Math 145, Problem Set 4. Due Friday, May 2.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. Let f and g be distinct irreducible polynomials in $k[X, Y]$ of degrees d and e. Show that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ intersect in at most de points:

(i) Let S be the set of intersection points. We proved in the last problem set that S is finite. Let n be the number of points in S. We would like to show that

$$n \leq de.$$

To begin, write

$$S = \{(a_1, b_1), \ldots, (a_n, b_n)\}.$$

Explain why there exists $\lambda \in k$ such that

$$\lambda a_i + b_i \neq \lambda a_j + b_j, \text{ for } i \neq j.$$

(ii) Let

$$F(X, Y) = f(X, Y - \lambda X), G(X, Y) = g(X, Y - \lambda X).$$

Show that F and G have degrees at most d and e. Show that F and G are distinct irreducible polynomials. Show that $\mathcal{Z}(F) \cap \mathcal{Z}(G)$ consists in n points

$$\{(a_1, c_1), \ldots, (a_n, c_n)\}$$

where

$$c_i \neq c_j \text{ for all } i \neq j.$$

(iii) Writing

$$F(X, Y) = F_d(Y)X^d + \ldots + F_0(Y), \ G(X, Y) = G_e(Y)X^e + \ldots + G_0(Y)$$

as polynomials in X with coefficients in $k[Y]$, observe that

$$\deg F_i \leq d - i \text{ for all } 0 \leq i \leq d, \text{ and } \deg G_j \leq e - j, \text{ for all } 0 \leq j \leq e.$$

Prove that the resultant $R_{F,G} \in k[Y]$ has degree at most de, hence it has at most de roots.

(iv) Conclude that if $(a, c) \in \mathcal{Z}(F) \cap \mathcal{Z}(G)$, then c can take on at most de values. Conclude that $n \leq de$.

2. Which of the following algebraic sets are isomorphic:

(i) $\mathbb{A}^1$
(ii) $\mathcal{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2$
(iv) $\mathcal{Z}(x^2 - y^5) \subset \mathbb{A}^2$
(v) $\mathcal{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2$.

3. Consider the cubic curve

$$y^2 = x(x - 1)(x - \lambda)$$

in $\mathbb{A}^2$, where $\lambda \neq 0, 1$. Show that there are no nonconstant morphisms

$$f : \mathbb{A}^1 \to X.$$

Hint: Write $f(t) = (g(t), h(t))$ and observe that

$$h(t)^2 = g(t)(g(t) - 1)(g(t) - \lambda)$$

must be a perfect square.

4. Show that $X = \mathbb{A}^2 \setminus \{(0,0)\}$ cannot be isomorphic to an affine algebraic set:

(i) Show that if $f : X \to \mathbb{A}^1$ is a regular function on X, then f must be a polynomial in $k[x, y]$. To see this, write f as a quotient g/h of two polynomials g, h without common factors. Observe that h cannot vanish on $\mathbb{A}^2 \setminus \{(0,0)\}$, and conclude that h must be constant.

(ii) If $\Phi : Y \to X$ is an isomorphism between an affine algebraic set Y and X, consider the composition $\Psi = \iota \circ \Phi : Y \to \mathbb{A}^2$, where $\iota : X \to \mathbb{A}^2$ is the inclusion. Show that the morphism Ψ induces an isomorphism on coordinate rings. Conclude that Ψ must be an isomorphism, hence ι must be an isomorphism, which must be a contradiction.

5. From the textbook, solve 4.8 and 4.10, page 77. Solve 5.1, page 90.