MATH 106 HOMEWORK 4 SOLUTIONS

1. Show directly from the definition that

$$\sin(2z) = 2\sin z\cos z$$

Solution:

$$\sin(2z) = \frac{e^{2zi} - e^{-2zi}}{2i} = 2\frac{(e^{zi} - e^{-zi})}{2i}\frac{(e^{zi} + e^{-zi})}{2} = 2\sin z\cos z$$

2. Write the following complex numbers in standard form:

(i) $(-1 + i\sqrt{3})^i$. What is the principal value?
(ii) $\tan^{-1}(2i)$,
(iii) $\tan(\frac{i\pi}{2})$,
(iv) solve the equation $\sin z = 2$.

Solution:

(i)

$$(-1+i\sqrt{3})^i = e^{i(\ln(2)+i(\frac{2\pi}{3}+2n\pi))} = e^{-(\frac{2\pi}{3}+2n\pi)}\cos(\ln 2)+ie^{-(\frac{2\pi}{3}+2n\pi)}\sin(\ln 2).$$

$$\text{P.V. } (-1 + i\sqrt{3})^i = e^{-\frac{2\pi}{3}}\cos(\ln 2) + ie^{-\frac{2\pi}{3}}\sin(\ln 2).$$

(ii)

$$\tan^{-1}(2i) = z \Leftrightarrow \tan(z) = 2i \Leftrightarrow \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -2 \Leftrightarrow$$

$$e^{iz} - e^{-iz} = -2e^{iz} - 2e^{-iz} \Leftrightarrow e^{-iz} = -3e^{iz} \Leftrightarrow e^{-2iz} = -3$$

$$z = -\frac{\pi}{2} + \frac{i\ln 3}{2} + n\pi.$$

(iii)

$$\tan(\frac{i\pi}{2}) = \frac{e^{i(\frac{i\pi}{2})} - e^{-i(\frac{i\pi}{2})}}{i(e^{i\frac{i\pi}{2}} + e^{-i(\frac{i\pi}{2})})} = i \cdot \frac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}.$$

(iv)

$$\sin z = 2 \Leftrightarrow e^{iz} - e^{-iz} = 4i \Leftrightarrow w - \frac{1}{w} = 4i, \text{ with } w = e^{iz} \Leftrightarrow$$

$$w^2 - 4iw - 1 = 0 \Leftrightarrow w = (2 \pm \sqrt{3})i \Leftrightarrow e^{iz} = (2 \pm \sqrt{3})e^{\frac{i\pi}{2}} \Leftrightarrow$$

$$z = \frac{\pi}{2} + 2k\pi + i\ln(2 \pm \sqrt{3}).$$

3. Evaluate the following integrals by parametrizing the contour:

(i) $\int_C x dz$ where C is the oriented line segment joining 1 to i,

(ii) $\int_C (z-1)dz$ where C is the semicircle joining 0 to 2,

(iii) $\int_C \cos(\frac{z}{2})dz$ where C is the line segment joining 0 to $\pi + 2i$.

Solution:

(i) Parametrize the contour by $z = (i-1)t + 1$ with $0 \leq t \leq 1$. Then

$$\int_C x dz = \int_0^1 (1-t)(i-1)dt = \frac{i-1}{2}.$$

(ii) Using the parametrization

$$z = 1 + e^{-i\theta},\ -\pi \leq \theta \leq 0,\ dz = -ie^{-i\theta}d\theta,$$

we compute

$$\int_C (z-1)dz = \int_{-\pi}^0 e^{-i\theta}(-ie^{-i\theta})d\theta = -i\int_{-\pi}^0 e^{-2i\theta}d\theta = \frac{1}{2}e^{-2i\theta}\Big|_{\theta=-\pi}^{\theta=0} = 0.$$

(iii) Writing $z = (\pi + 2i)t$ for $0 \leq t \leq 1$, we compute

$$\int_C \cos(\frac{z}{2})dz = \frac{\pi+2i}{2}\int_0^1 e^{i(\frac{\pi}{2}+i)t} + e^{-i(\frac{\pi}{2}+i)t}dt = -i\left(e^{i(\frac{\pi}{2}+i)t} - e^{-i(\frac{\pi}{2}+i)t}\right)\Big|_{t=0}^{t=1} =$$

$$= e + e^{-1}.$$

4. Evaluate

$$\int_C z^{-1+i}dz$$

where C is the positively oriented unit circle, and the integrand is defined by chosing the branch

$$0 < arg(z) < 2\pi.$$

What happens if we take $-\pi < arg(z) < \pi$?

Solution: For $z = e^{i\theta}$, we have

$$\int_C z^{-1+i}dz = \int_0^{2\pi} e^{i\theta(i-1)} \cdot ie^{i\theta}d\theta = i\int_0^{2\pi} e^{-\theta}d\theta = i(1-e^{-2\pi}).$$

If we take $-\pi < arg(z) < \pi$ we'll get $i(e^{\pi} - e^{-\pi})$.

5. Compute

$$\int_C z^a dz$$

where C is the counterclockwise unit circle and a is any real number. The principal value is used for the integrand. Do it in two ways: by picking a parametrization of C, and by using a suitable anti-derivative.

Solution: Let $z = e^{i\theta}$. If $a \neq -1$, we have

$$\int_C z^a dz = \int_{-\pi}^{\pi} e^{ia\theta} \cdot ie^{i\theta} d\theta = \frac{e^{i\pi(a+1)} - e^{-i\pi(a+1)}}{a+1} = \frac{2i}{a+1} \sin((a+1)\pi).$$

If $a = -1$, then

$$\int_C z^{-1} dz = \int_{-\pi}^{\pi} i d\theta = 2\pi i.$$

For the second method, let us assume first $a \neq -1$. Note that the principal value of $z^a = e^{a \operatorname{Log} z}$ is undefined at the negative reals. To compute the integral, we will split the circle into the upper and lower halves. For the integral across the upper half C_1, we will replace z^a by a different branch which is everywhere defined and holomorphic. We pick a branch cut which doesn't cross C_1, for instance

$$-\frac{\pi}{2} < arg(z) < \frac{3\pi}{2}.$$

This branch of z^a *agrees* with the principal branch along C_1. Then,

$$\int_{C_1} z^a dz = \frac{e^{(a+1)\log(z=-1)} - e^{(a+1)\log(z=1)}}{a+1} = \frac{e^{\pi i(a+1)} - 1}{a+1}$$

For the integral along C_2, we may branch cut along a half-line which avoids C_2, for instance

$$-\frac{3\pi}{2} < arg(z) < \frac{\pi}{2}. \tag{1}$$

The values of z^a for this branch coincide with the principal values along C_2. We evaluate

$$\int_{C_2} z^a dz = \frac{e^{(a+1)\log(z=1)} - e^{(a+1)\log(z=-1)}}{a+1} = \frac{1 - e^{-\pi i(a+1)}}{a+1}.$$

Putting things together

$$\int_C z^a\, dz = \frac{e^{\pi i(a+1)} - e^{-\pi i(a+1)}}{a+1}$$

just as before. The case $a = -1$ is done in similar way but using the anti-derivative of $\frac{1}{z}$ which is $\operatorname{Log}(z)$ and was done in class.

N.B. You should be careful when chosing your branch cut. It is especially easy to get a wrong answer when evaluating the integral along C_2. Picking a branch cut which avoids C_2 is not enough, *you have to make sure that the chosen branch coincides with the principal value*, so that you are not changing the integral. For instance the branch cut

$$\frac{\pi}{2} < arg(z) < \frac{5\pi}{2} \tag{2}$$

would not work in this case. This is because this branch of z^a *does not agree* with the principal value along C_2. This can be seen, for instance, at the point 1. There, the branch (2) gives the value $1^{a+1} = e^{2\pi i(a+1)}$ which differs from the principal value $1^{a+1} = 1$. However, you can convince yourself that the chosen branch (1) does work.

6. Show that
$$\int_{-1}^{1} z^i dz = \frac{(1+e^{-\pi})(1-i)}{2}$$
for any path joining -1 to 1 which lies above the real axis, endpoints excluded. The principal value is used for the integrand. Do it also for any path joining $-i$ and i which lies on the right of the imaginary axis.

Solution: Observe that the integrand does not exist at the endpoint $z = -1$. To fix this, let us consider a different branch cut along
$$-\frac{\pi}{2} < arg(z) < \frac{3\pi}{2}.$$
The branch of z^i above agrees *along the path of integration* with the principal branch (except possibly at $z = -1$ where the latter is undefined). Therefore, we may safely work with the new branch considered above. Now, note that z^i has an antiderivative in the upper half plane given by
$$\frac{z^{i+1}}{i+1}$$
where the new branch is used again in the definition of z^{i+1}. Therefore
$$\int_{-1}^{1} z^i dz = \frac{e^{(i+1)\log(z=1)} - e^{(i+1)\log(z=-1)}}{i+1} = \frac{1-e^{(i+1)(i\pi)}}{i+1} = \frac{(1+e^{-\pi})(1-i)}{2}.$$
The integral along the second path causes no problems since the path does not intersect the branch cut at the negative reals. Therefore,
$$\int_{-i}^{i} z^i dz = \frac{e^{(i+1)\log(z=i)} - e^{(i+1)\log(z=-i)}}{i+1} = \frac{e^{(i+1)\frac{i\pi}{2}} - e^{(i+1)\frac{-i\pi}{2}}}{i+1}$$
$$= \frac{(e^{-\frac{\pi}{2}} + e^{\frac{\pi}{2}})(i+1)}{2}.$$

7. What are the values of the following integrals:

(i) $\int_C \frac{z^2}{z-3} dz$ where C is the positively oriented unit circle,
(ii) $\int_C Log(z+2) dz$ where C is the positively oriented unit circle.

Solution: In both cases the integrands are holomorphic on and inside the unit circle so by Cauchy Theorem the integrals are 0. For the first function, the pole is at $z = 3$ which is clearly outside C. The second function is holomorphic everywhere except for the line $z = -2 + x$ with x is a negative real. This line also avoids the unit circle.

8. Determine the value of the integral

$$\int_C (z-1)^n dz$$

where n is any integer and C is a positively oriented square of side a, which doesn't go through 1.

Solution: If 1 is not inside the square, the integral is 0 because $(z-1)^n$ is holomorphic inside C no matter whether n is positive, negative or 0.

If 1 is inside the square, and $n \geq 0$ the same reasoning shows that the integral is 0.

So let us consider the case when $n < 0$, in which case the function $(z-1)^n$ has a pole at $z = 1$. We can consider a small circle around 1 that lies inside the square. Since $(z-1)^n$ is holomorphic in the area between the square and the circle, the integrals over these curves are equal. Consider parametrization of the circle $z = 1 + re^{i\theta}$, where r is the radius. Then

$$\int_C (z-1)^n dz = r^{n+1} \int_0^{2\pi} e^{in\theta} \cdot ie^{i\theta} d\theta = \frac{2ir^{n+1}}{n+1} e^{i(n+1)\theta} \Big|_{\theta=0}^{\theta=2\pi} = 0$$

The case $n = -1$ is special since then the denominator becomes 0. The integral can be computed by hand to be $2\pi i$.

9. Show that the area enclosed by a positively oriented simple closed curve C is given by

$$\frac{1}{2i} \int_C \bar{z} dz$$

Solution: We have

$$\int_C \bar{z} dz = \int_C (x - iy) \cdot (dx + idy) = \int_C (x\,dx + y\,dy) + i(x\,dy - y\,dx).$$

Let R be the region encolosed by C. We can apply Green's theorem to each of the two terms above to conclude

$$\int_C \bar{z} dz = \int \int_R 2i\,dx\,dy = 2i \text{ area } (R).$$