in Lemma 3.3, hence $\mu_{1,2} < 0$ for $\rho > \rho_0$. We conclude from (3.75) and (3.78) that

$$Q_a^\tau(a) \le (1 + 2(1 - \tau_1))g(x) + \frac{C\tau_1 K}{\rho - c_a^{\tau_1}} \le 3g(x) + \frac{C\tau_1 K}{\rho}, \tag{3.81}$$

for $\rho > \rho_0$. Going back to the explicit expression (3.46) for $g(x)$ and using (3.47) and (3.48), we see that

$$g(x) \le \frac{C}{\rho}(1 + e^x), \tag{3.82}$$

for $a > a_0$ and $\rho > \rho_0$. It follows from (3.81) that

$$Q_a^\tau(a) \le \frac{C}{\rho} e^x + \frac{CK}{\rho}, \text{ for all } -a < x < a. \tag{3.83}$$

This gives a contradiction to (3.54) if $K > C/\rho$, finishing the proof of Lemma 3.8. □

An upper bound for $\partial_x Q_a^\tau(x)$

Next, we show that the first derivative of Q_a^τ is bounded from above for all τ.

Lemma 3.11. *There exist $a_0 > 0$ and $\rho_0 > 0$, and $C > 0$ so that $Q'(x) = (Q_a^\tau)'(x)$ satisfies*

$$Q'(x) \le Ce^x + Ce^{-\lambda_2 a} \tag{3.84}$$

for all $a > a_0$, $x \in [-a, a]$ and all $\tau \in [0, 1]$, with $\lambda_2 > 0$ as in (3.47).

Proof. As we have already shown that $Q'(x) \ge 0$, it follows from (3.52) that

$$-\kappa Q'_{xx} + c_a^\tau Q'_x + (\rho - c_a^\tau)Q'(x) \le 2e^x, \tag{3.85}$$

with the boundary condition $Q'(-a) = 0$. To get the boundary condition at $x = a$ we use (3.12) and Lemma 3.8:

$$\kappa\partial_x Q'(a) = (\rho - c_a^\tau)Q_a^\tau(a) + c_a^\tau Q'(a) - \tau e^a H(R_a^\tau(a)) \le \rho Q_a^\tau(a) \le Ce^a. \tag{3.86}$$

If $\rho > \rho_0$, the function

$$p(x) = \frac{2C}{\kappa} e^x, \tag{3.87}$$

satisfies

$$-\kappa p_{xx} + c_a^\tau p_x + (\rho - c_a^\tau)p(x) \ge 2e^x, \tag{3.88}$$

with the boundary conditions

$$p(-a) > 0, \quad \kappa p_x(a) = 2Ce^a \ge Ce^a. \tag{3.89}$$

Then, the difference $\xi(x) = p(x) - Q_a^\tau(x)$ satisfies

$$\begin{aligned} & -\kappa\xi_{xx} + c_a^\tau \xi_x + (\rho - c_a^\tau)\xi(x) \geq 0, \\ & \xi(-a) > 0, \quad \kappa\xi_x(a) > 0. \end{aligned} \tag{3.90}$$

It follows that if $\rho > \rho_0$ then $\xi(x)$ can not attain a negative minimum inside $(-a, a)$ and, in addition, it can not attain a minimum at $x = a$. Thus, $\xi(x) > 0$ for all $x \in (-a, a)$, and

$$Q'(x) \leq p(x) = \frac{2C}{\kappa} e^x \text{ for all } x \in (-a, a), \tag{3.91}$$

and the proof of Lemma 3.11 is complete. □

A uniform gradient bound for $F_a^\tau(x)$

We first obtain a uniform bound for the derivative of $F_a^\tau(x)$. To simplify the notation, we drop the subscripts a and τ.

Lemma 3.12. *There exists $a_0 > 0$ and $C > 0$ such that*

$$\int_{-a}^{a} |F_x|^2 dx \leq C \textit{ for all } a > a_0 \quad \textit{and for all } \tau \in [0, 1]. \tag{3.92}$$

Proof. Integrating (3.17) from $-a$ to a gives

$$c = \kappa F_x(a) - \kappa F_x(-a) + \alpha(1-\tau)\int_{-a}^{a} F(1-F)dx + \alpha\tau \int_{-a}^{a} F(x)\int_{-a}^{x} s^*(y)(-F_y(y))dydx. \tag{3.93}$$

We also multiply both sides of (3.11) by F, integrate from $-a$ to a, and use (3.93):

$$\begin{aligned} \frac{c}{2} + \kappa\int_{-a}^{a} F_x^2 dx + \kappa F_x(-a) &= \alpha(1-\tau)\int_{-a}^{a} F^2(1-F)dx + \alpha\tau\int_{-a}^{a} F^2(x)\int_{-a}^{x} s^*(y)(-F_y(y))dydx \\ &\leq c - \kappa F_x(a) + \kappa F_x(-a), \end{aligned} \tag{3.94}$$

so that

$$\kappa\int_{-a}^{a} F_x^2 dx \leq \frac{c}{2} - \kappa F_x(a). \tag{3.95}$$

The uniform bounds on c in Lemma 3.3 allow to apply the standard elliptic regularity results to conclude that $|F_x(a)| \leq C$, with C that does not depend on a, and (3.92) follows. □

3.1.2 The degree argument

We have by now proved the a priori bounds in Proposition 3.1. We now use these a priori bounds to finish the proof of the existence part of Proposition 3.1 using a Leray-Schauder degree argument. Let us define the map $\mathcal{L}_\tau(c, F, Q) = (\theta, G, T)$ as the solution operator for the system

$$\begin{aligned} & -cG_x = \kappa G_{xx} + \int_{-a}^{x} [(1-\tau) + \tau s^*(y)](-F_y(y))dy \\ & (\rho - c)T + cT_x - \kappa T_{xx} = \tau e^x H(R(x)) \end{aligned} \tag{3.96}$$

with the boundary conditions

$$G(-a) = 1,\ G(a) = 0,\ T_x(-a) = 0 \ \text{ and } T_x(a) = T(a),$$

and with

$$R(x) = 1 - \tau + \tau\frac{\alpha}{2}e^x \int_x^a [Q(y) - Q(x)](-F_y^\tau(y))dy \tag{3.97}$$

and $s^*(x) = \min\{1, R(x)\}$. The constant θ is defined as

$$\theta = \frac{1}{2} - \max_{x\in[0,a]} F(x) + c. \tag{3.98}$$

This operator maps the Banach space $X = \mathbb{R} \times C^1([-a, a]) \times C^1([-a, a])$ with the norm

$$\|c, F, Q\|_X = \max\{|c|, \|F\|_{C^1}, \|Q\||_{C^1}\},$$

to itself, and its fixed points are solutions to (3.11). Therefore, it suffices to show that the operator $\mathcal{F}_\tau = \mathrm{Id} - \mathcal{L}_\tau$ has a nontrivial kernel for all $\tau \in [0, 1]$. Let B_M be a ball of radius M in X centered at the origin. Using the a priori bounds obtained above we can choose M sufficiently large to ensure that $\mathcal{F}_\tau$ does not vanish on the boundary ∂B_M. As the Leray-Schauder degree is homotopy invariant, it is enough to show that $\deg(\mathcal{F}_0, B_M, 0) \neq 0$. Note that

$$\mathcal{F}_0(c, F, Q) = \Big(\max_{x>0} F_0^c(x) - \frac{1}{2}, F - F_0^c, Q\Big), \tag{3.99}$$

where F_0^c solves

$$-cF_0' = \kappa F_0'' + \alpha F(1 - F), \quad F_0(-\infty) = 1,\ F_0(+\infty) = 0. \tag{3.100}$$

Hence, $\deg(\mathcal{F}_0, B_M, 0) = -1$, thus $\mathcal{F}_\tau$ has a nontrivial kernel. Therefore, a solution to (3.11) exists for all $\tau \in [0, 1]$, which proves the existence part of Proposition 3.1.

3.2 Identification of the limit

To complete the proof of Proposition 3.1 we get uniform bounds on the transition point x_0^a.

Lemma 3.13. *There exist constants x_0^+, x_0^- and a_0 such that for all $a > a_0$ we have*

$$x_0^- \le x_0^a \le x_0^+.$$

Proof. Recall that the point x_0^a is determined by $R(x_0^a) = 1$, so that

$$1 = \frac{\alpha}{2}e^{-x_0^a} \int_{x_0^a}^a Q_y^a(y)F^a(y)dy. \tag{3.101}$$

Using Lemma 3.9, we obtain

$$\exp\{-x_0^a\} \ge \frac{2}{\alpha B},$$

hence

$$x_0^a \le \log(\alpha B). \tag{3.102}$$

For a lower bound on x_0^a, let us assume that $x_0^a < 0$, and write, for any $z > 0$:

$$\begin{aligned}\frac{2}{\alpha} &= e^{-x_0^a}\int_{x_0^a}^a [Q^a(y) - Q^a(x_0^a)](-F_y^a(y))dy > e^{-x_0^a}\int_0^a [Q^a(y) - Q^a(x_0^a)](-F_y^a(y))dy \\ &> e^{-x_0^a}\int_0^a [Q^a(y) - Q^a(0)](-F_y^a(y))dy > e^{-x_0^a}\int_z^a [Q^a(y) - Q^a(0)](-F_y^a(y))dy \\ &\ge e^{-x_0^a}\int_z^a [Ae^y - B](-F_y^a(y))dy.\end{aligned} \tag{3.103}$$

We used Lemma 3.6 in the last step above to bound $Q^a(y)$ from below and Lemma 3.8 to bound $Q^a(0)$ from above. We may now choose $z > 0$ so that $Ae^y > 2B$ for all $y > z$, so that

$$\frac{2}{\alpha} \ge Be^{-x_0^a}\int_z^a (-F_y^a(y))dy = Be^{-x_0^a}F^a(z). \tag{3.104}$$

As z does not depend on a, and $F^a(0) = 1/2$, the Harnack inequality implies that there exists $s > 0$ that does not depend on a so that $F^a(z) > s$, so that

$$e^{x_0^a} > \frac{\alpha B s}{2},$$

finishing the proof of the lower bound for x_0^a. □

The a priori bounds obtained in Proposition 3.1 allows us to extract a subsequence $a_n \to +\infty$ such that the corresponding sequence $(c^{a_n}, F^{a_n}, Q^{a_n})$ converges to a limit (c, F, Q), in $C_{loc}^{2,\alpha}(\mathbb{R})$. Moreover, the functions F and Q are monotonic. The upper bound on $Q_a'(x)$ in Lemma 3.11 and the upper bound on $F_a(x)$ in Lemma 3.10 imply that

$$\int_x^{a_n} Q_{a_n}'(y)F_{a_n}(y)dy \to \int_x^\infty Q'(y)F(y)dy, \tag{3.105}$$

hence the corresponding sequences $s_{a_n}^*(x)$ and $R_{a_n}(x)$ converge as well to their respective limits $s^*(x)$ and $R(x)$ such that $R(x) \ge 0$ and $s^*(x) = \min(1, R(x))$, and

$$R(x) = \frac{\alpha}{2}e^{-x}\int_x^\infty Q'(y)F(y)dy. \tag{3.106}$$

In particular, as a consequence, the function Q satisfies the second equation in (2.30):

$$\rho Q = cQ - c\frac{\partial Q}{\partial x} + \kappa\frac{\partial^2 Q}{\partial x^2} + e^x H(R). \tag{3.107}$$

In order to see that $F(x)$ satisfies the first equation in (2.30), let us take $x_0^\pm$ as in Lemma 3.13 and write

$$\begin{aligned}\int_{-a_n}^x s_{a_n}^*(y)(-F_y^{a_n}(y))dy &= \int_{-a_n}^{x_0^-}(-F_y^{a_n}(y))dy + \int_{x_0^-}^{x_0^+} s_{a_n}^*(y)(-F_y^{a_n}(y))dy \\ &+ \int_{x_0^+}^{x} s_{a_n}^*(y)(-F_y^{a_n}(y))dy = 1 - F^{a_n}(x_0^-) + I_n + II_n.\end{aligned} \tag{3.108}$$

The bounded convergence theorem implies that

$$I_n \to \int_{x_0^-}^{x_0^+} s^*(y)(-F_y(y))dy, \tag{3.109}$$

while the Lebesgue dominated convergence theorem and (3.60) imply that

$$II_n \to \int_{x_0^+}^{x} s^*(y)(-F_y(y))dy. \tag{3.110}$$

It follows that F satisfies

$$-cF_x - \kappa F_{xx} = \alpha F(x) \int_{-\infty}^{x} s^*(y)(-F_y(y))dy. \tag{3.111}$$

It remains to show that the limit F satisfies the correct boundary conditions and that $Q(x)$ converges to a positive constant on the left and grows exponentially on the right, as in (2.31). Note that Lemma 3.10 implies that $F(x) \to 0$ as $x \to +\infty$. The next lemma takes care of the limit on the left.

Lemma 3.14. *The limiting function $F(x)$ converges to 1 as $x \to -\infty$.*

Proof. Let x_0^- be the lower bound on x_0^a as in Lemma 3.13. Then, for $a > a_0$ the function F^a satisfies

$$-c^a F_x^a - \kappa F_{xx}^a = \alpha F^a \int_{-a}^{x} (-F_y^a(y))dy = \alpha F^a(1 - F^a), \quad \text{for } x < x_0^-.$$

Integrating both sides from $(-a)$ to x_0^- gives

$$c^a(1 - F^a(x_0^-)) - \kappa F_x^a(-a) + \kappa F_x^a(x_0^-) = \int_{-a}^{x_0^-} \alpha F^a(1 - F^a)dx.$$

Note that c^a is bounded by Lemma 3.3, $F^a(x)$ is bounded for all x and $F_x(-a)$ and $F_x(x_0^-)$ are also bounded by elliptic regularity. Therefore, the left side is bounded independently of a for $a > a_0$, hence so is the integral in the right side. It follows that the integral

$$\int_{-\infty}^{x_0^-} F(1 - F)dx$$

is finite. As $F(x)$ is monotonically decreasing, and $F(x) \geq 1/2$ for $x \leq 0$, it follows that $F(x) \to 1$ as $x \to -\infty$. □

Next, we look at the left limit of $Q(x)$.

Lemma 3.15. *The limiting function $Q(x)$ converges to a positive constant q_- as $x \to -\infty$.*

Proof. Note that for $x < x_0^-$ we have $R(x) > 1$, so that $H(R(x)) = 2R(x)$, and (3.107) becomes

$$\rho Q(x) = cQ(x) - c\frac{\partial Q(x)}{\partial x} + \kappa\frac{\partial^2 Q(x)}{\partial x^2} + \alpha\int_x^\infty Q_y(y)F(y)dy. \tag{3.112}$$

As the function $Q(x)$ is monotonically increasing, and the derivatives $Q'(x)$ and $Q''(x)$ are uniformly bounded for $x < 0$, there exists a sequence $x_n \to -\infty$ such that both $Q'(x_n) \to 0$ and $Q''(x_n) \to 0$. Passing to the limit $n \to +\infty$ in (3.112) leads to

$$(\rho - c)q_- = \alpha\int_{-\infty}^\infty Q_y(y)F(y)dy, \tag{3.113}$$

where

$$q_- = \lim_{x\to-\infty} Q(x).$$

It follows that $q_- > 0$. $\square$

Finally, we look at the behavior of $Q(x)$ on the right.

Lemma 3.16. *The limit*

$$\lim_{x\to+\infty} Q(x)e^{-x} \tag{3.114}$$

exists and equals $1/(\rho - \kappa)$.

Proof. Lemmas 3.6 and 3.8 imply that there exist $0 < A_1 < A_2$ and $B > 0$ such that

$$A_1 e^x \le Q(x) \le A_2 e^x + B. \tag{3.115}$$

Consider the function $Z(x) = Q(x)e^{-x}$, which satisfies

$$\rho Z = -c\frac{\partial Z}{\partial x} + \kappa\frac{\partial^2 Z}{\partial x^2} + 2\kappa\frac{\partial Z}{\partial x} + \kappa Z + H(R). \tag{3.116}$$

Note that for $A_1 \le Z(x) \le A_2$ and for $x > x_0^+$ we have $R(x) < 1$, so that

$$H(R) = 1 + R^2,$$

and (3.116) becomes

$$(\rho - \kappa)Z = -(c - 2\kappa)\frac{\partial Z}{\partial x} + \kappa\frac{\partial^2 Z}{\partial x^2} + 1 + R^2, \text{ for } x > x_0^+. \tag{3.117}$$

Let us assume that $y_n \to +\infty$ such that $Z(y_n) \to \zeta$ as $n \to +\infty$. Since $Z(x)$ is uniformly bounded and positive, (3.116) implies that $\|Z\|_{C^{2,\alpha}} \le C$, hence the functions $Z_n(x) = Z(x + y_n)$ converge, after extracting a subsequence to a function $\bar{Z}$. As $R(x) \to 0$ as $x \to +\infty$, the function $\bar{Z}(x)$ is a bounded solution to

$$(\rho - \kappa)\bar{Z} = -(c - 2\kappa)\frac{\partial \bar{Z}}{\partial x} + \kappa\frac{\partial^2 \bar{Z}}{\partial x^2} + 1, \text{ for } x \in \mathbb{R}. \tag{3.118}$$

It follows that $\bar{Z}(x) \equiv 1/(\rho - \kappa)$, and, in particular, $\zeta = 1/(\rho - \kappa)$, finishing the proof. $\square$

This also completes the proof of Theorem 1.1, except for the strict inequality in the upper bound $c < 2\sqrt{\kappa\alpha}$.

The proof of Proposition 1.2

We prove the matching lower and upper bounds on c. First, exactly as in the proof of Lemma 3.3, using an exponential super-solution and the normalization at $x = 0$, we can show that for any $\varepsilon > 0$ there exists $a_0 > 0$ such that

$$-\varepsilon < c^a < 2\sqrt{\kappa \int_{-a}^{a} s_a^*(y)(-F_y^a(y))dy} + \varepsilon \text{ for all } a > a_0. \tag{3.119}$$

Passing to the limit $a \to +\infty$, we get an upper bound

$$c \leq 2\sqrt{\kappa \int_{-\infty}^{\infty} s^*(y)(-F_y(y))dy}. \tag{3.120}$$

For the lower bound, let $F(x)$ be the solution to the traveling wave equation

$$-cF_x - \kappa F_{xx} = \alpha F(x) \int_{-\infty}^{x} s^*(y)(-F_y(y))dy, \tag{3.121}$$

with $F(-\infty) = 1$ and $F(+\infty) = 0$ that we have just constructed, and set $F^n(x) = F(x+n)/F(n)$. The functions F_n satisfy

$$-cF_x^n - \kappa F_{xx}^n = \alpha\gamma_n(x)F^n, \quad \gamma_n(x) = \int_{-\infty}^{x+n} s^*(y)(-F_y(y))dy, \tag{3.122}$$

with $F_n(0) = 1$. The standard elliptic regularity estimates and the Harnack inequality imply that after extracting a subsequence, the functions $F_n(x)$ converge locally uniformly to a limit $G(x)$ that satisfies

$$-cG_x - \kappa G_{xx}^n = \alpha\gamma G, \quad \gamma = \int_{-\infty}^{\infty} s^*(y)(-F_y(y))dy > 0, \tag{3.123}$$

and $G(0) = 1$. In addition, the function $G(x)$ is positive and monotonically decreasing. As a consequence, since $c > 0$, we must have

$$c \geq 2\sqrt{\kappa\gamma}, \tag{3.124}$$

and the proof of Proposition 1.2 is complete. □

4 Numerical results

In this section, we describe numerical results obtained via an iterative finite differences scheme for the traveling wave system on a finite interval $[-a, a]$. The main idea is to solve the equations in the coupled system iteratively, one at a time while fixing the other. We stop the iterations when we observe convergence of the numerical speed.

We use the following algorithm to construct numerically the traveling wave:

- Start with an initial guess for $F(x)$ and $Q(x)$, compute $R(x)$ from (3.3), and $H(x)$ using (2.22), and set $s^*(x) = \min(1, R(x))$. A suitable initial guess for $F(x)$ is the traveling wave solution of the Fisher-KPP equation on $[-a, a]$. As we expect that the solution of the value function will have exponential behavior, we take $Q(x) = e^x$ as an initialization. We solve $R(x)$ using a finite difference scheme with terminal condition $R(a) = 0$.
- Given $H(x)$, we solve (3.2) for $Q(x)$ on $[-a, a]$ with the boundary conditions (3.5).
- Given $s^*(x)$ we solve (3.1) for $F(x)$ and c, with the boundary conditions (3.4) and normalization (3.6), using an iterative finite different scheme.

Recall that both $Q(x)$ and $R(x)$ grow exponentially, on the right and on the left, respectively. Accordingly, we rescale them, so that all functions involved are bounded. As the equation for F is non-linear, we use another iterative finite difference scheme to solve it, with a modified boundary condition that uses the solution of the linearized problem.

4.1 A rescaling for $Q(x)$ and $R(x)$

As $Q(x)$ approaches a positive constant on the left, simply rescaling it by e^{-x} to remove the exponential growth on the right would lead to an exponential growth on the left for the rescaled function. Instead, define

$$g(x) = \begin{cases} 1 + 2\tan^{-1}(1) - 2\tan^{-1}(x+1) & \text{for } x \le 0, \\ e^{-x} & \text{for } x \ge 0. \end{cases} \tag{4.1}$$

The function $g(x)$ is continuous, with continuous first three derivatives, converges to a constant on the left and decays exponentially on the right. The function $\tilde{Q}(x) = g(x)Q(x)$ satisfies

$$\phi_1 \tilde{Q} + \phi_2 \tilde{Q}_x - \kappa \tilde{Q}_{xx} = g e^x H(R(x)),$$

with

$$\phi_1 = \rho - c - c\frac{g_1}{g} - \frac{\kappa(2g_1^2 - g g_2)}{g^2} = \rho - \kappa \quad \text{for } x > 0,$$
$$\phi_2 = c + 2\frac{\kappa g_1}{g} = c - 2\kappa \quad \text{for } x > 0.$$

Here, g_1 and g_2 are, respectively, the first and the second derivative of g. Note that for $x \ge 0$ the functions ϕ_1 and ϕ_2 are constants and for $x < 0$ they are bounded.

The boundary conditions $Q'(-a) = 0$ and $Q'(a) = Q(a)$ transform to

$$\tilde{Q}'(-a) = \frac{g_1(-a)}{g(-a)}\tilde{Q}(-a) \quad \text{and } \tilde{Q}'(a) = 0.$$

We solve the equation for $\tilde{Q}$ instead, with a finite difference scheme. We use central differences for both the first and the second derivatives.

We use a similar strategy to rescale $R(x)$. Note that for $x < 0$ we have that $R(x) \sim e^{-x}$. The right boundary condition $R(a) = 0$ allows us to use simple rescaling by multiplying $R(x)$ by e^x. The function $\tilde{R}(x) = e^x R(x)$ satisfies

$$\tilde{R}'(x) = -\frac{\alpha}{2} Q'(x) F(x) \tag{4.2}$$

with the boundary condition $\tilde{R}(a) = 0$. We solve for $\tilde{R}$ using a finite difference scheme and approximate the first derivative using a backward difference. We use the numerical solution for F on the same finite grid. We approximate $Q'(x)$ in (4.2) as

$$Q'(x_i) = \frac{1}{g}\tilde{Q}_x - \frac{g_1}{g^2}\tilde{Q} \approx \frac{1}{g}\frac{\tilde{Q}_{i+1} - \tilde{Q}_{i-1}}{2h} - \frac{g_1}{g^2}\tilde{Q}_i,$$

where h is the grid spacing. The numerical equation for $\tilde{R}$ is

$$\frac{\tilde{R}_{i+1} - \tilde{R}_i}{h} = -\frac{\alpha}{2} F_i \left[\frac{1}{g}\frac{\tilde{Q}_{i+1} - \tilde{Q}_{i-1}}{2h} - \frac{g_1}{g^2}\tilde{Q}_i \right].$$

4.2 Numerical Solution for $F(x)$ on the interval $[-a, a]$

Let us first briefly explain an iterative finite difference scheme and relaxed boundary conditions for the Fisher-KPP equation

$$-cF_x = \kappa F_{xx} + \alpha F(1 - F), \tag{4.3}$$

on a finite interval $[-a, a]$, with $F(-a) = 1$, $F(a) = 0$ and $F(0) = 1/2$. This corresponds to $s^*(x) \equiv 1$. Near $x = a$, the solution to the Fisher-KPP equation is well approximated by the linearized equation

$$-cF_x - \kappa F_{xx} = \alpha F. \tag{4.4}$$

A solution to (4.4) with the initial condition $F(0) = 1/2$ is

$$F(x) = \frac{1}{2}\exp\{-\beta x\},$$

where β is given by

$$\kappa\beta^2 - c\beta + \alpha = 0, \qquad \beta(c) = \begin{cases} \dfrac{c}{2\kappa} & \text{if } 0 < c < 2, \\ \dfrac{c - \sqrt{c^2 - 4\kappa\alpha}}{2\kappa} & \text{if } c \geq 2. \end{cases} \tag{4.5}$$

We use

$$F(a) = \frac{1}{2}\exp\{-\beta(c)a\}, \tag{4.6}$$

as a new boundary condition for the Fisher-KPP equation on $[-a, a]$, instead of $F(a) = 0$. It will change at each step of the iterative algorithm to reflect the change in c. The modified boundary condition speeds up the rate of convergence of the solution to the iterative scheme described below.

The iterative algorithm solves

$$-cF_x^{k+1}-\kappa F_{xx}^{k+1}=\alpha F^k(1-F^k), \tag{4.7}$$

on the interval $[-a,a]$, with the boundary condition $F^{k+1}(-a)=1$, $F^{k+1}(a)=(1/2)\exp\{-\beta(c_k)a\}$ with the initialization $F_0(x)=(a-x)/(2a)$. The first and the second derivatives are approximated using the central differences. The speed c is updated after each iteration, to enforce $F^k(0)=1/2$ at each iteration step.

Solution for $F(x)$ for general $s^*(x)$

We solve numerically the equation:

$$-cF_x-\kappa F_{xx}=\alpha F\int_{-a}^{x}s^*(y)(-F_y(y))dy \tag{4.8}$$

on the interval $[-a,a]$ with the boundary condition as in the scheme for the Fisher-KPP equation. Take a partition with step h and let $n=2a/h$, $x_i=-a+ih$ and $F_i^k=F^k(x_i)$ be the value of the k-th approximation of the solution at x_i. At each step of the iterative scheme the speed c_k is updated, so that $F^k(0)=1/2$. We take F^0 to be the solution of the Fisher-KPP equation

$$-c_0F_x^0-\kappa F_{xx}^0=\alpha F^0(1-F^0)$$

on the interval $[-a,a]$, with c_0 chosen, so that $F^0(0)=1/2$. Using the central differences, we approximate the left side of (4.8) as

$$-c_k\frac{F_{i+1}^k-F_{i-1}^k}{2h}-\kappa\frac{F_{i+1}^k-2F_i^k+F_{i-1}^k}{h^2}.$$

To reduce the error in the approximation of the right side of (4.8), we integrate by parts

$$\begin{aligned}
&\int_{-a}^{x_i}s^*(y)(-F_y(y))dy=\int_{-a}^{x_0}s^*(y)(-F_y(y))dy+\int_{x_0}^{x_i}s^*(y)(-F_y(y))dy\\
&=1-F(x_0)+\int_{x_0}^{x}s^*(y)(-F_y(y))dy=1-F(x_0)+F(x_0)-F(x_i)s^*(x_i)-\int_{x_0}^{x_i}s_y^*(y)(-F(y))dy\\
&=1-F(x_i)R(x_i)+\int_{x_0}^{x_i}R_y(y)F(y)dy\approx 1-F_i^kR_i+\sum_{m}^{i-1}h\frac{R_y(x_j)F_j^k+R_y(x_{j+1})F_{j+1}^k}{2}.
\end{aligned}$$

In the last computation we take $x_0=\max\{x|s^*(x)=1\}$ and m such that $x_0=-a+mh$. We also use that $s^*(x)=\min\{1,R(x)\}$, so for $x>x_0$ we have that $s_x^*(x)=R_x(x)$. Thus, the discretized version of (4.8) is

$$-c_k\frac{F_{i+1}^k-F_{i-1}^k}{2h}-\kappa\frac{F_{i+1}^k-2F_i^k+F_{i-1}^k}{h^2}=\alpha F_i^{k-1}\Big(1-F_i^kR_i+\sum_{m}^{i-1}h\frac{R_y(x_j)F_j^k+R_y(x_{j+1})F_{j+1}^k}{2}\Big). \tag{4.9}$$

4.3 Summary of the numerical scheme

We summarize here the overall scheme used to construct numerically the traveling wave profile. We solve the mean field system on the interval $[-a, a]$ with a partition with step h. Let $n = 2a/h$, $x_i = -a + ih$ and $f_i = f_i(x_i)$ be the value a function f at point x_i.

We use the following iterative algorithm:

- Starting with an initial guesses for $F(x)$ and Q(x), compute $\tilde{R}(x) = e^x R(x)$ by

$$\frac{\tilde{R}_{i+1} - \tilde{R}_i}{h} = -\frac{\alpha}{2} F_i \left[\frac{1}{g} \frac{\tilde{Q}_{i+1} - \tilde{Q}_{i-1}}{2h} - \frac{g_1}{g^2} \tilde{Q}_i \right],$$

  with terminal condition $\tilde{R}(a) = 0$, where $\tilde{Q}(x) = g(x)Q(x)$ and g, g_1 and g_2 are defined as in Section 4.1. Compute $H(x)$ using (2.22), and set $s^*(x) = \min(1, R(x))$.

- Given $H(x)$, compute $\tilde{Q}(x) = g(x)Q(x)$ using the equation

$$\phi_1 \tilde{Q} + \phi_2 \tilde{Q}_x - \kappa \tilde{Q}_{xx} = g e^x H(R(x)),$$

  with

$$\phi_1 = \rho - c - c\frac{g_1}{g} - \frac{\kappa(2g_1^2 - gg_2)}{g^2} = \rho - \kappa \quad \text{for } x > 0,$$
$$\phi_2 = c + 2\frac{\kappa g_1}{g} = c - 2\kappa \quad \text{for } x > 0,$$

  and central differences for all of the derivatives, with boundary conditions:

$$\tilde{Q}'(-a) = \frac{g_1(-a)}{g(-a)} \tilde{Q}(-a) \quad \text{and } \tilde{Q}'(a) = 0.$$

- Given $s^*(x)$ we solve for $F(x)$ and the speed c using an iterative finite difference algorithm. Let F^k and c^k denote the solution for F and c after the $k - th$ iteration of the algorithm. Given F^{k-1} and c^{k-1}, we solve for F^k and c^k using the following equation

$$-c_k \frac{F_{i+1}^k - F_{i-1}^k}{2h} - \kappa \frac{F_{i+1}^k - 2F_i^k + F_{i-1}^k}{h^2} = \alpha F_i^{k-1} \Big(1 - F_i^k R_i + \sum_m^{i-1} h \frac{R_y(x_j) F_j^k + R_y(x_{j+1}) F_{j+1}^k}{2} \Big).$$

  Here $x_0 = \max\{x | s^*(x) = 1\}$ and m is such that $x_0 = -a + mh$. To compute the speed c^k we use the normalization $F^k(0) = \frac{1}{2}$. We use the following boundary condition for F^k:

$$F^k(-a) = 1 \quad F(a) = \frac{1}{2} exp\{-\beta(c^{k-1})a\},$$

  where

$$\beta(c) = \begin{cases} \dfrac{c}{2\kappa} & \text{if } 0 < c < 2, \\ \dfrac{c - \sqrt{c^2 - 4\kappa\alpha}}{2\kappa} & \text{if } c \geq 2. \end{cases}$$

  For F^0 and c^0 we take the last values for F and c from the outer iterative process. We repeat until we observe convergence of the speed up to the fourth decimal point.

- We repeat the last three steps until we observe convergence of the speed up to the third decimal point.

4.4 Discussion of the numerical results

We now discuss some conclusions one can draw from the numerical simulations. In particular, we compare the traveling wave profile of the knowledge distribution function to the traveling wave profile of the Fisher-KPP equation, and study numerically the dependence of the traveling wave solution of the mean field system on the parameters in the model.

Convergence as a increases

We first illustrate the convergence of the numerical scheme for the mean field system on a finite interval $[-a, a]$ as a increases. We fix the system parameters $\kappa = 1$, $\alpha = 2$ and $\rho = 10$, and the discretization step $h = 0.02$, and consider $a = \{15, 20, 25, 30, 35, 40\}$. We observe that F, Q and c stabilize for $a > 35$. As expected, we observe the solutions for F and Q converge pointwise as a grows. As the plots illustrating the convergence of F and Q do not seem very informative or surprising, we present below the convergence in a of various objects associated to these functions. Convergence of the transition point x_0 as a increases is illustrated in Figure 1.

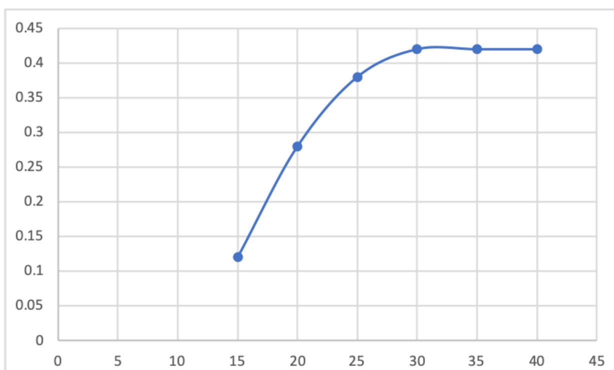

Figure 1: The transition point x_0 for $a \in \{15, 20, 25, 30, 35, 40\}$

Recall that, according to Theorem 1.1, the function $Q(x)$ approaches a positive limit $Q_- > 0$ on the left, and a multiple Q_+e^x of an exponential on the right, so that the rescaled function $\tilde{Q}$ approaches Q_- and Q_+ on the left and the right, respectively. This is illustrated in Figure 2.

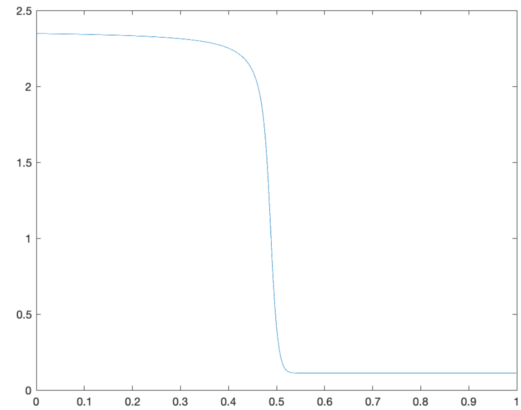

Figure 2: Rescaled Q on $[-40, 40]$, with $a = 40$.

Convergence of Q_- and Q_+ as a increases is illustrated in Figures 3, 4. Note that Q_+ is very close to the theoretical value $1/(\rho - \kappa)$, as in the third line of (1.4) in Theorem 1.1.

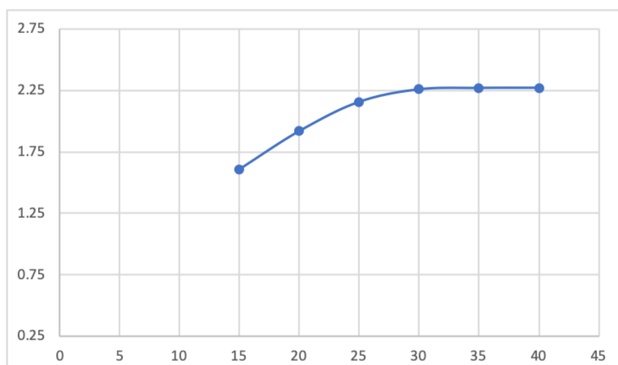

Figure 3: The left limit Q_- for $a \in \{15, 20, 25, 30, 35, 40\}$.

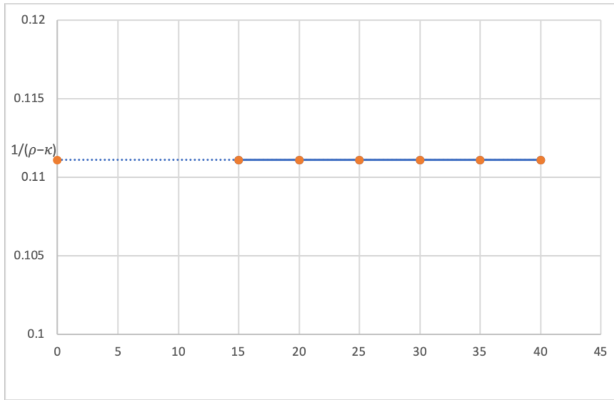

Figure 4: The right limit Q_+ for $a \in \{15, 20, 25, 30, 35, 40\}$.

The search function for $a = 40$ is plotted in Figure 5 below.

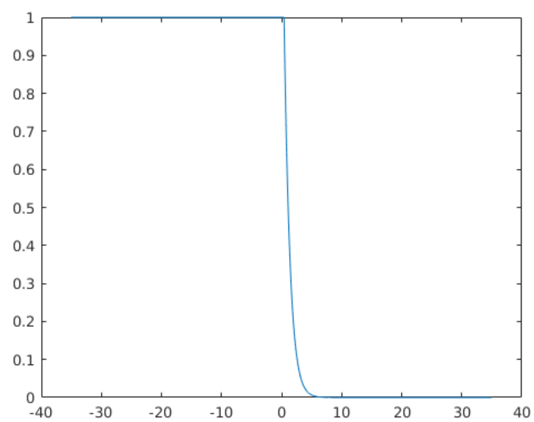

Figure 5: $s^*(x)$ on $[-40, 40]$

Comparison to a Fisher-KPP wave

Let us now compare the traveling wave solution F for the mean field system to a Fisher-KPP traveling wave with the same values of $\alpha = 2$ and $\kappa = 1$. Note that the Fisher-KPP speed with these parameters is $c_{FKPP} = 2\sqrt{2}$. The mean field system speed is smaller. The speed of the mean field system for different values of a is represented in the following plot:

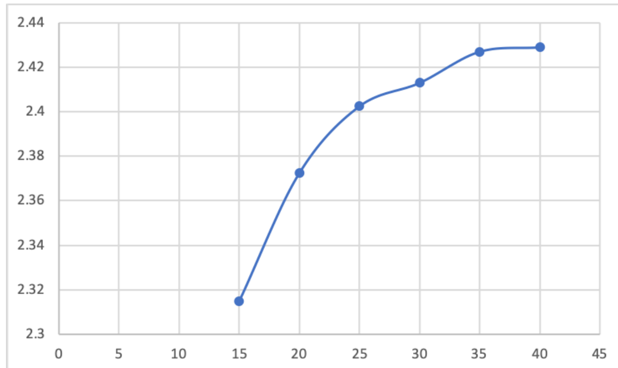

Figure 6: The traveling wave speed for $a \in \{15, 20, 25, 30, 35, 40\}$.

We plot the solution for F and the solution of the Fisher-KPP equation with $\alpha = 2$ Figure 7. As expected, F is above the Fisher-KPP solution on the left and below on the right, although they are very close. We will later see that the discrepancy becomes larger as ρ increases.

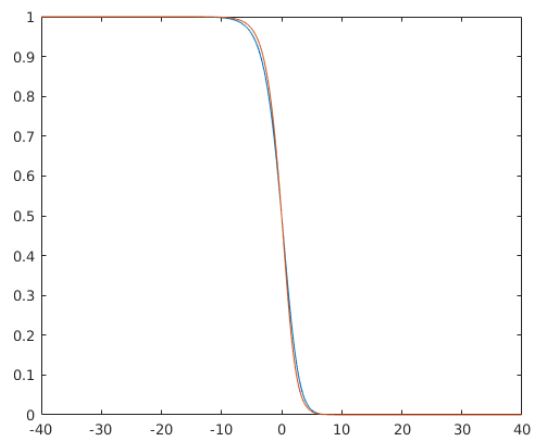

Figure 7: The Fisher-KPP solution vs F on $[-40, 40]$

The difference between the two profiles is better seen when the level slope of the function F is plotted vs the level slope of the solution of the Fisher-KPP equation with $\alpha = 2$ in the Figure 8. We observe that F is steeper for all values of $F(x)$.

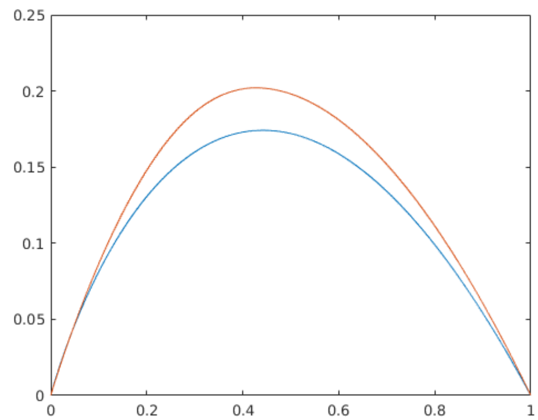

Figure 8: The level slope comparison - KPP vs F on $[-40, 40]$

In the rest of this section we study numerically how the three parameters in the model affect the wave profiles and the speed of the wave.

Dependence of the solution on α

Recall that the probability that an agent, who spends time s searching, meets another agent over a time interval Δt is $\alpha\sqrt{s}\Delta t$. Therefore, the parameter α corresponds to the effectiveness of the search – the larger α, the easier it is for agents to meet other agents and increase their productivity via learning. We now fix the values of $\kappa = 1$ and $\rho = 10$, the discretization $h = 0.02$ and $a = 40$ and illustrate numerically the dependence of the solution of the mean field system on α.

As seen from Figure 9, the speed of the traveling wave increases as α increases, but not as fast as the minimal speed for the Fisher-KPP equation given by $c_{FKPP} = 2\sqrt{\kappa\alpha}$. We also observe that as α approaches from above the critical value $\alpha_c = 1$, below which traveling waves do not exist, the speed tends to the Fisher-KPP speed, corresponding to $s^* \equiv 1$. This is as expected: as the search effectiveness decreases and approaches α_c, for the economy to grow along a balanced path, the agents need to spend more and more time searching, so that the transition point x_0 would tend to $+\infty$. As a reference, we also display in Figure 9 the lower bound $c_{low} = 2\kappa$ that holds for the traveling wave speed for any $\alpha > 0$.

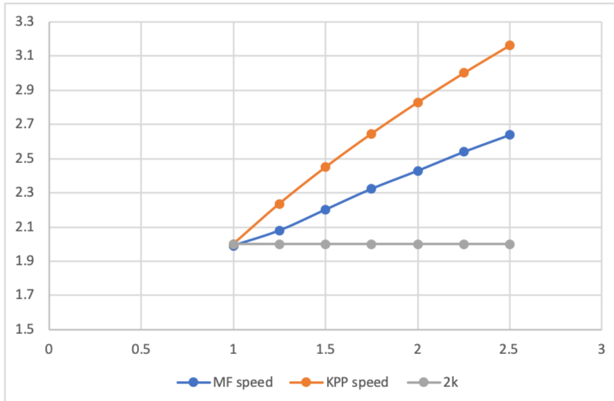

Figure 9: Dependence of the speed on α

The dependence of x_0 on α is also illustrated in Figure 10.

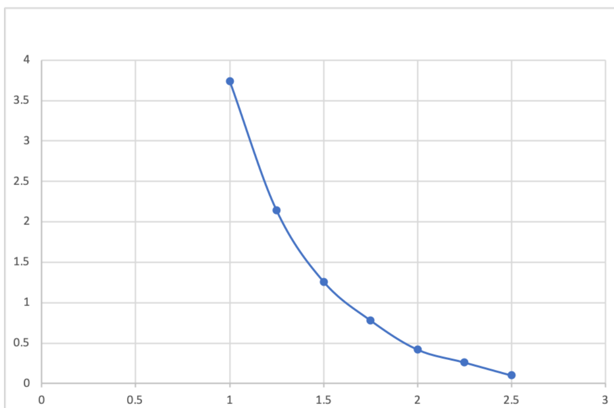

Figure 10: Dependence of x_0 on α

Dependence of the solution on ρ

Next, we consider numerically the dependence of the solution on the interest rate ρ. We fix the rest of the parameters in the simulation as $\kappa = 1$, $\alpha = 2$, $h = 0.02$ and $a = 40$. We observe that as ρ increases both the speed and x_0 decrease. This corresponds to a slower growth of the economy and to slower learning: agents tend to produce and not search as production is more profitable than the heavily discounted benefit of learning. In particular, we see that when ρ increases, the wave speed approaches its lower bound $c_{low} = 2\kappa$. On the other hand, we also see in Figure 11 that as ρ approaches from above the critical value $\rho_c = \kappa$, the speed approaches the Fisher-KPP speed, corresponding to $x_0 \to +\infty$. Numerically, we see in Figure 12 see that in this case x_0 also increases.

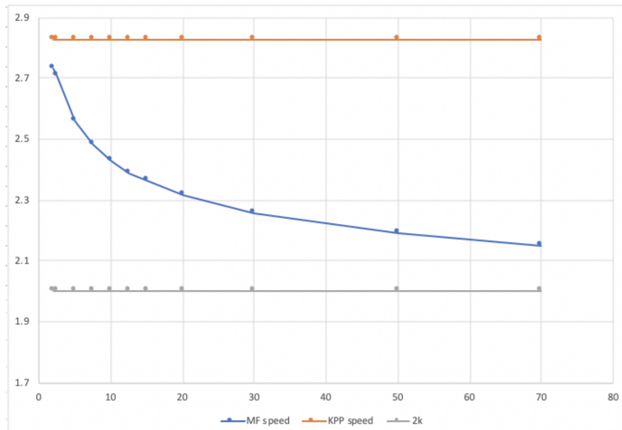

Figure 11: Dependence of the speed on ρ

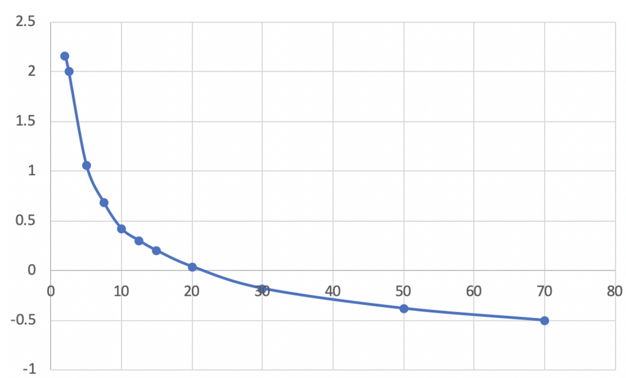

Figure 12: Dependence of x_0 on ρ

In Figures 13 and 14 we plot the numerical solution for F against the numerical solution for the Fisher-KPP equation for $a = 40$ and for different values of ρ. We only consider $x \in [-5, 5]$ as both solutions approach very fast the values 1 on the left and 0 on the right outside of this interval. We observe that the numerical solution of F gets closer to the numerical solution of the Fisher-KPP equation as ρ decreases.

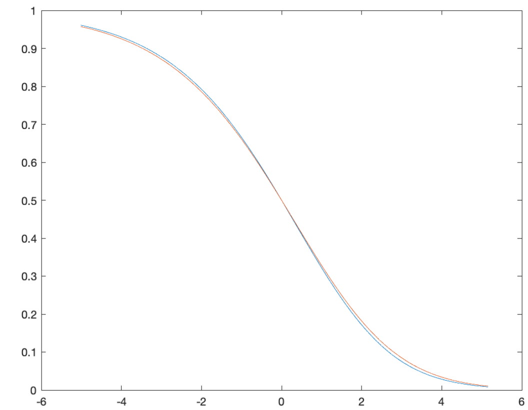

Figure 13: The traveling wave profile F vs the Fisher-KPP solution for $\rho = 2.2$.

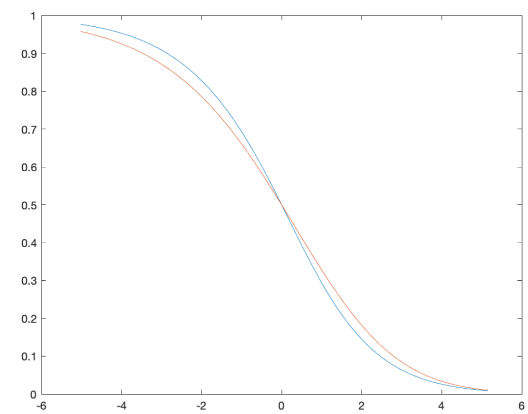

Figure 14: The traveling wave profile F vs the Fisher-KPP solution for for $\rho = 35$.

Dependence of the solution on κ

Finally, we look at the dependence of the solution on the diffusivity κ. We fix the parameters in the simulation as $\alpha = 2$, $\rho = 10$, $h = 0.02$, and $a = 40$. We observe that as κ increases the speed increases and the transition point x_0 moves to the right. This is intuitive from the economics point of view as large κ induces a fat tail of the distribution of knowledge, so agents will have higher incentive to search, as meeting an agent with a high' productivity and thus increasing your own productivity will be easier.

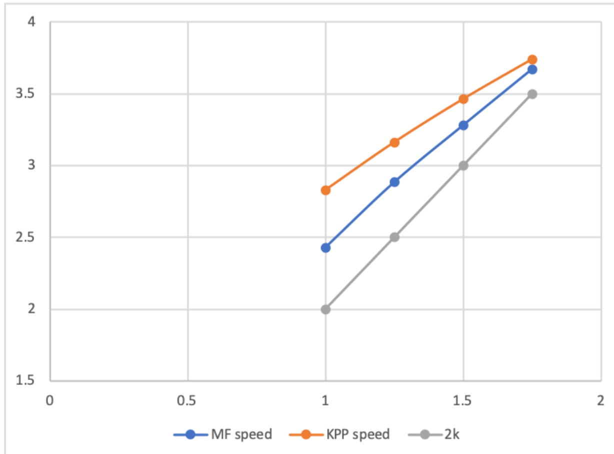

Figure 15: Dependence of the speed on κ.

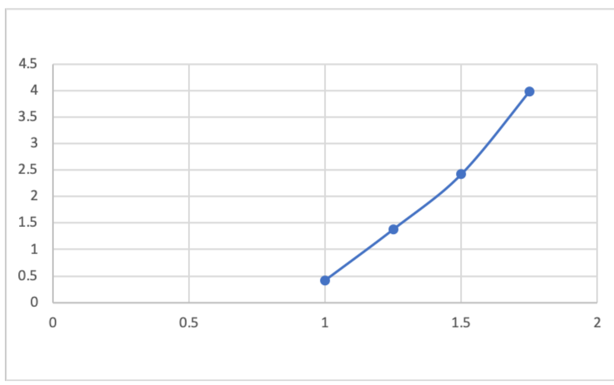

Figure 16: Dependence of x_0 on κ.

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