From the Implied Volatility Skew to a Robust Correction to Black-Scholes American Option Prices

Jean-Pierre Fouque* George Papanicolaou† K. Ronnie Sircar‡

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Abstract

We describe a robust correction to Black-Scholes American derivatives prices that accounts for uncertain and changing market volatility. It exploits the tendency of volatility to cluster, or fast mean-reversion, and is simply calibrated from the observed implied volatility skew. The two-dimensional free-boundary problem for the derivative pricing function under a stochastic volatility model is reduced to a one-dimensional free-boundary problem (the Black-Scholes price) plus the solution of a *fixed* boundary-value problem. The formal asymptotic calculation that achieves this is presented here. We discuss numerical implementation and analyze the effect of the volatility skew.

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*Department of Mathematics, North Carolina State University, Raleigh NC 27695-8205, *fouque@math.ncsu.edu*.

†Department of Mathematics, Stanford University, Stanford CA 94305, *papanico@math.stanford.edu*.
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‡Department of Mathematics, University of Michigan, Ann Arbor MI 48109-1109, *sircar@umich.edu*.
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1 Introduction

1.1 Derivative Pricing Methodology

In financial markets where an implied volatility skew is observed, stochastic volatility models have had tremendous success in improving upon the Black-Scholes theory for derivative pricing and hedging. Typically, the following "brute force" procedure is applied in practice:

- Choose a stochastic volatility model, preferably one such as the CIR model [7] for which there is an explicit or semi-explicit formula for European option prices.
- Estimate the parameters of the chosen model by fitting the formula (if there is one, or else by intensive simulation) to observed European option prices, or, by proxy, the implied volatility surface.
- Price other derivative securities such as Americans, Asians, exotics in this stochastic volatility environment using these estimated parameters.

The last part has received relatively little attention in the literature since one can merely "extend to an extra dimension" the pricing algorithms one would use under the Black-Scholes constant volatility model for those securities. For an American option, for example, this entails solving a free-boundary problem with two spatial dimensions, and while this introduces no new algorithmic problem, efficiency is greatly reduced, both in terms of speed and memory requirements.

This article describes a robust procedure to correct Black-Scholes American option prices to account for the observed nonflat implied volatility term-structure, modeled very generally to have arisen from random volatility. No specific model of the volatility process is required and there is no need to estimate the current level of the unobservable stock price volatility. The American option correction is computed numerically and is obtained through an easy extension of the method used to compute the Black-Scholes price. In fact the correction satisfies a *fixed* boundary problem.

The philosophy of using European option prices to price American (and other) derivatives consistently conforms to the "financial engineering" viewpoint that the modern-day strength of the Black-Scholes pricing *methodology* is not in the relationship between European option prices and historical stock price volatility, but rather in the relationship between the fair prices to be charged for more exotic contracts and implied volatility. In other words, European option prices, encapsulated in the skew surface, are part of the basic observables.

1.2 Volatility Clustering and Fast Mean-Reversion

The robustness to specific modeling and computational efficiency of the method we present comes from an asymptotic analysis of the derivative pricing problems that exploits the much-observed *clustering* property of market volatility. When volatility is high, it tends to stay high for a few days or so, before dropping to a lower level where it tends to stay for a few more days, and so on. As described in [4], this is closely related to *fast mean-reversion* in stochastic volatility models. That is, while volatility is fluctuating about its mean *slowly* compared to the tick-by-tick fluctuation of the stock price, it is fluctuating *fast* when looked at over the timescale of an options contract (typically a few months). We shall employ the latter description here, though often people refer to slow mean-reversion, meaning by comparison with the tick timescale.

In [4], we looked at the European option pricing problem under a large class of stochastic volatility models (reviewed in Section 3) in which volatility is “bursty” or fast mean-reverting. There it was found that implied volatility I (from European options) is well-approximated by a straight line in the composite variable LMMR, the log moneyness-to-maturity ratio:

$$\text{LMMR} = \frac{\log\left(\frac{\text{Strike Price}}{\text{Stock Price}}\right)}{\text{Time to Maturity}}.$$

That is,

$$I \approx a(\text{LMMR}) + b.$$

The parameters a and b are estimated from a linefit of observed implied volatility plotted as a function of LMMR, and they contain the original model parameters in the risk-neutral world (including the market price of volatility risk). These two plus the mean historical stock price volatility $\bar{\sigma}$ are all that is needed to price, under fast mean-reverting stochastic volatility, other European contracts, as shown in [4], barrier options [6], and many other “exotics”. Here we detail how the same is true for American options.

We begin with a brief review of the American option pricing theory when volatility is constant to introduce notation. In Section 3, we detail the class of stochastic volatility models we shall study and the limiting process that models volatility clustering. The main asymptotic analysis is presented in Section 4, where the fixed boundary problem for the correction is derived. This is illustrated with computations in Section 5, where the numerical issues of implementing the result are discussed. We also investigate the effect of skewness on the correction. We conclude in Section 6.

2 Review of the Constant Volatility Pricing Theory

We shall present the analysis for American put options which give the holder the right, but not the obligation, to sell one unit of the underlying stock at the strike price K at *any time before* the expiration date T. The techniques extend to any convex payoff structure equipped with the early exercise feature. In this section, we review briefly the classical American put pricing problem, mainly to highlight different formulations which are useful for different purposes. For more complete derivations, we refer to [1] or [9].

2.1 Black-Scholes Model and Optimal Stopping

The lognormal model for the stock price $(X_t)_{0\le t\le T}$ is defined on a probability space $(\Omega, \mathcal{F}, I\!P)$ on which $(W_t)_{t\ge 0}$ is a standard Brownian motion. It is defined by the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{1}$$

where μ is the modeler's subjective expected growth rate for the stock, and σ is the assumed-constant volatility. The filtration $(\mathcal{F}_t)_{0\le t\le T}$ represents information on (X_t) at time t.

The first way we shall characterize the American put option's price is as an optimal stopping problem in terms of the *a priori* unknown time at which it is exercised.

If the option is exercised, the chosen time, denoted by τ, is called the *exercise time*. As the market cannot be anticipated, the holder has to make his or her decision to exercise or not at time $t \le T$ with the information up to time t contained in the σ-algebra $\mathcal{F}_t$. In other words τ is a random time such that the event $\{\tau \le t\}$ (or its complement $\{\tau > t\}$) belongs to $\mathcal{F}_t$ for any $t \le T$. Such a random time is called a *stopping time* with respect to the filtration $(\mathcal{F}_t)$. The payoff function of the put option being $(K-x)^+$, its value at the exercise time τ is $(K-X_\tau)^+$ where X_τ is the stock price at the stopping time τ.

As is well-known, (1) describes a *complete* market model, and there is a unique pricing measure $I\!P^\star$ under which the discounted price of the stock is a martingale. We assume throughout a constant short-rate $r \ge 0$, so that

$$dX_t = rX_t dt + \sigma X_t dW_t^\star,$$

where $(W_t^\star)_{t\ge 0}$ is a standard Brownian motion under $I\!P^\star$. Using the theory of *optimal stopping* it can be shown that the *no arbitrage* price of an American put P is obtained by maximizing over all the stopping times the expected value of the discounted payoff under the risk-neutral probability. That is,

$$P(t,x) = \sup_{t\le\tau\le T} I\!E^\star \left\{ e^{-r(\tau-t)}(K-X_\tau)^+ | X_t = x \right\}, \tag{2}$$

is the price of the derivative at time $t < T$, when $X_t = x$ and where the supremum is taken over all the possible stopping times taking values in $[t, T]$.

The supremum in (2) is reached at the *optimal stopping time* $\tau^\star(t)$ defined by

$$\tau^\star(t) = \inf\left\{ t \le s \le T\,,\ P(s, X_s) = (K-X_s)^+ \right\}, \tag{3}$$

the first time that the price of the derivative drops down to its payoff. As one can see in order to determine $\tau^\star(t)$, one has to compute the price first.

2.2 Partial Differential Inequalities

A second characterization, in terms of partial differential equations, leads to a so-called *linear complementarity problem.*

Pricing functions for American derivatives satisfy partial differential *inequalities.* These follow from the Markovian model (1) we started with. The price of the American put is the solution of the system

$$
\begin{aligned}
\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial x^2}+rx\frac{\partial P}{\partial x}-rP &\le 0, &(4)\\
P &\ge (K-x)^+, &(5)\\
\left(\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 P}{\partial x^2}+rx\frac{\partial P}{\partial x}-rP\right)\left((K-x)^+-P\right) &= 0, &(6)
\end{aligned}
$$

to be solved in $\{(t,x) : 0 \le t \le T,\, x > 0\}$ with the final condition $P(T,x) = (K-x)^+$. The first inequality is linked to the supermartingale property of $e^{-rt}P(t,X_t)$ under $\mathbb{P}^\star$. This formulation is particularly convenient for numerical computation of $P(t,x)$ as it does not explicitly refer to the free boundary. We refer to [10] for further details, and return to this in Section 5.

2.3 Formulation involving the Exercise Boundary

A more visually appealing characterization is a system of equations for the pricing function and the free boundary in which the (t,x) plane is divided into an exercise region and a hold region, the labels describing the course of action of the *risk-neutral* investor following the optimal strategy when $X_t = x$. There is an increasing function $x^\star(t)$, the free boundary, to be determined, such that, at time t

$$
\begin{cases}
\text{for } x < x^\star(t),\ P(t,x) = K - x\\
\text{for } x > x^\star(t),\ \frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 P}{\partial x^2}+rx\frac{\partial P}{\partial x}-rP = 0\,,
\end{cases}
\tag{7}
$$

with

$$
\begin{aligned}
P(T,x) &= (K-x)^+ &(8)\\
x^\star(T) &= K. &(9)
\end{aligned}
$$

We also have that P and $\frac{\partial P}{\partial x}$ are continuous across the boundary $x^\star(t)$, so that

$$
\begin{aligned}
P(t,x^\star(t)) &= K - x^\star(t), &(10)\\
\frac{\partial P}{\partial x}(t,x^\star(t)) &= -1. &(11)
\end{aligned}
$$

The exercise boundary $x^\star(t)$ separates the *hold* region, where the option is not exercised, from the exercise region, where it is. This is illustrated in Figure 1.

In the corresponding Figure 2, we show the trajectory of the stock price and the optimal exercise time $\tau^\star$.

Notice that this is a system of equations and boundary conditions for $P(t,x)$ *and* the free boundary $x^\star(t)$.

This last formulation is the easiest to use for the formal asymptotic calculations of Section 4. For existence proofs, variational or penalization characterizations of the problem are often used, and we refer to [2] or [8] for details.

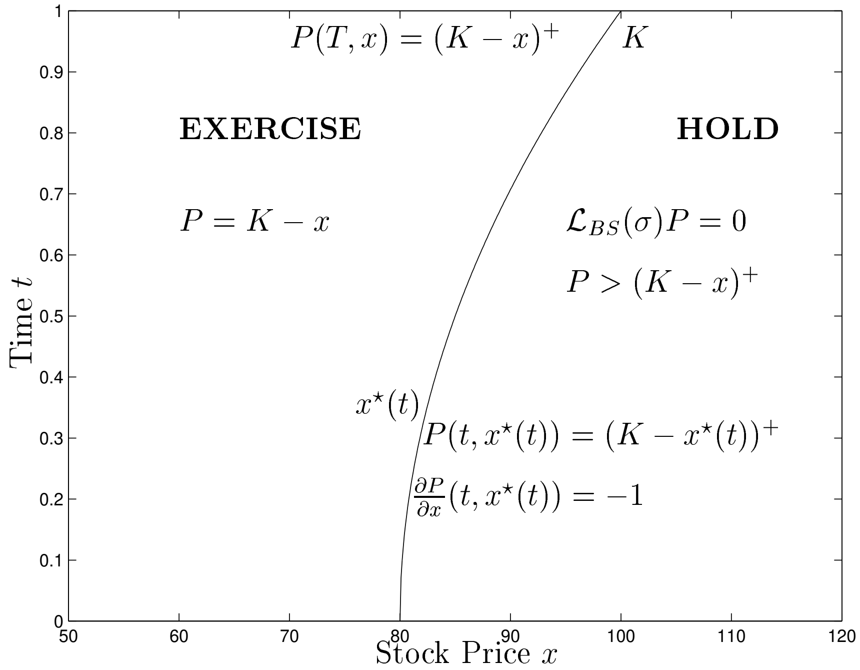

Figure 1: *The American put problem for* $P(t,x)$ *and* $x^\star(t)$, *with* $\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right)$.

3 Mean-Reverting Stochastic Volatility Models

We shall look at stochastic volatility models in which volatility (σ_t) is driven by an ergodic process (Y_t) that approaches its unique invariant distribution at an exponential rate α. The size of this rate captures clustering effects, and in particular we shall be interested in asymptotic approximations when α is large, which describes bursty volatility.

As explained in [4], it is convenient for exposition to take a specific simple example for (Y_t) and allow the generality of the modeling to be in the unspecified relation between volatility and this process: $\sigma_t = f(Y_t)$, where f is some positive (and sufficiently regular) function. Further, taking (Y_t) to be a Markovian Itô process allows us to simply model the asymmetry or fatter left-tails of returns distributions by incorporating a negative correlation between asset price and volatility shocks. We shall thus take (Y_t) to be a mean-reverting Ornstein-Uhlenbeck (OU) process, so that the stochastic volatility models we consider are

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma_t X_t dW_t, \qquad (12)\\ \sigma_t &= f(Y_t), \\ dY_t &= \alpha(m - Y_t)dt + \beta\left(\rho dW_t + \sqrt{1-\rho^2}\right) dZ_t. \end{aligned}$$

Here (W_t) and (Z_t) are independent standard Brownian motions and ρ is the correlation between asset price and volatility shocks that captures the skew, asymmetry or leverage effect. The asymptotic results as they are used are *not* specific to the choice of the OU diffusion process, nor do they depend on specifying f.

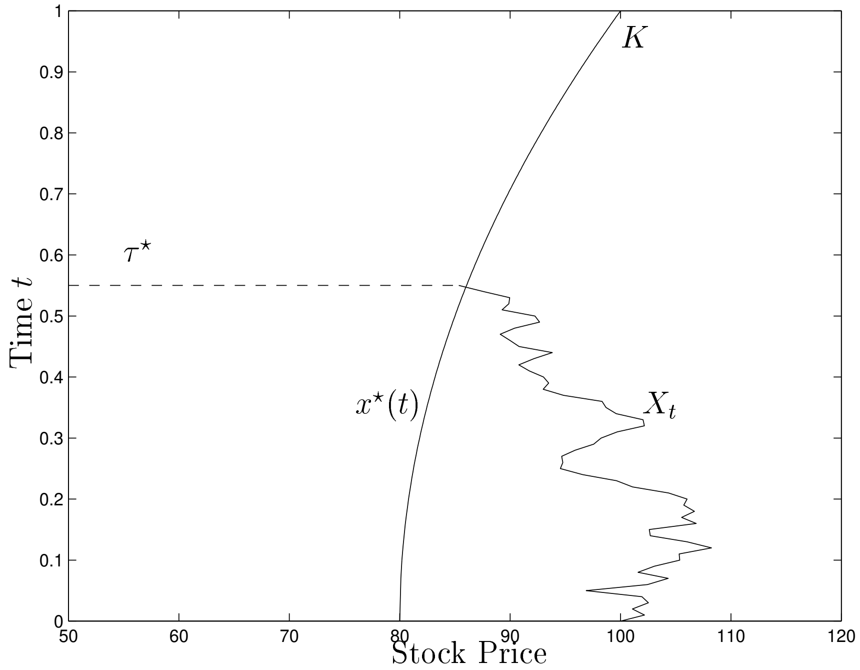

Figure 2: *Optimal exercise time $\tau^\star$ for an American put option.*

3.1 The Risk-Neutral Measure

We now consider the American put pricing problem. Recall that an American contract gives the holder the right of early exercise, and consequently the date that the contract is terminated is not known beforehand, unlike in the European case. As we reviewed in Section 2, for an American put option under the Black-Scholes model, the *no arbitrage* pricing function satisfies a free-boundary value problem characterized by the system of equations (7) with boundary conditions (8,9,10) and (11). This is a much harder problem than the European pricing problem, and there are no explicit solutions in general. It has to be solved numerically. Nevertheless, we show in Section 4 that the asymptotic method for correcting the Black-Scholes price for stochastic volatility can be extended to contracts with the early exercise feature, simplifying considerably the two-dimensional free boundary problems that arise in these models.

The model (12) describes an *incomplete* market meaning that not all contingent claims can be replicated. This has profound consequences for pricing, hedging and calibration problems for derivative securities. By standard no-arbitrage pricing theory, there is more than one possible equivalent martingale (or risk-neutral pricing) measure $\mathbb{P}^{\star(\gamma)}$ because the volatility is not a traded asset; the nonuniqueness is denoted by the dependence on γ, which we identify as the market price of volatility risk.

By Girsanov's theorem, $(W_t^\star, Z_t^\star)$ defined by

$$\begin{aligned} W_t^\star &= W_t + \int_0^t \frac{(\mu - r)}{f(Y_s)} ds, \\ Z_t^\star &= Z_t + \int_0^t \gamma_s ds, \end{aligned}$$

are independent Brownian motions under a measure $I\!P^{\star(\gamma)}$ defined by

$$\frac{dI\!P^{\star(\gamma)}}{dI\!P} = \exp\left(-\int_0^T \frac{(\mu-r)}{f(Y_s)}dW_s - \int_0^T \gamma_s dZ_s - \frac{1}{2}\int_0^T \left[\left(\frac{(\mu-r)}{f(Y_s)}\right)^2 + \gamma_s^2\right] ds\right),$$

assuming for instance that $(\frac{\mu-r}{f(Y_t)}, \gamma_t)$ satisfies the Novikov condition.

In particular, γ_t is the risk premium factor from the *second* source of randomness Z that drives the volatility. We shall assume that the market price of volatility risk γ_t is a function of the state Y_t: $\gamma = \gamma(Y_t)$. As explained in [4], we take the view that the market selects a pricing measure identified by a particular γ which is reflected in liquidly traded around-the-money European option prices. Prices of other derivative securities must be priced with respect to this measure, if there are to be no arbitrage opportunities. Under our assumption about the volatility risk premium, the process (Y_t) remains autonomous and Markovian under the pricing measure.

Under $I\!P^{\star(\gamma)}$,

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t dW_t^\star, && (13)\\ dY_t &= \left[\alpha(m-Y_t) - \beta\left(\rho\frac{(\mu-r)}{f(Y_t)} + \gamma(Y_t)\sqrt{1-\rho^2}\right)\right]dt \\ &\quad +\beta\left(\rho dW_t^\star + \sqrt{1-\rho^2}dZ_t^\star\right). && (14)\end{aligned}$$

The American put price $P(t,x,y)$ is given by

$$P(t,x,y) = \sup_{t\le\tau\le T} I\!E^{\star(\gamma)}\left\{e^{-r(\tau-t)}(K-X_\tau)^+ | X_t = x, Y_t = y\right\}, \qquad (15)$$

where the supremum is taken over all stopping times $\tau \in [t,T]$.

3.2 American Option Free Boundary Problem

The function $P(t,x,y)$ in (15) again satisfies a free boundary problem analogous to (7), with the additional spatial variable y, and the free boundary is now a surface which can be written $x = x_{fb}(t,y)$ and has to be determined as part of the problem:

$$\begin{aligned} P(t,x,y) &= K - x \text{ for } x < x_{fb}(t,y), && (16)\\ \frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial x\partial y} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2} & \\ +r\left(x\frac{\partial P}{\partial x} - P\right) + (\alpha(m-y) - \beta\lambda(y))\frac{\partial P}{\partial y} &= 0 \qquad \text{for } x > x_{fb}(t,y) && (17)\end{aligned}$$

where

$$\lambda(y) := \rho\frac{(\mu-r)}{f(y)} + \gamma(y)\sqrt{1-\rho^2}, \qquad (18)$$

with

$$\begin{aligned} P(T,x,y) &= (K-x)^+ && (19)\\ x_{fb}(T,y) &= K. \end{aligned}$$

Also, P, $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ are continuous across the boundary $x_{fb}(t, y)$, so that

$$\begin{aligned} P(t, x_{fb}(t,y), y) &= (K - x_{fb}(t,y))^+ , \\ \frac{\partial P}{\partial x}(t, x_{fb}(t,y), y) &= -1, \\ \frac{\partial P}{\partial y}(t, x_{fb}(t,y), y) &= 0. \end{aligned} \tag{20}$$

These are shown in Figure 3.

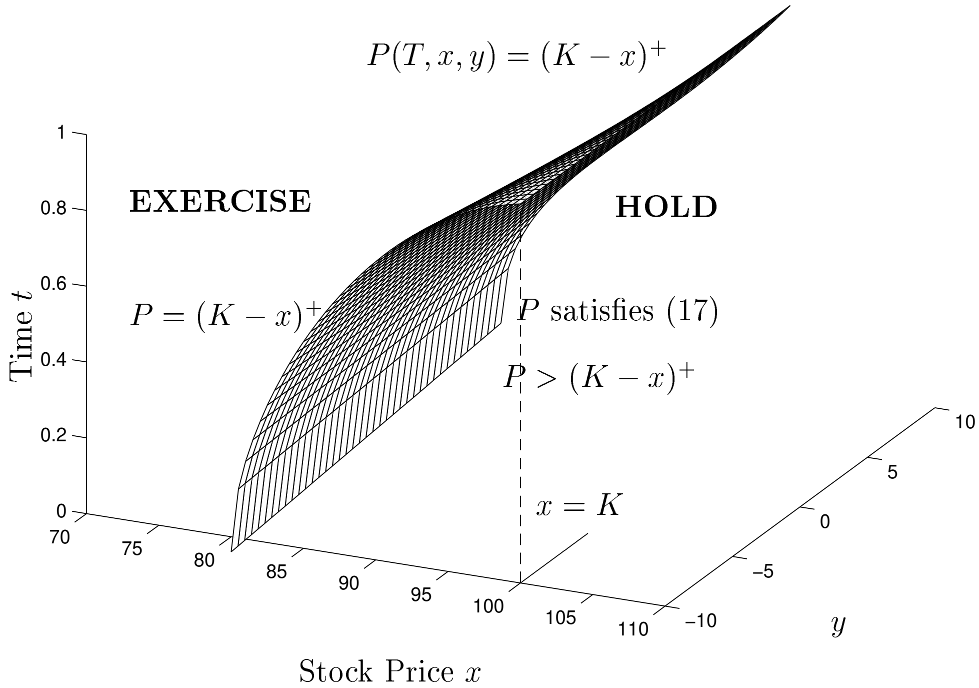

Figure 3: *The full problem for the American put under stochastic volatility. The free boundary conditions on the surface are given by (20).*

4 Stochastic Volatility Correction for American Put

The fast mean-reversion limit $\alpha \to \infty$ with $\nu^2 := \beta^2/2\alpha$ a fixed $\mathcal{O}(1)$ constant captures the volatility clustering behaviour we want to exploit. In the limit, volatility is like a constant and we return to the Black-Scholes theory. We are interested in the correction to Black-Scholes when α is large. A detailed study of high-frequency S&P 500 data that establishes this fast mean-reversion pattern is found in [5].

We use the notation of [4]:

$$\varepsilon = \frac{1}{\alpha},$$

$$
\begin{aligned}
\beta &= \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}, \\
P^{\varepsilon}(t,x,y) &= P(t,x,y), \\
\mathcal{L}^{\varepsilon} &= \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \\
\mathcal{L}_0 &:= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \\
\mathcal{L}_1 &:= \sqrt{2}\nu\rho x f(y)\frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\lambda(y)\frac{\partial}{\partial y}, \\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right),
\end{aligned}
\tag{21}
$$

where $\mathcal{L}_0$ is the infinitesimal generator of the mean-reverting OU process, $\mathcal{L}_1$ contains the mixed derivative (from the correlation) and the market price of risk γ, and $\mathcal{L}_2$ is the Black-Scholes partial differential operator $\mathcal{L}_{BS}(f(y))$, where

$$
\mathcal{L}_{BS}(\sigma) := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right). \tag{22}
$$

In Section 4.2, we present the formal asymptotic calculations for the American put problem.

4.1 Review of European Options Asymptotics

We give here for reference the main result of the asymptotic analysis of the European options problem under fast mean-reverting stochastic volatility that appears in [4].

If C^{E} is the European call option price satisfying

$$
\begin{aligned}
\frac{\partial C^{\mathrm{E}}}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 C^{\mathrm{E}}}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 C^{\mathrm{E}}}{\partial x \partial y} + \frac{1}{2}\beta^2\frac{\partial^2 C^{\mathrm{E}}}{\partial y^2} & \\
+r\left(x\frac{\partial C^{\mathrm{E}}}{\partial x} - C^{\mathrm{E}}\right) + (\alpha(m-y) - \beta\lambda(y))\frac{\partial C^{\mathrm{E}}}{\partial y} &= 0
\end{aligned}
$$

in $x > 0$, $t < T$, with $C^{\mathrm{E}}(T,x,y) = (x-K)^+$, then I defined by

$$
C^{\mathrm{E}} = C_{BS}(I),
$$

where C_{BS} is the Black-Scholes formula, is given by

$$
I = a\frac{\log(K/x)}{(T-t)} + b + \mathcal{O}(\alpha^{-1}). \tag{23}
$$

The parameters a and b are estimated as the slope and intercept of the linefit of observed implied volatilities plotted as a function of LMMR. In particular, b is the *at-the-money* implied volatility (because LMMR$= 0$ when $K = x$).

The price C^h of any other European derivative with payoff function $h(x)$, for example binary options (and barrier options with the addition of suitable boundary conditions as explained in [6]), is given by

$$C^h = C_0 + \tilde{C}_1 + \mathcal{O}(\alpha^{-1}).$$

Here $C_0(t,x)$ is the solution to the corresponding Black-Scholes problem with constant volatility $\bar{\sigma}$,

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})C_0(t,x) &= 0, \\ C_0(T,x) &= h(x), \end{aligned}$$

where $\mathcal{L}_{BS}$ is the Black-Scholes differential operator defined in (22). The correction $\tilde{C}_1(t,x)$ for stochastic volatility satisfies

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{C}_1 = V_3 x^3 \frac{\partial^3 C_0}{\partial x^3} + V_2 x^2 \frac{\partial^2 C_0}{\partial x^2}, \tag{24}$$

with

$$V_2 := \bar{\sigma}\left((\bar{\sigma} - b) - a(r + \frac{3}{2}\bar{\sigma}^2)\right) \tag{25}$$

$$V_3 := -a\bar{\sigma}^3, \tag{26}$$

and $\bar{\sigma}$ the long-run historical asset price volatility. The terminal condition is $\tilde{C}_1(T,x) = 0$. The explicit solution is given by

$$\tilde{C}_1(t,x) = -(T-t)\left(V_3 x^3 \frac{\partial^3 C_0}{\partial x^3} + V_2 x^2 \frac{\partial^2 C_0}{\partial x^2}\right).$$

Notice that to this order of approximation, $C_0 + \tilde{C}_1$ does not depend on the present level y of the unobservable driving process (Y_t). This will also be true in the American case.

The table below then distinguishes the model parameters from the parameters that are actually needed for the European theory. The latter can be written as groupings of the former by the formulas given in [4], but for practical purposes, there is no need to do so.

<table>
<tr><th>Model Parameters</th><th>Parameters that are needed</th></tr>
<tr><td>Growth rate of stock μ</td><td rowspan="2">Mean historical volatility of stock $\bar{\sigma}$</td></tr>
<tr><td>Long-run mean volatility m</td></tr>
<tr><td>Rate of mean-reversion of volatility α</td><td rowspan="2">Slope of implied volatility linefit a</td></tr>
<tr><td>Volatility of volatility β</td></tr>
<tr><td>Correlation between shocks ρ</td><td rowspan="2">At-the-money implied volatility b</td></tr>
<tr><td>Volatility risk premium γ</td></tr>
</table>

The three parameters on the right-side of the table are easily estimated and found to be quite stable from S&P 500 data in [4, 3].

We shall see in the next section that a similar equation to (24) is involved in the problem for the American option correction, but also that only the parameters V_2 and V_3 show up, as in the European case.

4.2 American Options Asymptotics

The main equation (17) can be written

$$\mathcal{L}^\varepsilon P^\varepsilon(t,x,y) = 0, \text{ in } x > x^\varepsilon_{fb}(t,y),$$

where we denote the free boundary surface by $x^\varepsilon_{fb}(t,y)$ to stress the dependence on ε. Note that the exercise boundary is, in general, a surface $F^\varepsilon(t,x,y) = 0$ which we write as $x = x^\varepsilon_{fb}(t,y)$.

We look for an asymptotic solution of the form

$$\begin{aligned} P^\varepsilon(t,x,y) &= P_0(t,x,y) + \sqrt{\varepsilon}P_1(t,x,y) + \varepsilon P_2(t,x,y) + \cdots, \quad (27) \\ x^\varepsilon_{fb}(t,y) &= x_0(t,y) + \sqrt{\varepsilon}x_1(t,y) + \varepsilon x_2(t,y) + \cdots, \quad (28) \end{aligned}$$

which converges as $\varepsilon \downarrow 0$. We have expanded the formula for the free-boundary surface as well.

Our strategy for constructing a solution will be to expand the equations and boundary conditions in powers of ε, substituting the expansions (27) and (28). We then look at the the equations at each order in both the hold and exercise regions and take the dividing boundary for each subproblem to be $x_0(t,y)$ which is accurate to principal order. Thus the extension or truncation of the hold region to the x_0 boundary is assumed to introduce only an $\mathcal{O}(\sqrt{\varepsilon})$ error into each term $P_j(t,x,y)$ of the expansion for the price. This will be true up to a region of width $\mathcal{O}(\sqrt{\varepsilon})$ about x_0. When the stock price is so close to the exercise boundary, we do not expect the asymptotics to be accurate because the contract likely does not exist long enough for the "averaging effects" of fast mean-reverting volatility to take hold. This is exactly as for a European option close to the expiration date, when the asymptotic approximation is not valid.

The expansion of the partial differential equation $\mathcal{L}^\varepsilon P^\varepsilon = 0$ in the hold region is as in the European case:

$$\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0\right) + \left(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right) + \sqrt{\varepsilon}\left(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right) + \cdots = 0. \quad (29)$$

Keeping terms up to $\sqrt{\varepsilon}$, we expand the free boundary conditions (20) as

$$\begin{aligned} P_0(t,x_0(t,y),y) + \sqrt{\varepsilon}\Big(x_1(t,y)\frac{\partial P_0}{\partial x}(t,x_0(t,y),y) & \\ + P_1(t,x_0(t,y),y)\Big) &= K - x_0(t,y) - \sqrt{\varepsilon}x_1(t,y) \quad (30) \\ \frac{\partial P_0}{\partial x}(t,x_0(t,y),y) + \sqrt{\varepsilon}\Big(x_1(t,y)\frac{\partial^2 P_0}{\partial x^2}(t,x_0(t,y),y) & \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\partial P_1}{\partial x}(t, x_0(t,y), y)\Bigg) &=& -1, &(31)\\
&\frac{\partial P_0}{\partial y}(t, x_0(t,y), y) + \sqrt{\varepsilon}\left(x_1(t,y)\frac{\partial^2 P_0}{\partial x \partial y}(t, x_0(t,y), y)\right. && &\\
&\quad \left. + \frac{\partial P_1}{\partial y}(t, x_0(t,y), y)\right) &=& 0, &(32)
\end{aligned}
$$

where the partial derivatives are taken to mean the one-sided derivatives into the region $x > x_0(t,y)$, because we expect, by analogy with the Black-Scholes American pricing problem, that the pricing function will not be smooth across the free boundary. However, it is smooth inside either region.

The terminal condition gives $P_0(T,x,y) = (K-x)^+$ and $P_1(T,x,y) = 0$, and the condition $P^\varepsilon = (K-x)^+$ in the exercise region gives that $P_0(t,x,y) = (K-x)^+$ and $P_1(t,x,y) = 0$ in that region.

4.3 First Approximation

To highest order in ε, we have the following problem:

$$
\begin{aligned}
\mathcal{L}_0 P_0(t,x,y) &= 0 && \text{in } x > x_0(t,y),\\
P_0(t,x,y) &= (K-x)^+ && \text{in } x < x_0(t,y),\\
P_0(t, x_0(t,y), y) &= (K - x_0(t,y))^+, &&\\
\frac{\partial P_0}{\partial x}(t, x_0(t,y), y) &= -1. &&
\end{aligned}
$$

Since $\mathcal{L}_0$ is the generator of an ergodic Markov process acting on the variable y, a standard argument implies that P_0 does not depend on y on each side of x_0. Consequently it cannot depend on y on the surface x_0 either, and so $x_0 = x_0(t)$ also does not depend on y.

Recall that $\mathcal{L}_1$ contains y-derivatives in both terms, so that $\mathcal{L}_1 P_0 = 0$, and the next order gives

$$
\begin{aligned}
\mathcal{L}_0 P_1(t,x,y) &= 0 && \text{in } x > x_0(t),\\
P_1(t,x,y) &= 0 && \text{in } x < x_0(t),\\
P_1(t, x_0(t), y) &= 0, &&\\
x_1(t,y)\frac{\partial^2 P_0}{\partial x^2}(t, x_0(t)) + \frac{\partial P_1}{\partial x}(t, x_0(t), y) &= 0. &&
\end{aligned}
$$

By the same argument, P_1 also does not depend on y: $P_1 = P_1(t,x)$.

From the $\mathcal{O}(1)$ terms in (29), we have

$$
\begin{aligned}
\mathcal{L}_0 P_2(t,x,y) + \mathcal{L}_2 P_0(t,x) &= 0 && \text{in } x > x_0(t),\\
P_2(t,x,y) &= 0 && \text{in } x < x_0(t),
\end{aligned}
\tag{33}
$$

since $\mathcal{L}_1 P_1 = 0$. In the region $x > x_0(t)$, this is a Poisson equation over $-\infty < y < \infty$, because the exercise boundary does not depend on y to principal order. There is no solution unless $\mathcal{L}_2 P_0$ has mean zero with respect to the invariant measure of the OU process Y_t:

$$
\langle \mathcal{L}_2 P_0 \rangle = 0,
$$

where

$$\langle g \rangle = \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^{\infty} e^{-(y-m)^2/2\nu^2} g(y)dy,$$

the expectation with respect to the invariant measure of the OU process. Since $\mathcal{L}_2$ only depends on y through the $f(y)$ coefficient, $\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle C_0$, and

$$\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma}) = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot \right),$$

where $\bar{\sigma}^2 = \langle f^2 \rangle$. Thus $P_0(t,x)$ and $x_0(t)$ satisfy the problem shown in Figure 4, which is exactly the Black-Scholes American put problem with constant volatility $\bar{\sigma}$.

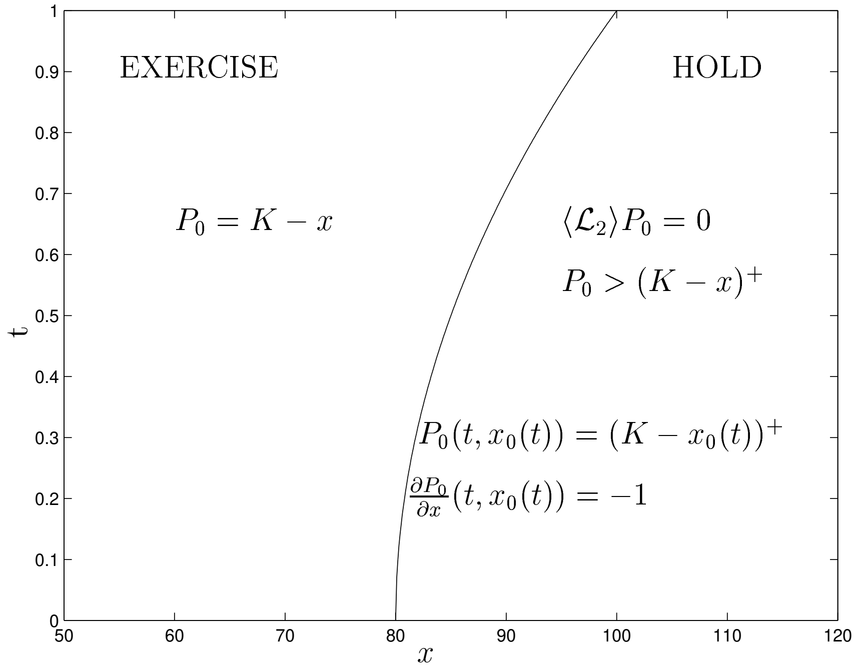

Figure 4: *The problem for $P_0(t,x)$, which is exactly the Black-Scholes American put pricing problem with average volatility $\bar{\sigma}$.*

There is no explicit solution for $P_0(t,x)$ or $x_0(t)$, and we discuss how to compute them numerically, along with the stochastic volatility correction in Section 5.

4.4 The Stochastic Volatility Correction

We now look for the function $\sqrt{\varepsilon}P_1$ which corrects the Black-Scholes American put pricing function P_0 for fast mean-reverting stochastic volatility.

The $\mathcal{O}(\sqrt{\varepsilon})$ terms in (29) give that

$$\begin{aligned} \mathcal{L}_0 P_3(t,x,y) + \mathcal{L}_1 P_2(t,x,y) + \mathcal{L}_2 P_1(t,x) &= 0 \qquad \text{in } x > x_0(t), \\ P_3(t,x,y) &= 0 \qquad \text{in } x < x_0(t). \end{aligned}$$

In the hold region $x > x_0(t)$, this is a Poisson equation for P_3 over $-\infty < y < \infty$. It has no solution unless

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0.$$

Substituting for $P_2(t,x,y)$ with

$$P_2 = -\mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \right) P_0,$$

from (33), this condition is

$$\left\langle \mathcal{L}_2 P_1 - \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \right) P_0 \right\rangle = 0,$$

where

$$\langle \mathcal{L}_2 P_1 \rangle = \langle \mathcal{L}_2 \rangle P_1 = \mathcal{L}_{BS}(\bar{\sigma}) P_1$$

since P_1 does not depend on y.

It is convenient to write the equation for

$$\tilde{P}_1(t,x) := \sqrt{\varepsilon} P_1(t,x),$$

so that ε will be absorbed in with the other parameters. Using the notation

$$\mathcal{A} = \sqrt{\varepsilon} \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \right) \right\rangle,$$

the equation determining $\tilde{P}_1$ in the hold region is

$$\mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_1 = \mathcal{A} P_0, \tag{34}$$

as P_0 does not depend on y.

The operator $\mathcal{A}$ is computed explicitly in [4] as

$$\mathcal{A} = V_3 x^3 \frac{\partial^3}{\partial x^3} + V_2 x^2 \frac{\partial^2}{\partial x^2},$$

where the parameters $V_2 = V_2(a,b,\bar{\sigma})$ and $V_3 = V_3(a,b,\bar{\sigma})$ are calibrated from the mean historical volatility and the slope and intercept of the European options implied volatility curve as a linear function of LMMR through the relations (25) and (26). Indeed it is exactly the operator that appears in the equation for the correction to European securities prices (24), barrier options [6] and other exotic options.

Thus in the region $x > x_0(t)$, $\tilde{P}_1(t,x)$ satisfies

$$\mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_1 = V_3 x^3 \frac{\partial^3 P_0}{\partial x^3} + V_2 x^2 \frac{\partial^2 P_0}{\partial x^2}, \tag{35}$$

where $P_0(t,x)$ is the Black-Scholes American put price. It can be shown that $P_0(t,x)$ is bounded with bounded derivatives inside the hold region $x > x_0(t), t < T$, and the discontinuity of its second x-derivative across $x_0(t)$ is not a difficulty for the $\tilde{P}_1$ problem (either analytically or numerically).

The complete problem for $\tilde{P}_1$ is then shown in Figure 5.

This is a *fixed* boundary problem for $\tilde{P}_1(t,x)$: the boundary $x_0(t)$ is the free boundary determined from the P_0 problem. However, this boundary is determined up to an error of $\sqrt{\varepsilon}$, so there is on $\mathcal{O}(\sqrt{\varepsilon})$ error in $\tilde{P}_1$ within an $\mathcal{O}(\sqrt{\varepsilon})$ neighbourhood of x_0. The asymptotic approximation is good outside this neighbourhood of $x_0(t)$. We have to solve for $\tilde{P}_1$ numerically after obtaining the numerical solution P_0.

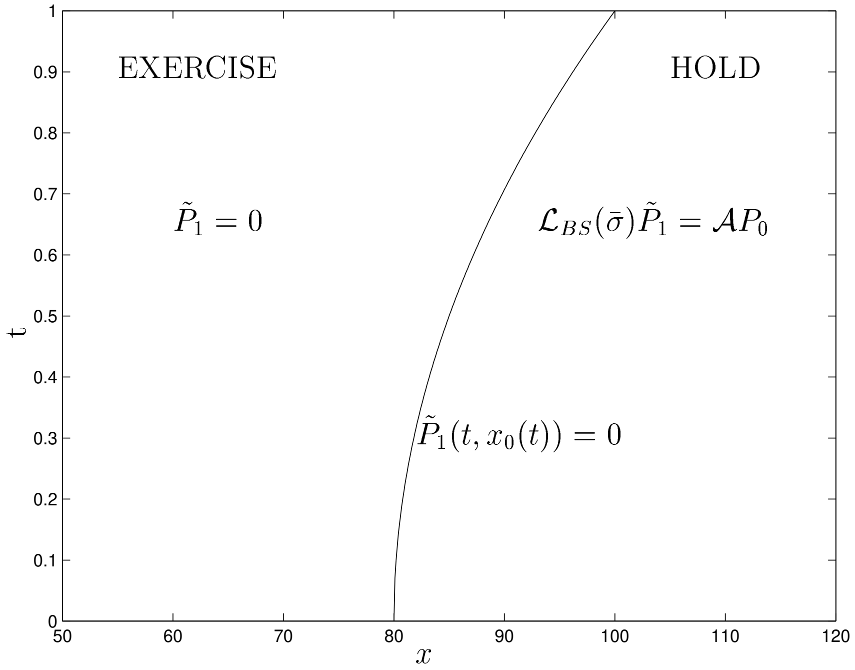

Figure 5: *The* fixed boundary *problem for* $\tilde{P}_1(t,x)$.

4.5 Uncorrelated Volatility

It is shown in [4] that V_2 and V_3 are related to the original model parameters (α, m, ν, ρ) and $f(\cdot)$ and $\gamma(\cdot)$ by

$$
\begin{aligned}
V_2 &= \frac{\nu}{\sqrt{2\alpha}}\left(2\rho\langle f\phi'\rangle - \langle\lambda\phi'\rangle\right), \\
V_3 &= \frac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle,
\end{aligned}
$$

where $\phi(y)$ is a solution of the Poisson equation

$$
\mathcal{L}_0\phi = f(y)^2 - \langle f^2\rangle.
$$

When volatility shocks are uncorrelated with stock price shocks, $\rho = 0$ and consequently $V_3 = 0$ and $a = 0$ by (26). In this case, in the hold region,

$$
\mathcal{L}_{BS}(\bar{\sigma})(P_0 + \tilde{P}_1) = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2},
$$

so that

$$
\mathcal{L}_{BS}(\sqrt{\bar{\sigma}^2 - 2V_2})(P_0 + \tilde{P}_1) = -V_2 x^2 \frac{\partial^2 \tilde{P}_1}{\partial x^2} = \mathcal{O}(\varepsilon),
$$

where we have used $V_2 = \mathcal{O}(\sqrt{\varepsilon})$ and assumed sufficient smoothness in $\tilde{P}_1$ away from $x_0(t)$ such that $\frac{\partial^2 \tilde{P}_1}{\partial x^2} = \mathcal{O}(\sqrt{\varepsilon})$.

It follows that the corrected American price $\tilde{P} = P_0 + \tilde{P}_1$ is, up to $\mathcal{O}(\varepsilon)$, the solution of the Black-Scholes American put pricing problem with *corrected effective volatility*

$$
\tilde{\sigma} = \sqrt{\bar{\sigma}^2 - 2V_2}.
$$

That is, in the absence of correlation, the first-order effect of fast mean-reverting stochastic volatility is simply a volatility level correction and since $V_2 < 0$ whenever at-the-money implied volatility $b > \bar{\sigma}$, this shift is typically upwards.

Solving this problem also gives the corrected exercise boundary which is exactly the Black-Scholes boundary associated with $\tilde{\sigma}$.

4.6 Probabilistic Representation

From Figure 5, $\tilde{P}_1$ can also be represented as an expectation of a functional of the geometric Brownian motion $\bar{X}_t$ defined by

$$d\bar{X}_t = r\bar{X}_t dt + \bar{\sigma}\bar{X}_t d\bar{W}_t \,,$$

where $\bar{W}$ is a standard Brownian motion under the probability $\bar{\mathbb{P}}$. The process $\bar{X}_t$ is *stopped* at the boundary $x_0(t)$, so that

$$\tilde{P}_1(t,x) = \bar{\mathbb{E}}\left\{ -\int_t^T e^{-r(s-t)} \mathcal{A} P_0(s,\bar{X}_s) \mathbf{1}_{\{\bar{X}_u > x_0(u) \text{ for all } t \leq u \leq s\}} ds | \bar{X}_t = x \right\}.$$

5 Numerical Computations

We are interested in computing numerically the stochastic volatility corrected American put price $P_0(t,x) + \tilde{P}_1(t,x)$ away from the exercise boundary $x_0(t)$. To do this, we will use implicit finite-differences, first to determine the Black-Scholes price $P_0(t,x)$ satisfying the system (4,5,6). Then, we find the constant volatility exercise boundary $x_0(t)$ as the largest value of x where $P(t,x)$ coincides with the payoff function $(K-x)^+$.

We shall then solve the fixed boundary problem (35) on the same grid in the region $x > x_0(t), t < T$.

5.1 Numerics for Black-Scholes Problem

We follow the procedure detailed in [10], using the backward Euler finite-difference stencil on a uniform grid (after a change to logarithmic stock price co-ordinates). The constraint that the function $P_0(t,x)$ lies above the payoff function is enforced using the projected successive-over-relaxation (PSOR) algorithm in which an iterative method is used to solve the implicit time-stepping equations, while preserving the constraint between iterations.

Explicit tree-like methods are popular in the industry, but we prefer the stability of implicit methods that allow us to take a relatively large time-step. In addition, we recover the whole pricing function that allows us to visualise the quality of the solution, and the effect of changing parameters.

We use the change of variables described in [10], and look for a function $u(\tau,\xi)$ where

$$\begin{aligned} P_0(t,x) &= K e^{-\frac{1}{2}(k-1)\xi - \frac{1}{4}(k+1)^2\tau} u_0(\tau,\xi), \\ \xi &= \log(x/K), \\ \tau &= \frac{1}{2}\bar{\sigma}^2(T-t), \end{aligned} \tag{36}$$

and $k := 2r/\bar{\sigma}^2$. Then, for computational implementation, we restrict to a finite domain $\mathcal{D} = \{(\tau, \xi) : L \le \xi \le R, 0 \le \tau \le \frac{1}{2}\bar{\sigma}^2 T\}$, where R and L are suitably chosen not to effect the required accuracy of the numerical solutions. In practice, we found $L = \log(0.1/K)$ and $R = \log 2$ to be adequate. The transformed problem for $u_0(\tau, \xi)$ corresponding to (4,5,6) is

$$\begin{aligned} -\frac{\partial u_0}{\partial \tau} + \frac{\partial^2 u_0}{\partial \xi^2} &\le 0, \\ u_0(\tau, \xi) &\ge g(\tau, \xi), \\ \left(-\frac{\partial u_0}{\partial \tau} + \frac{\partial^2 u_0}{\partial \xi^2}\right)(u_0(\tau, \xi) - g(\tau, \xi)) &= 0, \end{aligned} \tag{37}$$

where $g(\tau, \xi) := e^{\frac{1}{2}(k+1)^2 \tau}\left(e^{\frac{1}{2}(k-1)\xi} - e^{\frac{1}{2}(k+1)\xi}\right)^+$. The initial and boundary conditions are

$$\begin{aligned} u_0(0, \xi) &= g(0, \xi), \\ u_0(\tau, L) &= g(\tau, L), \\ u_0(\tau, R) &= 0. \end{aligned}$$

We introduce a uniform grid on $\mathcal{D}$ with nodes $\{(\tau_n, \xi_j) : j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ and look for discrete approximations $U_j^n \approx u_0(\tau_n, \xi_j)$ where $\xi_j = j\Delta\xi, \tau_n = n\Delta\tau$, and $\Delta\xi = (R - L)/J$ and $\Delta\tau = \frac{1}{2}\bar{\sigma}^2 T/N$ are the grid spacings.

We approximate the ξ-derivatives by central differences

$$\begin{aligned} \frac{\partial^2 u_0}{\partial \xi^2}(\tau_n, \xi_j) &\approx \delta_\xi^2 U_j^n := \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta\xi)^2}, \\ \frac{\partial u_0}{\partial \xi}(\tau_n, \xi_j) &\approx \Delta_0 U_j^n := \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta\xi}, \end{aligned}$$

and the time-derivative by the backward Euler scheme. In other words, we solve the discretized system corresponding to (37):

$$\begin{aligned} -\frac{U_j^{n+1} - U_j^n}{\Delta\tau} + \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta\xi)^2} &\le 0, \\ U_j^{n+1} &\ge g(\tau_{n+1}, \xi_j), \\ \left(-\frac{U_j^{n+1} - U_j^n}{\Delta\tau} + \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta\xi)^2}\right)\left(U_j^{n+1} - g(\tau_{n+1}, \xi_j)\right) &= 0, \end{aligned}$$

for $j = 1, \cdots, J-1$ and $n = 0, \cdots, N-1$.

We obtain the numerical approximation $\{U_j^n\}$ on the uniform (τ, ξ)-grid using the PSOR algorithm. An alternative, the projected LU method, is described in [9]. From this we can reverse the transformations (36) to extract a numerical approximation to $P_0(t, x)$, the Black-Scholes American put pricing function on a nonuniform (t, x)-grid. This is illustrated in Figure 6.

Finally, we find the exercise boundary $x_0(t)$ as approximated by the grid point at which P_0 is closest to, but not above, the ramp function. The numerically-estimated free-boundary (in the transformed (τ, ξ) co-ordinates) is shown in Figure 7.

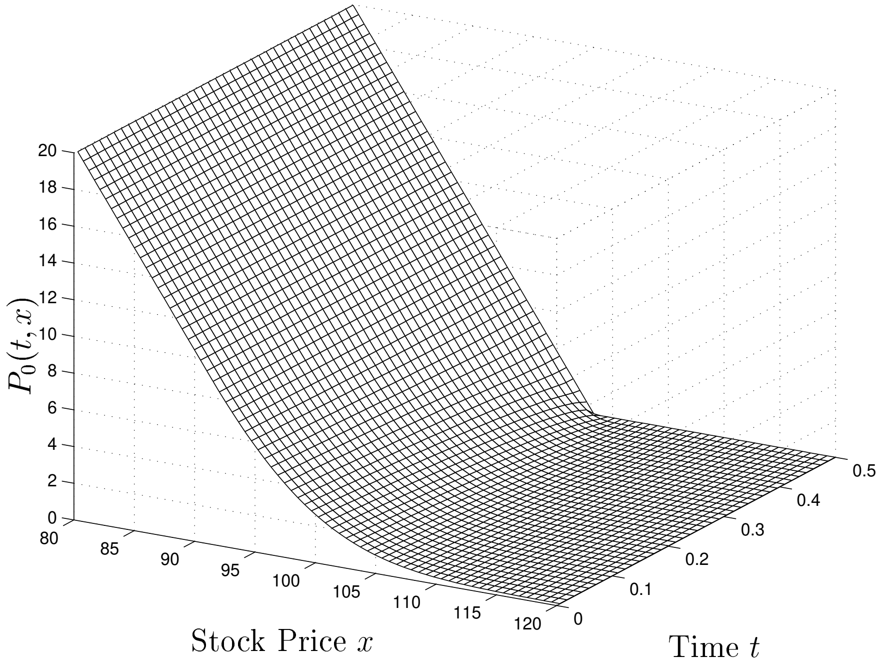

Figure 6: *Numerical solution for $P_0(t,x)$, the Black-Scholes American put pricing function with $\bar{\sigma} = 0.1$, and $K = 100$, $T = 0.5$, $r = 0.02$, using the backward Euler-PSOR method. The number of grid points are $J = 4000$ in the ξ-direction and $N = 1385$ timesteps.*

5.2 Computation of the Correction

To compute the correction for stochastic volatility $\tilde{P}_1(t,x)$ to the Black-Scholes price we have found, we need to solve the fixed-boundary problem (35) in the region $\{(t,x) : x > x_0(t), 0 \leq t \leq T\}$. Outside this region, the correction is zero. We are most interested in $\tilde{P}_1$ away from the exercise boundary, particularly around the money $x \approx K$. Close to the curve $x_0(t)$, or the final time T, we do not expect the asymptotic approximation to be valid because the contract will likely expire soon, and volatility will not look fast mean-reverting over such a short timescale, as explained in Section 4.2.

We continue to perform the computations in (τ, ξ) co-ordinates, working in the domain $\mathcal{D}_0 := \{(\tau,\xi) : \xi_{fb}(\tau) \leq \xi \leq R, 0 \leq \tau \leq \frac{1}{2}\bar{\sigma}^2 T\}$. Thus we look for a numerical approximation to the function $u_1(\tau,\xi)$, where

$$\tilde{P}_1(t,x) = e^{-\frac{1}{2}(k-1)\xi - \frac{1}{4}(k+1)^2\tau} u_1(\tau,\xi), \tag{38}$$

so that (35) transforms to

$$\begin{aligned}
-\frac{\partial u_1}{\partial \tau} + \frac{\partial^2 u_1}{\partial \xi^2} &= \frac{2K}{\bar{\sigma}^2}\left(V_3 \frac{\partial^3 u_0}{\partial \xi^3} + c_1 \frac{\partial^2 u_0}{\partial \xi^2} + c_2 \frac{\partial u_0}{\partial \xi} + c_3 u_0\right), \\
c_1 &= V_2 - \frac{3}{2}(k+1)V_3, \\
c_2 &= \frac{1}{4}(3k^2 + 6k - 1)V_3 - kV_2, \\
c_3 &= \frac{1}{8}(3 + k - 3k^2 - k^3)V_3 + \frac{1}{4}(k^2 - 1)V_2,
\end{aligned}$$

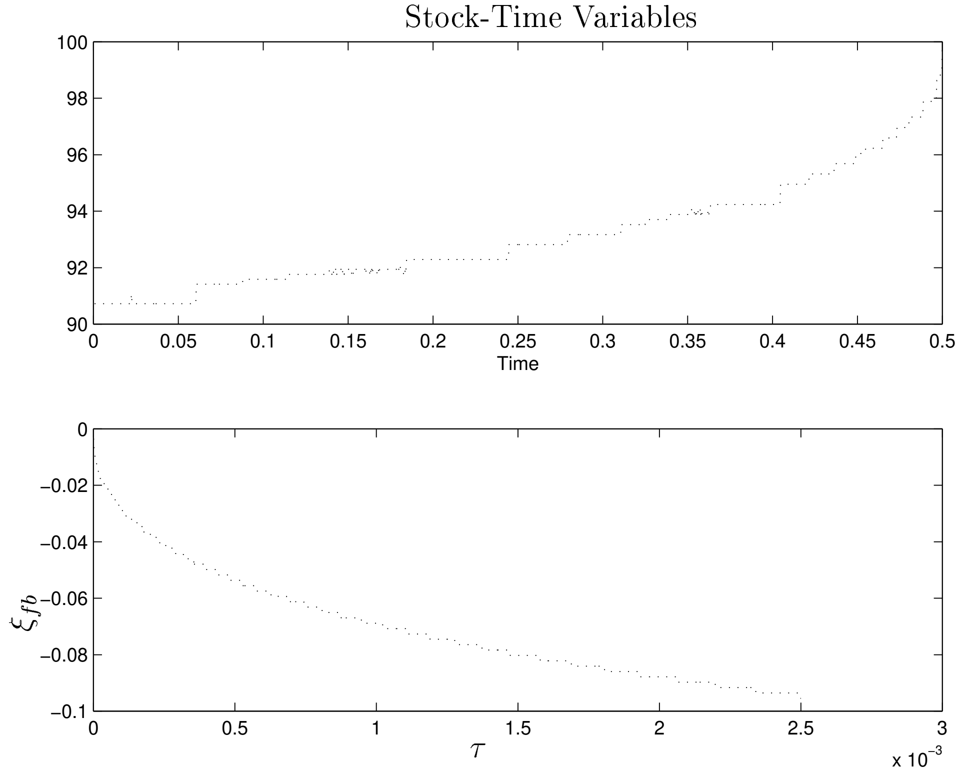

Figure 7: *Numerically-estimated free boundary in original and transformed co-ordinates. In the bottom graph, each dot is the largest grid point at each τ_n such that $U_j^n \leq g(\tau_n, \xi_j)$. The parameters are as in Figure 6. Notice how with even 4000 spatial grid points, it is hard to be very accurate about the boundary's location.*

with zero boundary and initial conditions

$$u_1(\tau, \xi_{fb}(\tau)) = 0 = u_1(0, \xi).$$

Our approximation $\tilde{U}_j^n \approx u_1(\tau_n, \xi_j)$ is computed using backward Euler on the same grid as we used to find $\{U_j^n\}$. The third derivative is approximated by

$$\frac{\partial^3 u_0}{\partial \xi^3}(\tau_n, \xi_j) \approx \Delta_+ \delta_\xi^2 U_j^n = \frac{U_{j+2}^n - 3U_{j+1}^n + 3U_j^n - U_{j-1}^n}{(\Delta\xi)^3}.$$

At each time-step, the system of equations to be solved is generated by

$$-\frac{\tilde{U}_j^{n+1} - \tilde{U}_j^n}{\Delta\tau} + \delta_\xi^2 \tilde{U}_j^{n+1} = \frac{2K}{\bar{\sigma}^2}\left(V\Delta_+\delta_\xi^2 U_j^{n+1} + c_1\delta_\xi^2 U_j^{n+1} + c_2\Delta_+ U_j^{n+1} + c_3 U_j^{n+1}\right),$$

for $j = j_{\min}^{n+1} + 1, \cdots, J-1$, where $j_{\min}^n$ is the location of the free-boundary: $\xi_{fb}(\tau_n) = \xi_{j_{\min}^n}$. Finally the transformations (36) and (38) are inverted to obtain the approximation to $\tilde{P}_1(t, x)$.

In Figure 8, we show $\tilde{P}_1(t, x)$ using parameter values estimated from the 1994 S&P 500 implied volatility surface in [3]. The effect of the correction is most around the strike price, and Figure 9 shows the Black-Scholes $P_0(0, x)$ and corrected $P_0(0, x) + \tilde{P}_1(0, x)$

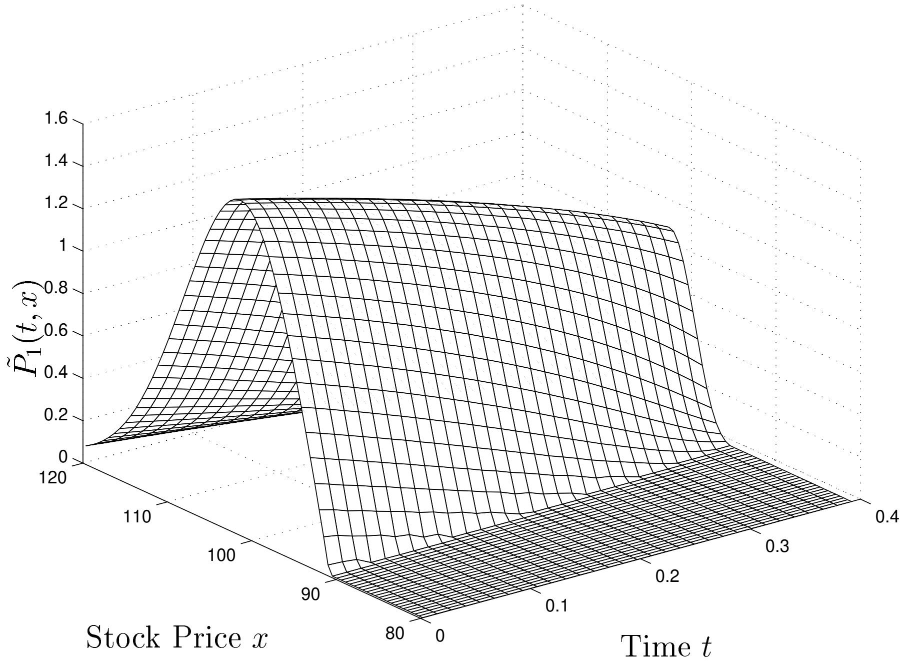

Figure 8: *The correction $\tilde{P}_1(t,x)$ to the Black-Scholes American put price to account for fast mean-reverting stochastic volatility, using the parameters estimated from S&P 500 implied volatilies: $a = -0.154$, $b = 0.149$ and from historical index data, $\bar{\sigma} = 0.1$. It is computed using backward Euler using the numerical solution shown in Figures 6 and 7.*

5.3 Effect of the Skew

We now examine the effect of the slope of the skew on the premium charged for the American put option. The full relation of a in (23) to the original model parameters $(\alpha, \beta, m, \rho, \gamma)$ as well as the function f in (12) is given in [4] and shows that, all else remaining the same,

$$a \propto \rho.$$

This is consistent with the industry jargon of using skew to mean the slope of the implied volatility surface and the leverage effect ($\rho < 0$) interchangeably.

The formula for b also contains ρ, but we shall investigate how changing the slope a effects stochastic volatility American put prices with the at-the-money implied volatility b fixed. A large negative skew indicates a premium is being charged in the market for out-of-the-money European puts ($K < x$) suggesting an expectancy for the stock price to drop, increasing the value of a put option to its holder.

Figure 10 shows that making the skew a more negative, while fixing the at-the-money implied volatility b *does* increase the price of the American put option in the neighbourhood of the strike price $K = 100$. In other words, as expected, a buyer must pay more for an American put if the skew is sharper, for x within 10% either side of the strike price.

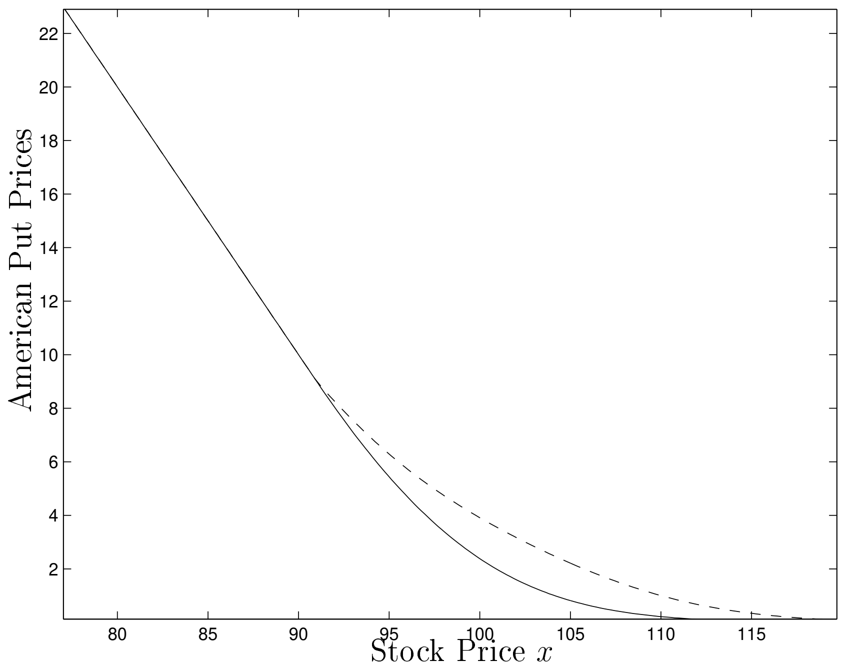

Figure 9: *Effect of the stochastic volatility correction on American put prices at time $t = 0$. The solid line shows the Black-Scholes American put price $P_0(0,x)$ with the constant historical volatility $\bar{\sigma} = 0.1$, and the dotted line shows the corrected price $P_0(0,x) + \tilde{P}_1(0,x)$ using the S&P 500 parameters described in the caption of Figure 8.*

6 Conclusion

We have presented a method for directly using the observed implied volatility skew to correct American option prices to account for random volatility. The procedure is robust in that it does not rely on a specific model of stochastic volatility, and it involves solving a numerical problem that is a minor extension of the method used to compute Black-Scholes American prices. We have outlined an algorithm for computation of the correction, and investigated how a steeper negatively sloping skew increases the premium charged for American puts in a stochastic volatility environment.

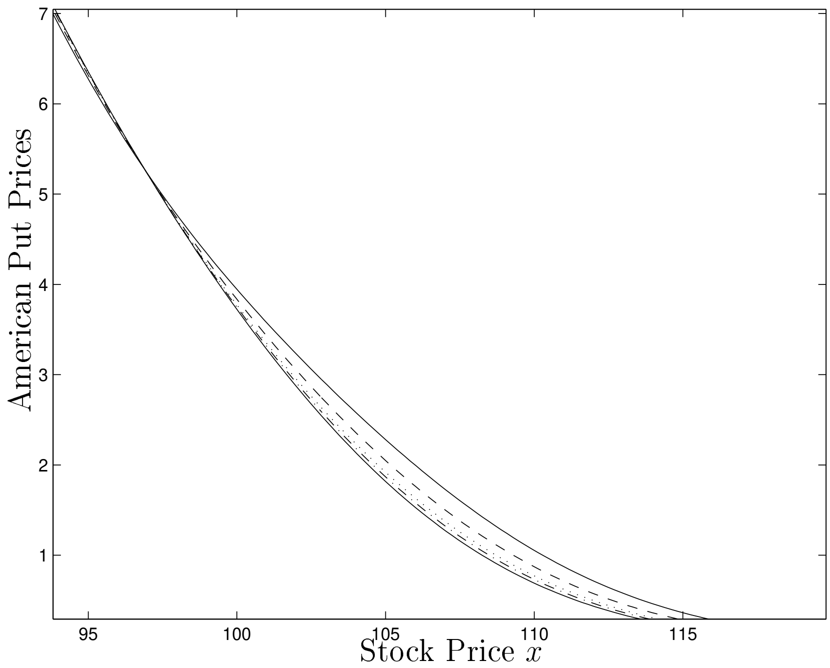

Figure 10: *Effect of changing the slope of the skew a on American put prices at time $t = 0$. The strike price of the contract is $K = 100$, and expiration date is $T = 0.5$. Making a more negative* increases *the price curves around-the-money. The values of a reading upwards from the bottom curve are* $0, -0.02, -0.04, -0.09$ *and* -0.18*. At-the-money implied volatility is fixed at $b = 0.149$ and $\bar{\sigma} = 0.1$.*

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