MATH 106 HOMEWORK 2 SOLUTIONS

1. Let $z = x + iy$ and

$$f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$$

Write $f(z)$ as a function of only z and $\bar{z}$.

Solution: Using the formulas

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i},$$

we compute

$$\begin{aligned} f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) = \\ &= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 - 2 \cdot \frac{z-\bar{z}}{2i} + i\left(2 \cdot \frac{z+\bar{z}}{2} - 2 \cdot \frac{z+\bar{z}}{2} \cdot \frac{z-\bar{z}}{2i}\right) \\ &= \frac{z^2 + \bar{z}^2 + 2z \cdot \bar{z}}{4} - \frac{z^2 + \bar{z}^2 - 2z \cdot \bar{z}}{-4} - \frac{z - \bar{z}}{i} + i(z + \bar{z}) - \frac{z^2 - \bar{z}^2}{2} \\ &= \bar{z}^2 + 2iz. \end{aligned}$$

2. Let f and g be two complex valued functions with the properties
(i) $\lim_{z \to a} f(z) = 0$
(ii) there exists a positive number M such that $|g(z)| \leq M$ for all z
Prove that

$$\lim_{z \to a} f(z)g(z) = 0$$

Solution: Since $\lim_{z \to a} f(z) = 0$, it follows that

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \; (|z - a| < \delta \Rightarrow |f(z)| < \epsilon) \tag{1}$$

We would like to show that

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \; (|z - a| < \delta \Rightarrow |f(z)g(z)| < \epsilon)$$

Fix $\epsilon > 0$. By (1) we can find $\delta > 0$ such that

$$|f(z)| < \frac{\epsilon}{M} \text{ whenever } |z-a| < \delta.$$

Therefore,

$$|f(z)g(z)| = |f(z)||g(z)| < \frac{\epsilon}{M} \cdot M = \epsilon \text{ whenever } |z-a| < \delta.$$

This shows that

$$\lim_{z \to a} f(z)g(z) = 0.$$

3. Suppose f and g are two holomorphic functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$. Explain why

$$\lim_{z \to 0} \frac{f(z)}{g(z)} = \frac{f'(0)}{g'(0)}.$$

Solution: The lefthand side equals

$$\lim_{z \to 0} \frac{f(z)}{g(z)} = \lim_{z \to 0} \frac{\frac{f(z)}{z}}{\frac{g(z)}{z}}.$$

The numerator and denominator can be evaluated as follows. Using $f(0) = g(0) = 0$, we compute

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = f'(0)$$

and similarly

$$\lim_{z \to 0} \frac{g(z)}{z} = \lim_{z \to 0} \frac{g(z) - g(0)}{z} = g'(0).$$

Therefore

$$\lim_{z \to 0} \frac{f(z)}{g(z)} = \frac{f'(0)}{g'(0)}.$$

4. Using only the definitions, check that the function $f(z) = \mathrm{Re}(z) + 2\bar{z}$ is not holomorphic. Explain why $f(z)$ is continuous.

Solution: It suffices to check that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

does not exist. First, observe that

$$f(z+h)-f(z)=\text{Re }(z+h)+2\cdot\overline{z+h}-\text{Re }z-2\bar{z}=\text{Re}h+2\bar{h}.$$

Letting $h\to 0$ on the real axis we have

$$f'(z)=\lim_{h\to 0}\frac{\text{Re}h+2\bar{h}}{h}=\frac{h+2h}{h}=3.$$

Letting $h\to 0$ on the imaginary axis, we have

$$f'(z)=\lim_{h\to 0}\frac{\text{Re}h+2\bar{h}}{h}=\lim_{h\to 0}\frac{0-2h}{h}=-2.$$

This discrepancy shows that f cannot be holomorphic.

We check continuity at an arbitrary point $z_0=x_0+iy_0$. Whenever $|z-z_0|<\delta$ we have

$$|f(z)-f(z_0)|=|x-x_0+2(\bar{z}-\bar{z_0})|\leq|x-x_0|+2|\overline{z-z_0}|\leq$$

$$\leq|z-z_0|+2|z-z_0|<3\delta$$

Now if $\epsilon>0$ is fixed, taking $\delta=\frac{\epsilon}{3}$, we have

$$|z-z_0|<\delta\Rightarrow|f(z)-f(z_0)|<\epsilon.$$

5. Compute the derivatives of the following functions
(i) $f(z)=(1-z)^5$ at $z_0=1+i$
(ii) $f(z)=\frac{1-2z}{1-z}$ at $z_0=2+i$.

Solution: Using the rules of differentiation, we have

(i) $f'(z)=5(1-z)^4(-1)\Rightarrow f'(1+i)=-5(-i)^4=-5.$

(ii) $f'(z)=\frac{(1-z)(-2)-(1-2z)(-1)}{(1-z)^2}=-\frac{1}{(1-z)^2}$. Therefore

$$f'(2+i)=-\frac{1}{(1+i)^2}=-\frac{1}{2i}.$$

6. Check that the real and the imaginary parts of the function $f(z) = (z+1)^3$ satisfy the Cauchy-Riemann equations.

Solution: Let $z = x + iy$. Then

$$\begin{aligned}(z+1)^3 &= (x+1+iy)^3 \\ &= (x+1)^3 + 3(x+1)^2(iy) + 3(x+1)(iy)^2 + (iy)^3 \\ &= (x+1)^3 - 3(x+1)y^2 + i\left[3(x+1)^2y - y^3\right]\end{aligned}$$

Hence,

$$u = (x+1)^3 - 3(x+1)y^2, \text{ and } v = 3(x+1)^2y - y^3.$$

The partial derivatives satisfy the Cauchy-Riemann equations, since

$$\begin{aligned}u_x &= 3(x+1)^2 - 3y^2 = v_y \\ u_y &= -6(x+1)y = -v_x\end{aligned}$$

7. Do the real and imaginary parts of the function in Pr. 1 satisfy the Cauchy-Riemann equations? What does this imply about $f(z)$?

Solution: The real and imaginary parts of f are

$$u = x^2 - y^2 - 2y, \text{ and } v = 2x - 2xy.$$

The partial derivatives are

$$\begin{aligned}u_x &= 2x \neq -2x = v_y \\ u_y &= -2y - 2 \neq -2 + 2y = -v_x\end{aligned}$$

with the only possible exception $x = y = 0$. This shows that the function f is not holomorphic.

8. Let $f : \mathbb{C} \to \mathbb{R}$ be a real valued non-constant function. Using the Cauchy-Riemann equations, show that f cannot be holomorphic.

Solution: Suppose $f = u + iv$ is holomorphic. Since f is real-valued, $v = 0$. The Cauchy-Riemann equations show that

$$u_x = v_y = 0, \text{ and } u_y = -v_x = 0.$$

Therefore $u_x = 0$ which shows u is a function of y alone, $u = \phi(y)$. Since $u_y = 0$, we have $\phi'(y) = 0$. Therefore, ϕ, and also u, must be constant. This is a contradiction.