Math 145, Problem Set 2. Due Friday, April 18.

1. Let X_1, X_2 be affine algebraic sets in $\mathbb{A}^n$. Show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,

(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, $i.e.$ find affine algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$.

Remark: The sum $\mathfrak{a} + \mathfrak{b}$ of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is the ideal consisting in all sums $f + g$ where $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$.

Hint: For part (ii), start by writing $X_1 = \mathcal{Z}(\mathfrak{a})$ and $X_2 = \mathcal{Z}(\mathfrak{b})$ for two ideals $\mathfrak{a}$ and $\mathfrak{b}$. Why can you assume that $\mathfrak{a}$ and $\mathfrak{b}$ are *radical* ideals?

2. Let $\mathbb{A}^3$ be the 3-dimensional affine space with coordinates x, y, z. Find the ideals of the following algebraic sets:

(i) The union of the (x, y)-plane with the z-axis.

(ii) The image of the map $\mathbb{A}^1 \to \mathbb{A}^3$ given by $t \to (t, t^3, t^5)$.

3. Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map *i.e.* $f(p) = (f_1(p), \ldots, f_m(p))$ for $p \in \mathbb{A}^n$, where $f_1, \ldots, f_m$ are polynomials in n variables. Are the following true or false:

(i) The image $f(X) \subset \mathbb{A}^m$ of an affine algebraic set $X \subset \mathbb{A}^n$ is an affine algebraic set.

(ii) The inverse image $f^{-1}(X) \subset \mathbb{A}^n$ of an affine algebraic set $X \subset \mathbb{A}^m$ is an affine algebraic set.

(iii) If $X \subset \mathbb{A}^n$ is an affine algebraic set, then the graph $\Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m}$ is an affine algebraic set.

4. Show that $\mathbb{A}^n$ is a Noetherian topological space in the Zariski topology.

5. (*Extra credit*) Let X be the union of the three coordinate axes in $\mathbb{A}^3$. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements.

Remark: Note that X has dimension 1 even though it is cut out by 3 equations.