

MATH 106 HOMEWORK 6 SOLUTIONS

1. (i) Give the Laurent expansion (in powers of z), representing the function

$$f(z) = \frac{1}{z^2 - 4z}$$

in certain domains, and specify these domains. Moreover, for a counterclockwise simple closed curve C inside each of the domains you found, determine the value of the integral

$$\int_C \frac{dz}{z^2 - 4z}$$

(ii) Give the Laurent expansion in powers of $z - 1$ for the same function $f(z)$, indicating the regions these expansions are valid. Furthermore, evaluate the integrals

$$\int_C \frac{dz}{z^2 - 4z}$$

for positively oriented simple closed curves C , contained in each of these regions.

Solution: (i) In the region $0 < |z| < 4$, we can write

$$f(z) = -\frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}} = -\frac{1}{4z} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = -\frac{1}{4z} - \frac{1}{4^2} - \frac{z}{4^3} - \dots - \frac{z^n}{4^{n+2}} - \dots$$

The value of the integral is equal to the coefficient in front of $\frac{1}{z}$ multiplied by $2\pi i$. The answer is $-\frac{\pi i}{2}$.

In the region $|z| > 4$, we have

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1 - \frac{4}{z}} = \frac{1}{z^2} \cdot \sum_{n=0}^{\infty} \left(\frac{4}{z}\right)^n = \frac{1}{z^2} + \frac{4}{z^3} + \dots + \frac{4^n}{z^{n+2}} + \dots$$

The value of the integral is 0, since there is no $\frac{1}{z}$ term.

(ii)

$$f(z) = \frac{1}{z(z-4)} = -\frac{1}{4} \left(\frac{1}{1+(z-1)} + \frac{1}{3-(z-1)} \right)$$

We have 3 regions: $|z-1| < 1$, $1 < |z-1| < 3$, $|z-1| > 3$. In the first,

$$f(z) = -\frac{1}{4} \left(\sum_{n=0}^{\infty} (-1)^n (z-1)^n + \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z-1}{3}\right)^n \right) = -\frac{1}{4} \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{3^{n+1}} \right) (z-1)^n.$$

In the second,

$$f(z) = -\frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(z-1)^{n+1}} + \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z-1}{3}\right)^n \right)$$

In the third,

$$\begin{aligned} f(z) &= -\frac{1}{4} \left(\frac{1}{z-1} \sum_{n=0}^{\infty} \left(-\frac{1}{z-1} \right)^n - \frac{1}{z-1} \sum_{n=0}^{\infty} \left(\frac{3}{z-1} \right)^n \right) = \\ &= -\frac{1}{4} \sum_{n=0}^{\infty} ((-1)^n - 3^n) \left(\frac{1}{z-1} \right)^{n+1} \end{aligned}$$

The integrals are found via the coefficient of $\frac{1}{z-1}$. They equal to 0, $-\frac{\pi i}{2}$ and 0 respectively.

2. (i) Find the first 4 terms of the Laurent expansion in powers of z , valid in the unit disc around the origin, for the function

$$f(z) = \frac{e^z}{z(z^2 + 1)}$$

(ii) Find the first 3 terms of the Laurent expansion in powers of z , valid in the region $0 < |z| < \pi$, for the function

$$f(z) = \frac{1}{z^2 \sin z}.$$

Compute the integral

$$\int_C \frac{dz}{z^2 \sin z}$$

over the positively oriented unit disc.

Solution: (i)

$$f(z) = \frac{1}{z} \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots \right) (1 - z^2 + z^4 - \dots) = \frac{1}{z} + 1 - \frac{z}{2} - \frac{5z^2}{6} + \dots$$

(ii)

$$f(z) = \frac{1}{z^2 \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)} = \frac{1}{z^3 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right)}.$$

Then,

$$\phi(z) = \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots}$$

is analytic in $|z| < \pi$, and therefore it has a Taylor expansion

$$\phi(z) = d_0 + d_1 z + d_2 z^2 + \dots$$

Then,

$$1 = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots \right) (d_0 + d_1 z + d_2 z^2 + \dots).$$

Solving for d_i , we obtain

$$\begin{aligned} \phi(z) &= 1 + \frac{z^2}{3!} + \frac{7z^4}{360} + \dots, \text{ so} \\ f(z) &= \frac{1}{z^3} + \frac{1}{6z} + \frac{7z}{360} + \dots \end{aligned}$$

The integral equals $\frac{\pi i}{3}$.

3. Find the residue at $z = 0$ of the following functions, and indicate the type of singularities they have at 0:

(a) $\frac{1}{z+z^2}$; (b) $z \cos\left(\frac{1}{z}\right)$; (c) $\frac{z-\sin z}{z}$; (d) $\frac{\cot z}{z^4}$

Solution:

$$(a) \frac{1}{z+z^2} = \frac{1}{z(1+z)} = \frac{1}{z}(1-z+z^2-\dots) = \frac{1}{z} - 1 + z + \dots \Rightarrow \text{Res}_{z=0} f(z) = 1.$$

The function has a simple pole at zero.

$$(b) z \cos\left(\frac{1}{z}\right) = z \left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} + \dots\right) = z - \frac{1}{2z} + \frac{1}{24z^3} + \dots \Rightarrow \text{Res}_{z=0} f(z) = -\frac{1}{2}.$$

The function has an essential singularity at $z = 0$.

$$(c) \frac{z - \sin z}{z} = \frac{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)}{z} = \frac{z^2}{3!} - \frac{z^4}{5!} + \dots \Rightarrow \text{Res}_{z=0} f(z) = 0.$$

The function has a removable singularity at $z = 0$.

$$(d) \frac{\cos z}{z^4 \sin z} = \frac{1 - \frac{z^2}{2!} + \dots}{z^4 \left(z - \frac{z^3}{3!} + \dots\right)} = \frac{1}{z^5} \left(1 - \frac{z^2}{2!} + \dots\right) \left(1 + \frac{z^2}{3!} + \frac{7z^4}{360} + \dots\right)$$

$$= \frac{1}{z^5} - \frac{1}{3z^3} - \frac{1}{45z} + \dots \Rightarrow \text{Res}_{z=0} f(z) = -\frac{1}{45}.$$

The function has a pole of order 5 at $z = 0$.

4. (i) Differentiating the Taylor expansion

$$\frac{1}{1-w} = 1 + w + w^2 + \dots + w^n + \dots$$

in the region $|w| < 1$, find the Taylor expansion of the function

$$\frac{1}{(1-w)^2}$$

in the same region $|w| < 1$.

(ii) Determine the different Laurent expansion of the function

$$f(z) = \frac{1}{(2z-1)^2}$$

in powers of $z - 1$.

Solution: (i) Differentiating the equality we get

$$\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \dots + nw^{n-1} + \dots$$

(ii) Consider

$$g(z) = \frac{1}{2z-1} = \frac{1}{1+2(z-1)} \Rightarrow f(z) = -\frac{g'(z)}{2}.$$

When $2|z-1| < 1$, we have

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} (-2(z-1))^n \Rightarrow g'(z) = \sum_{n=0}^{\infty} n(-2)^n (z-1)^{n-1} \\ \Rightarrow f(z) &= \sum_{n=0}^{\infty} n(-2)^{n-1} (z-1)^{n-1} = 1 - 4(z-1) + 12(z-1)^2 - \dots \end{aligned}$$

When $2|z-1| > 1$, we have

$$\begin{aligned} g(z) &= \frac{1}{2(z-1)} \sum_{n=0}^{\infty} \left(-\frac{1}{2(z-1)}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n (z-1)^n} \Rightarrow g'(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-n)}{2^n (z-1)^{n+1}} \\ \Rightarrow f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{2^{n+1}(z-1)^{n+1}} = \frac{1}{4(z-1)^2} - \frac{1}{4(z-1)^3} + \dots \end{aligned}$$

5. (i) Using the Taylor expansion of the function e^z and the formula

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

confirm the Taylor expansion of $\cos z$ in powers of z derived in class.

(ii) What is the Taylor expansion of $\cos z$ in powers of $z - \frac{\pi}{2}$?

Solution: (i) We have

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots \\ e^{-iz} &= 1 + (-iz) + \frac{(-iz)^2}{2} + \frac{(-iz)^3}{3!} + \frac{(-iz)^4}{4!} + \dots \end{aligned}$$

The odd terms cancel, so

$$\frac{e^{iz} + e^{-iz}}{2} = 1 + \frac{(iz)^2}{2!} + \frac{(iz)^4}{4!} + \dots = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots = \cos z.$$

(ii) Using Taylor's theorem, we need to compute the derivatives of $\cos z$ at $\frac{\pi}{2}$. The derivatives of $\cos z$ are $\cos z, -\sin z, -\cos z, \sin z, \dots$ and then they repeat. At $z = \frac{\pi}{2}$, the values of the derivatives are $0, -1, 0, 1, \dots$. Therefore

$$\cos z = -\left(z - \frac{\pi}{2}\right) + \frac{\left(z - \frac{\pi}{2}\right)^3}{3!} - \frac{\left(z - \frac{\pi}{2}\right)^5}{5!} + \dots + (-1)^{n+1} \frac{\left(z - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} + \dots$$

6. Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$

(a) $\frac{e^{-z}}{z^2}$; (b) $\frac{e^{-z}}{(z-1)^2}$; (c) $z^2 e^{\frac{1}{z}}$; (d) $\frac{z+1}{z^2-2z}$;

Solution:

- (a) $\frac{e^{-z}}{z^2} = \frac{1}{z^2} \left(1 - z + \frac{z^2}{2} + \dots \right) \Rightarrow \int_{|z|=3} f(z) dz = 2\pi i \operatorname{Res}_{z=0} f(z) = -2\pi i.$
- (b) $\frac{e^{-z}}{(z-1)^2} = e^{-1} \cdot \frac{e^{-(z-1)}}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2} \left(1 - (z-1) + \frac{(z-1)^2}{2!} + \dots \right) = \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{z-1} + \dots$
 $\Rightarrow \int_{|z|=3} f(z) dz = 2\pi i \operatorname{Res}_{z=1} f(z) = -2\pi i e^{-1}.$
- (c) $z^2 e^{\frac{1}{z}} = z^2 \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots \right) \Rightarrow \int_{|z|=3} f(z) dz = 2\pi i \operatorname{Res}_{z=0} f(z) = \frac{2\pi i}{3!} = \frac{\pi i}{3}.$
- (d) $\frac{z+1}{z^2-2z} = \frac{-1/2}{z} + \frac{3/2}{z-2} \Rightarrow \operatorname{Res}_{z=0} f(z) = -\frac{1}{2}$ since the second function is holomorphic at 0.
Similarly $\operatorname{Res}_{z=2} f(z) = \frac{3}{2}.$ By the residue theorem,

$$\int_{|z|=3} f(z) dz = 2\pi i (\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=2} f(z)) = 2\pi i.$$