Math 145, Problem Set 3. Due Friday, April 25.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. Find the irreducible components of the affine algebraic set $x^2 - yz = xz - x = 0$ in $\mathbb{A}^3$.

2. Find the irreducible components of the affine algebraic set $xz - y^2 = z^3 - x^5 = 0$ in $\mathbb{A}^3$.

3. Prove that the irreducible components of a Noetherian topological space are unique. That is, if

$$X = \bigcup X_i = \bigcup Y_i$$

such that all sets X_i and Y_i are irreducible, assumed to be irredundant (e.g. $X_i \not\subset X_j$ and similarly $Y_i \not\subset Y_j$ for $i \neq j$) then X_i's are a permutation of the Y_i's.

4. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be distinct algebraic subsets of $\mathbb{A}^2$ given as $\mathcal{Z}_1 = \mathcal{Z}(f)$ and $\mathcal{Z}_2 = \mathcal{Z}(g)$ where $f, g \in k[X, Y]$ are irreducible polynomials. Show that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ intersect in finitely many points, or equivalently that f and g have finitely many common zeros:

(i) Consider the resultant $R_{f,g} \in k[X]$, viewing f and g as polynomials in Y with coefficients in $k[X]$. Explain why the resultant cannot be the zero polynomial.

(ii) Note that if $(a, b) \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, then $f(a, Y)$ and $g(a, Y)$ have a common factor $Y - b$. Conclude that a must be a root of $R_{f,g}$. This shows that a can only take on only finitely many values. Conclude the same about b, and complete the proof.

5. An algebraic set $\mathcal{Z} \subset \mathbb{A}^2$ defined by an irreducible polynomial f of degree 2 is called an irreducible conic. Show that any irreducible conic can be written in the form

$$Y - X^2 = 0 \text{ or } XY - 1 = 0$$

after an affine change of coordinates in $\mathbb{A}^2$.

Remark: An affine change of coordinates taking (x, y) into (X, Y) is a transformation of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b},$$

where A is a 2×2 invertible matrix and $\mathbf{b} \in \mathbb{A}^2$ is a vector.

For instance, the conic

$$x^2 + y^2 = 1$$

becomes $XY - 1 = 0$ after the change of coordinates

$$X = x + \sqrt{-1}y, Y = x - \sqrt{-1}y.$$

Hint: Let $f(x, y) = ax^2 + by^2 + cxy + dx + ey + f$. If $a = b = 0$, show that the conic can be written as $XY = 1$ after changing coordinates. Otherwise, assume that $a \neq 0$ and complete the square in $ax^2 + cxy$. In suitable coordinates the polynomial f changes to $X^2 + b'Y^2 + d'X + e'Y + f'$. Continue in this fashion, completing the squares a few more times if necessary.

6. (*Baby Bezout's Theorem*) Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using the previous problem, show that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ intersect in at most 4 points.