MATH 106 HOMEWORK 5 SOLUTIONS

1. Let C_1 denote the positively oriented circle $|z| = 4$ and C_2 the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. With the aid of Cor. 2 in Sec 46, point out why

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \text{ when}$$

$$(a)\, f(z) = \frac{1}{3z^2+1} \quad (b)\, f(z) = \frac{z+2}{\sin\frac{z}{2}} \quad (c)\, f(z) = \frac{z}{1-e^z}$$

Solution: We need to show the given functions are holomorphic in the area between and on the two curves, call it A.

For (a), the zeros of $3z^2 + 1$ are $\pm\frac{i}{\sqrt{3}}$, they both lie in the area enclosed by C_1, so the function is holomorphic in A. For part (b), we observe that

$$\sin\frac{z}{2} = 0 \Leftrightarrow z = 2k\pi \Rightarrow f(z) \text{ is holomorphic in } A.$$

Finally, for (c), the denominator is zero when

$$1 - e^z = 0 \Leftrightarrow z = 2ik\pi \Rightarrow f(z) \text{ is holomorphic in } A.$$

2. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2, y = \pm 2$. Evaluate each of these integrals:

$$(a) \int_C \frac{e^{-z}}{z - \frac{i\pi}{2}}dz, \quad (b) \int_C \frac{\cos z}{z(z^2+8)}dz, \quad (c) \int_C \frac{z}{2z+1}dz,$$

$$(d) \int_C \frac{\cosh z}{z^4}dz, \quad (e) \int_C \frac{\tan\frac{z}{2}}{z - x_0}dz, -2 < x_0 < 2.$$

Solution: We use the Cauchy's integral formula.

(a) $\displaystyle\int_C \frac{e^{-z}}{z - \frac{i\pi}{2}}dz = 2i\pi e^{-\frac{i\pi}{2}} = 2\pi$

(b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}dz = 2i\pi\frac{\cos(0)}{0^2+8} = \frac{i\pi}{4}$

(c) $\displaystyle\int_C \frac{z}{2z+1}dz = \int_C \frac{z/2}{z+\frac{1}{2}}dz = 2i\pi \cdot \frac{-1}{4} = -\frac{i\pi}{2}$

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}dz = 2i\pi\frac{\sinh(0)}{3!} = 0$

(e) $\displaystyle\int_C \frac{\tan\frac{z}{2}}{z - x_0}dz = 2i\pi \cdot \frac{1}{2\cos^2\frac{x_0}{2}} = \frac{i\pi}{\cos^2\frac{x_0}{2}}$

3. Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ in the positive sense when

$$(a)\, g(z)=\frac{1}{z^2+4}, \quad (b)\, g(z)=\frac{1}{(z^2+4)^2}$$

Solution:

(a) $\displaystyle\int_{|z-i|=2}\frac{1}{z^2+4}dz=\int_{|z-i|=2}\frac{1}{(z-2i)(z+2i)}dz=2i\pi\cdot\frac{1}{(2i+2i)}=\frac{\pi}{2}$

(b) $\displaystyle\int_{|z-i|=2}\frac{1}{(z^2+4)^2}dz=\int_{|z-i|=2}\frac{1}{(z-2i)^2(z+2i)^2}dz=2i\pi\cdot\frac{-2}{(2i+2i)^3}=\frac{\pi}{16}$

4. Let C be the circle $|z|=3$, described in the positive sense. Show that if

$$g(w)=\int_C\frac{2z^2-z-2}{z-w}dz \quad (|w|\neq 3),$$

then $g(2)=8i\pi$. What is the value of $g(w)$ when $|w|>3$.

Solution:

$$g(2)=\int_C\frac{2z^2-z-2}{z-2}dz=2i\pi(2z^2-z-2)\bigg|_{z=2}=8i\pi$$

When $|w|>3$, the integrand is holomorphic on and inside C (since $z=w$ cannot happen on the circle C), therefore it is zero.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C\frac{f'(z)}{z-z_0}dz=\int_C\frac{f(z)}{(z-z_0)^2}dz$$

Solution: f holomorphic implies f' holomorphic inside and on C. If z_0 is outside C, both sides will be zero since the integrands will be holomorphic inside and on C. If z_0 is inside C, the left-hand side will equal to $2i\pi f'(z_0)$ by Cauchy formula for f'. By the formula for derivatives, the right-hand side will be $2i\pi f'(z_0)$ also.

6. Let f be an entire function such that $|f(z)|\leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z)=az$ where a is a complex number.

Solution: Let $z_0\in\mathbb{C}$ and C_R be the circle of radius R around z_0. Then, the maximum value of $|f(z)|$ on C_R can be estimated

$$|f(z)|\leq A|z|\leq A(R+|z_0|).$$

Just as in the proof of the Liouville's theorem, we have

$$\left|f''(z_0)\right|=\left|\frac{1}{i\pi}\int_{C_R}\frac{f(z)}{(z-z_0)^3}dz\right|\leq\frac{1}{\pi}\cdot\frac{A(|z_0|+R)}{R^3}.2\pi R=\frac{2A(|z_0|+R)}{R^2}$$

for every R. Making $R\to\infty$, we obtain $f''(z_0)=0$. Since z_0 was arbitrary, we get $f''(z)\equiv 0$, hence $f(z)=az+b$, where a,b are complex constants. From $|f(z)|\leq A|z|$ when $z=0$ we get $f(0)=0$, so $b=0$. Therefore $f(z)=az$.

7. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = Re[f(z)]$ has an upper bound u_0. Show $u(x, y)$ must be a constant.

Solution: The function $g(z) = e^{f(z)}$ is entire, and

$$|g(z)| = |e^{u+iv}| = |e^u(\cos v + i \sin v)| = |e^u| \leq e^{u_0}.$$

By Liouville's theorem $g(z)$ is constant so $g'(z) = 0$. Now, $g'(z) = e^{f(z)} f'(z)$ so $f'(z) = 0$ since the exponential cannot be 0, implying that $f(z)$ is constant.