Problem 23, page 418.

Find the area of the surface cut from the paraboloid \( z = 2x^2 + 2y^2 \) by the planes \( z = 2 \) and \( z = 8 \).

Solution: The surface is the graph of the function \( f(x, y) = 2x^2 + 2y^2 \) over the region in the \( xy \) plane bounded by \( 2 = z = 2x^2 + 2y^2 \) and \( 8 = z = 2x^2 + 2y^2 \). In polar coordinates, this is the region \( 1 \leq r \leq 2 \). The surface area is then

\[
\int_0^{2\pi} \int_1^2 \sqrt{1 + f_x^2 + f_y^2} \, r \, dr \, d\theta = \int_0^{2\pi} \int_1^2 \sqrt{1 + (4x)^2 + (4y)^2} \, r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_1^2 \sqrt{1 + 16r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{48} (1 + 16r^2)^{3/2} \right]_{r=1}^{r=2} d\theta = \int_0^{2\pi} \frac{1}{48} (65^{3/2} - 17^{3/2}) d\theta
\]

\[
= \frac{\pi}{24} (65^{3/2} - 17^{3/2}).
\]

Problem 24(b), page 418.

Calculate the surface area of the portion of the plane \( x + y + z = a \) cut out by the cylinder \( x^2 + y^2 = a^2 \) by considering it as a graph.

Solution: The surface is the graph of the function \( f(x, y) = a - x - y \) over the region in the \( xy \) plane determined by \( x^2 + y^2 \leq a^2 \). In polar coordinates, this is the region \( 0 \leq r \leq a \). The surface area is then

\[
\int_0^{2\pi} \int_0^a \sqrt{1 + f_x^2 + f_y^2} \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \sqrt{1 + (-1)^2 + (-1)^2} \, r \, dr \, d\theta = \int_0^{2\pi} \frac{\sqrt{3}a^2}{2} d\theta
\]

\[
= \frac{\pi \sqrt{3}a^2}{2}.
\]

Problem 3, page 438.

Find the flux of \( \mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) across the surface \( S \) consisting of the triangular region of the plane \( 2x - 2y + z = 2 \) that is cut out by the coordinate planes. Use an upward-pointing normal to orient \( S \).

Solution: The surface is the graph of the function \( f(x, y) = 2 - 2x + 2y \) over the region in the fourth quadrant of the \( xy \) plane where \( f(x, y) \) is nonnegative. The latter region is bounded by the lines \( x = 0 \), \( y = 0 \), and \( y = x - 1 \). The flux is then

\[
\int_0^1 \int_{x-1}^0 \mathbf{F}(f(x, y)) \cdot \mathbf{N}(x, y) \, dy \, dx = \int_0^1 \int_{x-1}^0 \mathbf{F}(f(x, y)) \cdot (-f_x(x, y), -f_y(x, y), 1) \, dy \, dx
\]

\[
= \int_0^1 \int_{x-1}^0 (x, y, z) \cdot (2, -2, 1) \, dy \, dx = \int_0^1 \int_{x-1}^0 2y - 2y + z \, dy \, dx = \int_0^1 \int_{x-1}^0 2y \, dy \, dx
\]
(because the integration is over the plane $2x - 2y + z = 2$)

$$= \int_0^1 2(1 - x) \, dx = \left[ 2x - x^2 \right]_0^1 = 1$$

**Problem 7, page 438.**

Let $S$ be a sphere of radius $a$. (a) Find $\int_S x^2 + y^2 + z^2 \, dS$.

Solution: We use spherical coordinates:

$$\int_0^{2\pi} \int_0^\pi (x^2 + y^2 + z^2) a^2 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi (a^2) a^2 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left[ -a^4 \cos \phi \right]_0^\pi \, d\theta$$

$$= \int_0^{2\pi} 2a^4 \, d\theta = 4\pi a^4.$$

(b) Use symmetry and part (a) to easily find $\int_S y^2 \, dS$.

Solution: We discovered that $4\pi a^4 = \int_S x^2 + y^2 + z^2 \, dS = \int_S x^2 \, dS + \int_S y^2 \, dS + \int_S z^2 \, dS$. Since the sphere is symmetric, all three integrals are the same. So $4\pi a^4 = 3 \int_S y^2 \, dS$, and $\int_S y^2 \, dS = \frac{4}{3}\pi a^4$.

**Problem 9, page 438.**

Let $Z$ denote the surface of the cylinder $x^2 + y^2 = 4$, $-2 \leq z \leq 2$, and consider the surface integral

$$\int \int_Z z - x^2 - y^2 \, dS.$$

(a) Calculate the value of the integral.

Solution: We use cylindrical coordinates. The radius is 2:

$$\int_{-2}^2 \int_0^{2\pi} (z - x^2 - y^2) r \, d\theta \, dz = \int_{-2}^2 \int_0^{2\pi} (z - 4) 2 \, d\theta \, dz = \int_{-2}^2 4\pi(z - 4) \, dz$$

$$= \left[ 2\pi z^2 - 16\pi z \right]_{-2}^2 = -64\pi.$$

(b) Now use geometry and symmetry to evaluate the integral with a minimum of labor.

Solution: The surface is symmetric across the $xy$ plane. We can split the surface $Z$ into the portion $Z_+$ above the $xy$ plane and the portion $Z_-$ below the $xy$ plane:

$$\int \int_Z z - x^2 - y^2 \, dS = \int \int_Z z - 4 \, dS = \int \int_{Z_+} z - 4 \, dS + \int \int_{Z_-} z - 4 \, dS.$$

Because of the symmetry, when we apply the change of variable $z \mapsto -z$ to the second integral, $Z_-$ becomes $Z_+$:

$$= \int \int_{Z_+} z - 4 \, dS + \int \int_{Z_+} -z - 4 \, dS = \int \int_{Z_+} -8 \, dS$$
which is $-8$ times the surface area of $Z$. (This is the geometry part):

$$\frac{1}{8} \cdot 8 \times (24\pi) = -64\pi$$

which agrees with part (a). Good!

**Problem 14, page 438.**

Let $Z$ denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by the equation $x^2 + y^2 = 9$. Orient $Z$ with outward normals. Determine

$$\int \int_Z (x \mathbf{i} + y \mathbf{j}) \cdot d\mathbf{S}.$$

Solution: The quick way to do this is to notice that $x \mathbf{i} + y \mathbf{j}$ is perpendicular to the surface and parallel to the normal at every point, and has a constant magnitude of $\sqrt{x^2 + y^2} = 3$. Thus the integral becomes $\int \int_Z 3 \, dS$, which is just $3$ times the surface area, or $3(24\pi) = 72\pi$.

**Problem 20, page 438.**

Find the flux of $\mathbf{F} = y \mathbf{i} - x \mathbf{j}$ across the upper hemisphere of $x^2 + y^2 + z^2 = a^2$.

Solution: The normal vector is $\mathbf{n} = \frac{1}{a} \langle x, y, z \rangle$ so that

$$\mathbf{F} \cdot \mathbf{n} = \frac{z}{a} = \cos \phi.$$

The flux becomes

$$\int \int_D a \cos \phi(a^2 \sin \phi) \, d\phi \, d\theta = 2\pi a^3 \int_0^\pi \sin \phi \cos \phi \, d\phi = 0.$$

**Problem 22, page 438.**

Find the flux of $\mathbf{F} = x^2 \mathbf{i} + xy \mathbf{j} + xz \mathbf{k}$ across the upper hemisphere of $x^2 + y^2 + z^2 = a^2$.

Solution: As before $\mathbf{n} = \frac{1}{a} \langle x, y, z \rangle$, so

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{a} x^3 + xy^2 + xz^2 = \frac{x}{a}(x^2 + y^2 + z^2) = ax.$$

The flux becomes

$$\int \int_S ax \, dS = 0$$

by symmetry.

**Problem 6, page 453.**

Verify Gauss’s theorem for the field $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ over the region

$$D = \{(x, y, z) | 0 \leq z \leq 9 - x^2 - y^2\}.$$
Solution: We use cylindrical coordinates for the volume:

$$\nabla \cdot \mathbf{F} = 3 \Rightarrow \iiint_D 3 \, dV = 3 \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} dz \, r \, dr \, d\theta$$

$$= 6\pi \int_0^3 (9r - r^3) \, dr = \frac{243\pi}{2}$$

Now, $\partial D$ consists of the disk of radius 3 and the paraboloid. So,

$$\iiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\text{Disk}} \mathbf{F} \cdot d\mathbf{S} + \iiint_{\text{Paraboloid}} \mathbf{F} \cdot d\mathbf{S}$$

$$= \int_0^{2\pi} \int_0^3 \int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x,y,0) \cdot (0,0,-1) \, dy \, dx + \int_0^{2\pi} \int_0^3 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x,y,9-x^2-y^2) \cdot (2x,2y,1) \, dx \, dy$$

$$= 0 + \int_0^3 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (9 + x^2 + y^2) \, dy \, dx = \int_0^{2\pi} \int_0^3 (9r + r^3) \, dr \, d\theta$$

$$= 2\pi \left[ \frac{9}{2} r^2 + \frac{r^4}{4} \right]_0^3 = \frac{243\pi}{2}$$

**Problem 14, page 453.**

Let $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + (2 - 2z) \mathbf{k}$. Compute

$$\iiint_S \mathbf{F} \cdot d\mathbf{S}$$

over the surface given by $z = e^{1-x^2-y^2}$ for $z \geq 1$.

Solution: Since $\nabla \cdot \mathbf{F} = 0$, by Gauss’s Theorem we have

$$\iiint_S \mathbf{F} \cdot d\mathbf{S} = -\iiint_D \mathbf{F} \cdot d\mathbf{S},$$

where $D = \{(x,y,1)| x^2 + y^2 \leq 1 \}$. The integral becomes

$$-\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x,y,2-2) \cdot (0,0,-1) \, dy \, dx = 0$$

So that

$$\iiint_S \mathbf{F} \cdot d\mathbf{S} = 0$$

**Problem 4, page Handout.**

Let $\mathbf{F} = 3y \mathbf{i} - 3x \mathbf{j} + 2z \mathbf{k}$.

Find the divergence of $\mathbf{F}$

Solution:

$$\nabla \cdot \mathbf{F} = 0 + 0 + 2 = 2$$

Let $S$ be the sphere of radius $a$ centered at the origin, with an outward facing normal. Find the flux of $\mathbf{F}$ through $S$
Solution: We have $\mathbf{n} = \frac{1}{a} \langle x, y, z \rangle$ so that

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{a} \langle 3y, -3x, 2z \rangle \cdot \langle x, y, z \rangle = \frac{2z^2}{a} = 2a \cos^2 \phi.$$ 

The flux becomes

$$\int_0^{2\pi} \int_0^\pi (2a \cos^2 \phi)(a^2 \sin \phi d\phi d\theta) = 4\pi a^3 \int_0^\pi \sin \phi \cos^2 \phi d\phi = 4\pi a^3 \left[ -\frac{\cos^3 \phi}{3} \right]_0^\pi = \frac{8\pi a^3}{3}.$$

Evaluate the flux using Gauss’s Theorem

Solution:

$$\int \int \int_{\text{Ball}} \nabla \cdot \mathbf{F} \, dV = 2 \int \int \int_{\text{Ball}} \, dV = 2V = 2 \frac{4\pi a^3}{3} = \frac{8\pi a^3}{3}.$$