[1] Using Cauchy - Riemann:

$$u_x = 2x-2y,\ v_y = -2y+2x \Rightarrow u_x = v_y$$
$$u_y = -2y-2x,\ v_x = 2x+2y \Rightarrow u_y = -v_x$$
$\Rightarrow f$ entire.

$f'(z) = u_x + i v_x$. For $z=i$, $x=0, y=1, u_x=-2, v_x = 2 \Rightarrow \boxed{f'(i) = -2+2i}$

[2] $-1-i\sqrt{3} = e^{-\frac{2\pi i}{3}}$ (for principal parts we need the angle in $(-\pi,\pi)$). $\Rightarrow$

$\Rightarrow \frac{e}{2}(-1-i\sqrt{3}) = e^{1-\frac{2\pi i}{3}} \Rightarrow \left[\frac{e}{2}(-1-i\sqrt{3})\right]^{3\pi i} = e^{3\pi i\left(1-\frac{2\pi i}{3}\right)} = e^{3\pi i + 2\pi^2} = \boxed{-e^{2\pi^2}}$

[3] $u_x = -(e^y+e^{-y})\sin x + (e^y-e^{-y})\cos x$ | $u_y = (e^y-e^{-y})\cos x + (e^y+e^{-y})\sin x$

$u_{xx} = -(e^y+e^{-y})\cos x - (e^y-e^{-y})\sin x$ | $u_{yy} = (e^y+e^{-y})\cos x + (e^y-e^{-y})\sin x$

$\Rightarrow \boxed{u_{xx}+u_{yy}=0} \Rightarrow u$ harmonic.

We find the harmonic conjugate v. We have $v_x = -u_y = -(e^y-e^{-y})\cos x - (e^y+e^{-y})\sin$

$\Rightarrow v = -(e^y-e^{-y})\sin x + (e^y+e^{-y})\cos x + \phi(y)$. Since $v_y = u_x \Leftrightarrow$

$\Leftrightarrow -(e^y+e^{-y})\sin x + (e^y-e^{-y})\cos x + \phi'(y) = -(e^y+e^{-y})\sin x + (e^y-e^{-y})\cos x$

$\Rightarrow \phi'(y) = 0 \Rightarrow \phi = c,\ c\in\mathbb{R}$. Since $f(0) = 2i \Rightarrow v(0) = 0 \Rightarrow c+2=0 \Rightarrow \boxed{c=-2}$

But ~~we~~ $\boxed{f(z) = u+iv} = \underbrace{(e^y+e^{-y})\cos x + (e^y-e^{-y})\sin x}_{u} + i\underbrace{(e^y+e^{-y})\cos x -}_{v}$

$-i(e^y-e^{-y})\sin x - 2i = (1+i)(e^y+e^{-y})\cos x + (1-i)(e^y-e^{-y})\sin x - 2i =$

$= (1+i)\left[(e^y+e^{-y})\cos x - i(e^y-e^{-y})\sin x\right] - 2i =$

$= (1+i)\left[e^y(\cos x - i\sin x) + e^{-y}(\cos x + i\sin x)\right] - 2i =$

$= (1+i)\left(e^{y-ix} + e^{-y+ix}\right) - 2i = (1+i)(e^{-iz}+e^{iz}) - 2i = \boxed{2(1+i)\cos z - 2i}$

[4] Since u, v are harmonic conjugates $\Rightarrow f = u+iv$ is holomorphic. We need to check

$F = u(x^2-y^2, 2xy) + iv(x^2-y^2, 2xy)$ is holomorphic. But $z^2 = (x+iy)^2 = x^2-y^2+2xyi$ so

$F = u(z^2) + iv(z^2) = f(z^2)$ which is clearly holomorphic.

[5] Clearly, if $z=0$, $f(0)=0$ so $|f(z)|$ is minimized at $z=0$. (and $\forall z\in R, f(z)\neq 0$ so

this min is unique).

For the max, we know it must occur on the boundary. We have to check the