Math 145, Problem Set 6. Due Friday, May 23.

You may assume that the ground field is $k = \mathbb{C}$.

1. (*Hyperelliptic curves.*) Let $a_1, \ldots, a_5$ be pairwise distinct constants. Find the singularities of the projective *hyperelliptic curve of genus* 2*:*

$$y^2z^3 = (x - a_1z) \ldots (x - a_5z).$$

2. (*Singularities of arbitrary algebraic sets.*) Let $X \subset \mathbb{P}^n$ be a projective variety of dimension d cut out by the homogeneous equations

$$f_1 = \ldots = f_r = 0.$$

We say that X is singular at p if the rank of the $r \times (n+1)$ Jacobi matrix of partial derivatives

$$\left(\frac{\partial f_i}{\partial x_j}(p) \right), \; 1 \leq i \leq r, \; 0 \leq j \leq n,$$

strictly less than $n - d$.

Remark: When $r = 1$, the dimension of X is $d = n - 1$ (this is intuitively clear: one equation cuts the dimension down by 1, but this fact is hard to prove using our definition of dimension). Thus in this case, we recover the definition we gave in class for singular points.

Remark: The same definition applies to algebraic varieties in $\mathbb{A}^n$.

Check that the twisted cubic

$$X = \{(x_0 : x_1 : x_2 : x_3) : x_1^2 - x_0x_2 = x_2^2 - x_1x_3 = x_0x_3 - x_1x_2 = 0\}$$

is nonsingular. You may want to convince yourselves that the dimension of X is 1 (even though it is cut out by 3 equations), possibly using that X is the image of the Veronese isomorphism $\mathbb{P}^1 \to X \subset \mathbb{P}^3$.

3. (*Nonsingular projective curves are irreducible.*)

(i) Prove that if $C = \mathcal{Z}(f)$ and $D = \mathcal{Z}(g)$ are curves in $\mathbb{P}^2$, then

$$\text{Sing}\,(C \cup D) = \text{Sing}\,(C) \cup \text{Sing}\,(D) \cup (C \cap D).$$

To prove this, note first that $C \cup D = \mathcal{Z}(fg)$.

(ii) Deduce that a nonsingular curve $\mathcal{Z}(F)$ in $\mathbb{P}^2$ is irreducible. Indeed, argue that if F has at least two distinct factors f and g, then f and g must be homogeneous, and thus f and g must have at least a common zero which is a singular point for $\mathcal{Z}(F)$. Is this statement true for affine plane curves?

4. Show that a *general* hypersurface of degree d in $\mathbb{P}^n$ is non-singular:

(i) For any hypersurface $\mathcal{Z}(f) \subset \mathbb{P}^n$ of degree d, view the coefficients of f as a point p_f in a large dimensional projective space $\mathbb{P}^N$ (This projective space is called *the moduli space* of degree d hypersurfaces). Let

$$X = \{(f, p) \in \mathbb{P}^N \times \mathbb{P}^n : p \text{ is a singular point of } f\}.$$

Show that X is a projective algebraic set in $\mathbb{P}^N \times \mathbb{P}^n$.

(ii) Conclude that the image $\pi(X)$ of X under the projection onto $\mathbb{P}^N$ is a projective algebraic set. What is $\pi(X)$? Conclude that the subset of $\mathbb{P}^N$ corresponding to smooth hypersurfaces is open and *nonempty.*

Remark: This will prove in particular that f is singular provided that the coefficients of f satisfy certain polynomial relations. (In fact, one such polynomial relation will suffice but this is harder to see.) Therefore, if you pick your coefficients of f randomly, these polynomial relations will most likely not be satisfied and your hypersurface is non-singular. This is the explanation of the word *general.*

5. *Blowups.* Solve problem 6.6 part (a) in the textbook.

6. *Analytic singularities.* Consider the singular plane curves Z and W given by the equations

$$y^2 - x^2(x+1) = 0 \text{ and } xy = 0$$

respectively.

(i) Explain briefly why Z and W are not isomorphic. Explain that $(0,0)$ is an ordinary double point for both of these curves. What are the tangent directions at $(0,0)$ for Z and W? Sketch (the real points of) Z and W. Do Z and W look *alike* near the origin?

(ii) Show that there are *formal power series*

$$\tilde{x} = f_1 + f_2 + f_3 + \ldots \text{ and}$$

$$\tilde{y} = g_1 + g_2 + g_3 \ldots$$

in the variables x and y such that the equation of Z becomes

$$\tilde{x}\tilde{y} = 0.$$

Hint: Construct the degree i homogeneous parts f_i and g_i inductively. Show you can pick

$$f_1 = y - x, g_1 = x + y.$$

Next, you would need

$$f_2(x+y) + g_2(y-x) = -x^3.$$

Why can you construct f_2 and g_2? Continue in this fashion.

Remark: If we work over an arbitrary field k it doesnt make sense to ask if the power series $\tilde{x}$ and $\tilde{y}$ converge, hence the terminology *formal power series.* Convergence may be arranged if you work over the complex numbers, but you don't have to prove it.

Remark: It turns out the assignment

$$(x, y) \to (\tilde{x}, \tilde{y})$$

is invertible $e.g.$ you can solve for x, y in terms of formal power series in $\tilde{x}, \tilde{y}$. In fact, this statement is generally true about any power series

$$\tilde{x} = ax + by + \ldots, \tilde{y} = cx + dy + \ldots$$

provided that $ad - bc \neq 0$. Therefore, the assignment

$$(x, y) \to (\tilde{x}, \tilde{y})$$

is a *formal* change of coordinates, establishing a *formal isomorphism* between Z and W. We say that Z and W are *analytically equivalent.*

Remark: Over the complex numbers, convergence may be arranged near the origin, if x, y are small, and thus the word *formal* may be replaced by *local analytic isomorphism* near the origin.

(iii) Explain briefly why any ordinary double point singularity in $\mathbb{A}^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

Remark: It can be shown that any double point is analytically equivalent to the singularity $\tilde{y}^2 = \tilde{x}^r$, for some r. The case $r = 2$ corresponds to the case which concerned us above.

7. *Extra credit.* (*Dual conics.*) Let $C \subset \mathbb{P}^2$ be a non-singular curve, given as the zero locus of a homogeneous polynomial $f \in k[x, y, z]$. Consider the morphism

$$\Phi : C \to \mathbb{P}^2, p \mapsto \left[\frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p)\right].$$

The image $\Phi(C) \subset \mathbb{P}^2$ is called the dual curve to C.

(i) Why is Φ a well-defined morphism? Find a geometric description of Φ. For instance, what does it mean geometrically if $\Phi(P) = \Phi(Q)$ for two distinct points $P, Q \in C$?

(ii) If C is an irreducible conic, prove that its dual $\Phi(C)$ is also an irreducible conic. One way to prove this is to linearly change coordinates and assume the conic C is $x^2 + y^2 + z^2 = 0$. How does the morphism Φ change when we change coordinates?

(iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.