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On the global well-posedness theory for a class of PDE models for criminal activity

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ABSTRACT

We study a class of 'reaction–advection–diffusion' system of partial differential equations, which can be taken as basic models for criminal activity. This class of models are based on routine activity theory and other theories, such as the 'repeat and near-repeat victimization effect' and were first introduced in Short et al. (2008) [11]. In these models the criminal density is advected by a velocity field that depends on a scalar field, which measures the appeal to commit a crime. We refer to this scalar field as the *attractiveness field*. We prove local well-posedness of solutions for the general class of models. Furthermore, we prove global well-posedness of solutions to a fully-parabolic system with a velocity field that depends logarithmically on the attractiveness field. Our final result is the global well-posedness of solutions the fully-parabolic system with velocity field that depends linearly on the attractiveness field for small initial mass.

1. Introduction

The use of mathematical models to obtain insight into sociological phenomena is a fairly new idea; however, one that has generated vast interest. In particular, the study of crime patterns via the use of partial differential equations has resulted in a lot of compelling research. In this work we study systems of partial differential equations of the form

$$A_t = \eta\Delta A - A + f(A, \rho) + \bar{A}(x) \tag{1a}$$

$$\rho_t = \nabla\cdot\big(\nabla\rho - \rho\nabla\chi(A)\big) - A\rho + \bar{B}(x), \tag{1b}$$

on Ω, a bounded subset of $\mathbb{R}^2$. The scalar $A(x, t)$ is a field that influences the movement of criminals. The criminal density at a point $x \in \Omega$ at time $t \geq 0$ is denoted by $\rho(x, t)$. The system (1) can be seen as a basic model for residential burglary *hotspots*, which are spatio-temporal areas of high crime density. This system is related to models for gravitational interacting particles, chemotaxis (Keller–Segel model), and electrolytes (Debye system) for which there is an extensive mathematical literature. See for example [1–7].

Although crime has been present throughout all times and societies it is still a phenomenon that is little understood. One can observe that while crime is everywhere, certain neighborhoods have a higher propensity to crime than others [8,9]. In fact, data shows that there are areas with high crime rates surrounded by areas with low crime rates. These areas of high density of crime are known as *hotspots* and they can change temporally and spatially or might cease to exist altogether. This observation led Short and collaborators to develop a PDE model for residential burglaries based on *routine activity theory* [10], with the aim of understanding the dynamics of hotspots. Routine activity theory states that criminal acts occur when a motivated agent encounters an appropriate opportunity. This implies that to prevents crime there should be a significant focus on factors that create opportunity. There are other factors that can also affect criminal activity, such as socio-economic and psychological factors, however; these are assumed to be random noise. This has the advantage of making the system mathematically tractable. More specifically, the theory modeled in [11] is the 'repeat and near-repeat victimization effect' [12–16]. This effect is seen in actual residential burglary data and states that a house, and its neighbors, have a higher probability of being burglarized soon after a burglary has occurred. See [17] for a discussion on measuring and modeling this effect. It is worth noting that the interest of studying a continuum system to model criminal activity is not as a means to predict crime, but rather to obtain some insight into this phenomena. These studies provide a useful, different, and interesting perspective. The model obtained in [11] is a fully-parabolic system of reaction–advection–diffusion PDEs. The 'repeat and near-repeat victimization effect' leads to a nonlinear increase of the attractiveness value, as $f(A, \rho) = A(x, t)\rho(x, t)$, which is the expected number of crimes. Note that this leads to an approximate quadratic growth in the attractiveness value. Furthermore, the criminal density is being advected by a

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velocity field given by $\chi(A) = \nabla \log A$. The model is given by

$$A_t = \eta\Delta A - A\rho + \rho + \bar{A}(x) \tag{2a}$$

$$\rho_t = \Delta\rho - \nabla\cdot(\rho\nabla\log(A)) - A\rho + \bar{B}(x). \tag{2b}$$

In [11], stability analysis of the flat steady-state leads to the specification of parameter regimes that lead to hotspots for (2). In an effort to explore the robustness of this model we studied the local well-posedness of classical solutions in [18]. A continuation argument that provides a sufficient condition for the extension of classical solutions in time was also proved. However, the lack of a free energy functional, due to the nonlinearities of the system, has prevented progress on the global well-posedness theory. In this work we study the case when $f(A, \rho) = \rho$, which models similar criminology theory. We assume that the presence of criminals in a neighborhood automatically increases the scalar field. Hence, we focus on the model

$$A_t = \eta\Delta A - A + \rho + \bar{A}(x) \tag{3a}$$

$$\rho_t = \Delta\rho - \nabla\cdot(\rho\nabla\chi(A)) - A\rho + \bar{B}(x). \tag{3b}$$

Note that this is in contrast with the model in developed in [11] (see (2)), which assumes that the attractiveness value increases with each crime (given by $A\rho$). The attractiveness value is diffused with coefficient of diffusivity $\eta \in [0, 1]$. However, if no crimes occur the attractiveness value decays to its base value given by $\bar{A}(x)$. The criminal density is being diffused and advected by the velocity field $\nabla\chi(A)$. The function $\bar{B}(x)$ is the rate at which criminals are being created at a location x. The functions $\bar{A}(x)$ and $\bar{B}(x)$ have a significant affect on the long term behavior of the solutions, which can have extremely interesting interpretations. However, for the results of of this paper these functions do not play a significant role; hence, not much more will be said about them. We will explore the effects of $\bar{A}(x)$ in the long time behavior of the solutions in future work. Following the nomenclature from the Keller–Segel model literature we will refer to $\chi(A)$ as the *sensitivity function* [19]. In particular, we are interested in the cases where the sensitivity function is *linear*, that is $\chi(A) = A$, and the when the sensitivity function is *logarithmic*, that is $\chi(A) = \log A$. However, the local theory provided here is valid for any function $\chi(A)$ that satisfies the following inequalities

$$\|\nabla\chi(A)\|_p \lesssim C(\bar{A}(x), A_0(x))\,\|\nabla A\|_p \tag{4a}$$

$$\begin{aligned}\left\|\nabla\big(\chi(A_1) - \chi(A_2)\big)\right\|_p &\lesssim C(\bar{A}(x), A_0(x))\,\|A_1\|_\infty\,\|\nabla(A_1 - A_2)\|_p \\ &\quad + \|A_1 - A_2\|_\infty\,\|\nabla A_2\|_p\,.\end{aligned} \tag{4b}$$

Criminals are removed when they commit a burglary and are created at constant (in time) rate. The local existence theory for the linear and logarithmic sensitivity function is similar. However, the logarithmic sensitivity leads to global solutions without use of the quadratic decay of criminals and for all initial data. On the other hand, if the criminals are advected with a linear sensitivity function we only prove global existence for small enough initial data. This issue seems to be simply technical and we believe that the quadratic decay in (3b) is necessary to prevent blow-up for large initial mass. Essentially, both the logarithmic sensitivity function and the quadratic decay seen in (3) are both mechanisms that prevent blow-up. The system developed in [11] contains both of these mechanisms. Indeed, we conjecture, from preliminary numerical simulations ran by Martin B. Short, that to model the 'repeat and near-repeat victimization effect' one needs both mechanisms to prevent finite time blow-up.

One key difference in the logarithmic sensitivity function and the linear sensitivity function is that the former always guarantees global control of the entropy,

$$\mathcal{S}(t) := \int \rho(x,t)\log\rho(x,t)\;dx. \tag{5}$$

This control is obtained from the following energy functional

$$\mathcal{F}_{\log}(t) := \int \rho(x,t)\,(\log\rho(x,t) - \gamma\log A(x,t))\,dx, \tag{6}$$

for some $\gamma > 0$ to be specified later, which is bounded from above for any solution of (3). However, we seek a bound on the entropy and to obtain this we will use of the fact that $\int|\rho\log A|\,dx$ is always bounded from below. Note that $\mathcal{F}_{\log}(t)$ is not a dissipated quantity; indeed, the lack of conservation of mass in addition to the growth in the system give, at best, a bound that increases in time. Nevertheless, this suffices to obtain control of the entropy on any finite time interval. The system with linear sensitivity function has a corresponding functional, that is also always bounded from above

$$\begin{aligned}\mathcal{F}_L(t) &= \int \rho(\log\rho - A)dx + \frac{1}{2}\int A^2dx \\ &\quad + \frac{\eta}{2}\int|\nabla A|^2\,dx - \int A\bar{A}dx \\ &:= \mathcal{S}_L(t) + \mathcal{W}_L(t),\end{aligned} \tag{7}$$

where $\mathcal{S}_L(t) := \int\rho(\log\rho - A)dx$ and the remaining part is defined by $\mathcal{W}_L(t)$. Note that a global upper bound on $\mathcal{F}_L(t)$ is not sufficient to obtain global existence. As noted earlier we seek a global upper and lower bounds for the entropy; however, we do not obtain this unless $\int|\rho A|\,dx$ is bounded. In contrast to the logarithmic case, additional tools and more care are needed to bound such quantity. In particular, the Moser–Trudinger inequality is the natural inequality to help obtain control of $\int A\rho$. Similar inequalities has been extensively used for similar reasons, see for example [20]. However, as our system does not conserve mass the Moser–Trudinger inequality is not sufficient to prove global existence, except for the case when $\bar{B}(x) \equiv 0$. Physically, this means that criminals are not being created. In this case, we prove global existence of solutions to the fully-parabolic case when the velocity field depends linearly on the sensitivity function, provided the initial mass $\int\rho_0(x)\,dx$ is small enough.

Outline: In Section 2.1 we state the definitions and notation that will be used throughout the paper. Following, in Section 2.2 we provide a summary of the results. In Section 3 the local well-posedness theory is discussed. The global well-posedness theory with logarithmic sensitivity function is discussed in Section 4. The global well-posedness theory for the parabolic–parabolic for the system with linear sensitivity function and small initial mass is found in Section 5. Finally, we conclude with a discussion on the fully-nonlinear model developed in [11] in Section 6.

2. Definitions, notation, and summary of results

2.1. Definitions and notation

In this section we establish the notation and state the definitions necessary for the rest of the paper. We study (3) on a bounded domain $\Omega \subset \mathbb{R}^2$, with smooth boundary, with no-flux boundary condition on $\rho(x,t)$ and a *free boundary condition* on $A(x,t)$. More specifically,

$$\begin{aligned}&A(x,t) = \mathcal{G}_\eta(x,t) * (\rho(x,t) + \bar{A}(x)) \quad \text{and} \\ &(-\nabla\rho + \rho\nabla\chi(A))\cdot\vec{\nu}|_{\partial\Omega} = 0,\end{aligned} \tag{8}$$

where $\mathcal{G}_\eta(x,t)$ is the fundamental solution to the $\partial_t - \eta\Delta + 1$ in $\mathbb{R}^2 \times \mathbb{R}^+$, which will be made precise below in Definition 1 where we state the formal definition of weak solutions. By *free boundary condition* we mean that the boundary has no effect on the attractiveness value. Hence, we can define $A(x,t)$ in all of $\mathbb{R}^2$. Consider initial conditions

$$A(0,x) = A_0(x), \qquad \rho(0,x) = \rho_0(x), \tag{9}$$

such that $A_0(x)$, $\rho_0(x)$ are smooth. Furthermore, we also assume that $\bar{A}(x)$, $\bar{B}(x)$ are smooth with support in Ω and $\inf \bar{A}(x) > \underline{A} > 0$. This amount of regularity is excessive and can be reduced significantly; however, as this is not the aim of the paper we do not discuss this issue further. We seek to prove the existence of weak solutions, which use the heat kernel K_η on $\mathbb{R}^2$, given by

$$K_\eta(x,t) = \frac{1}{(4\eta\pi t)} e^{-\frac{|x|^2}{4\eta t}}. \tag{10}$$

Definition 1 (*Weak Solutions*)**.** Let $\rho(x,t) \in L^\infty(0,T; L^2\Omega) \cap L^2(0,T; H^1(\Omega))$, then (A,ρ) is a weak solution to (3) if

$$\int_\Omega \rho\psi = \int_\Omega \rho_0\psi(x,0) + \int_0^t \int_\Omega \rho\psi_t - \int_0^t \int_\Omega (\nabla\rho - \rho\nabla\chi(A)) \cdot \nabla\psi + \int_0^t \int_\Omega \left(\bar{B} - A\rho\right)\psi, \tag{11a}$$

$$A(x,t) = e^{-t}K_\eta * A_0(x) + \int_0^t e^{\tau - t} K_\eta(t-\tau) * \left(\rho(\tau,x) + \bar{A}(x)\right) ds, \tag{11b}$$

holds for all $\psi \in H^1(\Omega \times (0,T))$.

It is useful to define the following notation:

$$M_v(t) = \int v\, dx \quad \text{and} \quad \|v\|_p = \int_\Omega v^p dx.$$

When no confusion is possible we will omit the domain of integration. For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$, we define the $H^m(\Omega)$-norm as follows:

$$\|v\|_{H^m} := \left(\sum_{|\alpha| \le m} \|D^\alpha v\|_0^2\right)^{\frac{1}{2}}. \tag{12}$$

Since we are working extensively with different bounds and the constants are not always important, we introduce the notation $A \lesssim_{a,b,\dots} B$ to mean that there exists a positive constant c, which depends on $a, b, \dots$, such that $A \le cB$. This notation will be used when the constants are irrelevant and become tedious. Furthermore, we will suppress the dependence on the quantities $a, b, \dots$ when convenient and appropriate. Finally, as we will be working vastly with quantities that are controlled, but not conserved, we define $\mathcal{P}(t)$ to denote an arbitrary bounded function in time.

2.2. Summary of results

We begin by stating the local existence of weak solutions that satisfy boundary conditions (8). These boundary conditions imply that $A(x,t)$ is defined in all of $\mathbb{R}^2$ and so Duhamel's formulation for $A(x,t)$, given by (11b), is sensible. Our first result holds for both $\chi(A) = A$ and $\chi(A) = \log A$.

Theorem 1 (*Local Existence*)**.** *Given initial data (9) such that $A_0(x), \rho_0(x) \in H^1(\Omega)$ and $A_0(x) \ge \underline{A}$, there exist non-negative weak solutions $A(x,t)$, $\rho(x,t)$, in the sense of Definition 1, to the system (3), with $\chi(A)$ satisfying (4), on $[0,T]$ for some $T > 0$.*

We also know that the energy functional, $\mathcal{F}_{\log}(t)$, remains bounded for all solutions obtained in Theorem 1. More precisely, the following lemma provides an upper bound for the energy.

Proposition 1 (*Energy Control*)**.** *Let $A(x,t)$, $\rho(x,t)$ be the local solutions to the system (3) provided by Theorem 1, with $\chi(A) = \log A$, and with positive and finite initial energy, $\mathcal{F}_{\log}(0)$, such that*

$$(1 + \gamma + \gamma\eta)^2 \le 4\gamma(1+\eta), \tag{13}$$

with $\gamma > 0$. Then for all $t > 0$,

$$\mathcal{F}_{\log}(t) \le \mathcal{P}(t, A_0(x), \rho_0(x), \bar{A}(x), \bar{B}(x)), \tag{14}$$

where $\mathcal{P}$ remains bounded for all t. In particular, $\mathcal{F}_{\log}(t)$ remains bounded from above for all time.

This proposition is essential to obtain global control of the entropy, which proves to be key for the global well-posedness theory. We now state the global existence results for the logarithmic sensitivity function, $\chi(A) = \log A$.

Theorem 2 (*Global Existence for Logarithmic Sensitivity*)**.** *Let $A(x,t)$, $\rho(x,t)$ be the weak solution to (3) obtained in Theorem 1, with $\chi(A) = \log A$, then the solution is global in time.*

As mentioned earlier there is a relation between the system (1) the Keller–Segel model for chemotaxis. In two-dimensions when $\Omega = \mathbb{R}^d$ the Keller–Segel model (with linear sensitivity function) exhibits a critical mass phenomena: if the initial mass of the cell density is less than the sharp critical value, 8π, then the solution is global; on the other hand, if the initial mass is greater than 8π then the solutions are expected to blow-up in finite time. The proof of global existence of solutions to the Keller–Segel model relies on the fact that there is conservation of mass. Our next result is for the linear sensitivity function when $\bar{B}(x) \equiv 0$, as this enables us to obtain a global bound on the mass of the criminal density. Before stating this result we recall that Moser–Trudinger inequality for bounded domains.

Lemma 1 (*Generalized Moser–Trudinger Inequality [21, Proposition 2.3]*)**.** *Let $v \in H^1(\Omega)$, Ω with piecewise C^2 boundary and minimal interior angle of corners θ then*

$$\int_\Omega e^v \le C \exp\left(\frac{1}{|\Omega|}\left|\int_\Omega v dx\right| + \frac{1}{8\theta}\int_\Omega |\nabla v|^2\, dx\right).$$

Theorem 3 (*Global Existence for Linear Sensitivity*)**.** *There exist global solutions, $(\rho(x,t), A(x,t))$, to the system (3), with $\chi(A) = A$, and $\bar{B}(x) \equiv 0$, for initial data (9) that satisfies*

$$\int \rho_0(x)\, dx < 4\theta\eta, \tag{15}$$

with $\theta(\Omega)$ defined as in Lemma 1.

Finally, we state a local existence result for the parabolic–elliptic case with linear sensitivity function for reference in our future work. The proof of this result is standard and can a summary of the proof can be found in Appendix B.

Theorem 4 (*Local Existence for the Parabolic–Elliptic System*)**.** *Let $\rho_0(x) \in H^s(\Omega)$ there exists a unique solution $\rho(x,t) \in C(0, T_{\max}; H^s)$ to (B.1) on some maximal interval of existence. Furthermore, if $T_{\max} < \infty$ then*

$$\lim_{t \to T_{\max}} \|\rho(t)\|_\infty = \infty. \tag{16}$$

3. Local existence

The local existence theory heavily relies on the regularity properties of $A(x,t)$ obtained from the fact that $\mathcal{G}_\eta(x,t)$ is a smoothing operator. In particular, the following lemma states the L^p regularity estimates for the heat kernel.

Lemma 2. *In dimension n, the heat kernel, K_η, defined by (10) satisfies the following estimates for $p \geq 1$*

$$\|K(\cdot,t)\|_{L^p(\mathbb{R}^2)} \leq C_{p,\eta}\, t^{-\frac{n(p-1)}{2p}} \tag{17a}$$

$$\|\nabla K(\cdot,t)\|_{L^p(\mathbb{R}^2)} \leq C_{p,\eta}\, t^{-\frac{n(p-1)}{2p}-\frac{1}{2}}. \tag{17b}$$

The above estimates are obtained by direct computation and its proof can be found in Appendix C. After choosing a suitable functional space to work in, our aim is to use a fixed point theorem. Specifically, we will use Theorem 5, which is equivalent to the Schauder fixed point theorem (see [22]) to obtain a solution. Compactness will be obtained via the use of the Aubin–Lions Lemma, (Lemma 7) see also [23]. For completeness, the statements of these two results can be found in Appendix A. We now discuss the necessary a priori bounds. First, note that we have a polynomial bounds on the growth of the mass of both $A(x,t)$ and $\rho(x,t)$. Indeed, it is true that

$$M_\rho(t) \leq M_{\overline{B}}\, t + M_{\rho_0}, \tag{18a}$$

$$M_A(t) \leq C(M_{\overline{B}})t^2 + C(M_{\overline{A}}, M_{\rho_0})t + M_{A_0}. \tag{18b}$$

Formally, the bound (18a) can be obtained by integrating (3b) and taking its time derivative. One can then plug in (18a) into the evolution equation of the mass of $A(x,t)$, once again obtained by integrating (3a), and get (18b). The second lemma provides additional a priori bounds for the attractiveness value necessary for the proof of local existence. Fix $\alpha^* \in (5/6, 1)$ and

$$p' = \frac{6\alpha^*}{6\alpha^* - 5}, \tag{19}$$

note that $p' \in (6, \infty)$.

Lemma 3 (*A priori Estimates for the Attractiveness Value*)**.** *Let $A(x,t)$ satisfy (11b) with initial data $A(x,0) > \overline{A}$ for $x \in \Omega$ with $\rho(x,t) \in L^{p'}(0,T;L^2(\Omega))$ then the following a priori estimates hold.*

(i) $A(x,t) \geq \overline{A}(1+t)e^{-t}(1-e^{\frac{-R^2}{4\eta t}})\ \forall\ (t,x) \in \mathbb{R}^+ \times \Omega$.
(ii) $\sup_{0\leq t\leq T} \|\nabla A(\cdot,t)\|_6 \leq C + \mathcal{P}(T)\,\|\rho\|_{L^{p'}(0,T;L^2)}$.
(iii) $\sup_{0\leq t\leq T} \|A(\cdot,t)\|_\infty \leq C + \mathcal{P}(T)\,\|\rho\|_{L^3(0,T;L^2)}$.
(iv) $\sup_{0\leq t\leq T} \|\nabla \log A\|_6 \leq C(A_{\min}(T)) + C(A_{\min}(T))\mathcal{P}(T)\,\|\rho\|_{L^{p'}(0,T;L^2)}$

for all $t > 0$. Where, C a constant, $R > 0$ to be fixed later, $\mathcal{P}(t)$ represents an arbitrary function which is bounded in time, and $A_{\min}(t)$ a bounded function to be defined later.

Proof. Due to translation invariance it suffices to prove this for $x = 0$. Let $B = B(0,R)$ be the largest ball inscribed in Ω with radius R. Duhamel's formulation for the attractiveness value gives

$$\begin{aligned} A(0,t) &\geq e^{-t}\frac{1}{(4\eta\pi t)}\int_{\mathbb{R}^2} e^{\frac{-|y|^2}{4\eta t}} A_0(y)dy \\ &\quad + \int_0^t \frac{e^{\tau-t}}{(4\pi\eta(t-\tau))}\int_{\mathbb{R}^2} e^{\frac{-|y|^2}{4\eta(t-\tau)}}\overline{A}(y)dyd\tau \\ &\geq e^{-t}\frac{\overline{A}}{(4\pi\eta t)}\int_B e^{\frac{-|y|^2}{4\eta t}}dy \\ &\quad + \overline{A}\int_0^t \frac{e^{\tau-t}}{(4\pi\eta(t-\tau))}\int_B e^{\frac{-|y|^2}{4\eta(t-\tau)}}dyd\tau. \end{aligned}$$

Now, consider the change of variables $z = y/(\sqrt{4\eta t})$ then

$$\begin{aligned} A(0,t) &\geq e^{-t}\frac{\overline{A}}{\pi}\int_{|z|\leq \frac{R}{\sqrt{(4\eta t)}}} e^{-|z|^2}\,dz \\ &\quad + \overline{A}\int_0^t \frac{e^{\tau-t}}{\pi}\int_{|z|\leq\frac{R}{\sqrt{(4\eta(t-\tau))}}} e^{-|z|^2}\,dzd\tau \\ &= \overline{A}e^{-t}\left(1-e^{-\frac{R^2}{4\eta t}}\right) + \overline{A}\int_0^t e^{\tau-t}\left(1-e^{-\frac{R^2}{4\eta(t-\tau)}}\right)d\tau. \end{aligned}$$

On the interval $[0,t]$ we have that $e^{\tau-t} \geq e^{-t}$ and $e^{-\frac{R^2}{4\eta(t-\tau)}} \leq e^{-\frac{R^2}{4\eta t}}$; hence, we have that

$$\begin{aligned} A(0,t) &\geq \overline{A}(1+t)e^{-t}\left(1-e^{\frac{-R^2}{4\eta t}}\right) \\ &:= A_{\min}(t). \end{aligned} \tag{20}$$

Note that $A_{\min}(t) > 0$ for all t and so we obtain (i). For the next bounds we use the regularity of $K_\eta(x,t)$. Indeed,

$$\begin{aligned} \|\nabla A(x,t)\|_6 &\leq e^{-t}\left\|\nabla K_\eta(t) * A_0(x)\right\|_6 \\ &\quad + \int_0^t e^{\tau-t}\left\|\nabla K_\eta(t-\tau) * \left(\rho(\tau,x)+\overline{A}(x)\right)\right\|_6 ds, \\ &\leq C\left(\left\|\nabla K_\eta(t)\right\|_{\frac{3}{2}}, \|A_0(x)\|_2, \left\|\overline{A}(x)\right\|_2\right) \\ &\quad + \int_0^t e^{\tau-t}\left\|\nabla K_\eta(t-\tau)\right\|_{\frac{3}{2}}\|\rho\|_2\,d\tau \\ &\overset{(17b)}{\leq} C + \int_0^t (t-\tau)^{-5/6}\|\rho(\tau)\|_2\,d\tau \\ &\leq C + \left(\int_0^t (t-\tau)^{-\alpha^*}d\tau\right)^{1/\alpha^*}\|\rho\|_{L^{p'}(0,t;L^2)} \\ &\leq C + Ct^{1-\alpha^*}\|\rho\|_{L^{p'}(0,T;L^2)} \end{aligned}$$

where, p' is defined by (19). This proves (ii). The next bound is proved similarly,

$$\begin{aligned} \|A(x,t)\|_\infty &\leq e^{-t}\left\|K_\eta(t) * A_0(x)\right\|_\infty \\ &\quad + \int_0^t e^{\tau-t}\left\|K_\eta(t-\tau) * \left(\rho+\overline{A}(x)\right)\right\|_\infty d\tau \\ &\overset{(17a)}{\leq} C\left(\|A_0(x)\|_\infty, \left\|\overline{A}(x)\right\|_\infty\right) \\ &\quad + C\int e^{\tau-t}(t-\tau)^{-1/2}\|\rho\|_2\,d\tau \\ &\leq C_1 + C\left(\int_0^t (\tau-t)^{-\frac{p'}{2(p'-1)}}d\tau\right)^{\frac{p'-1}{p'}}\left(\int \|\rho(\tau)\|_2^{p'}\,d\tau\right)^{1/p'} \\ &\leq C_1 + \mathcal{P}(t)\,\|\rho\|_{L^{p'}(0,t;L^2)}. \end{aligned}$$

Note that that the above bound is finite as $\frac{p'}{2(p'-1)} < 1$ since $p' > 6$. The last bound (iv) follows from (i) and (ii). □

The a priori estimates proved in the previous theorem are determined from classical theory as will be seen in the proof of local existence. This also defines the functional framework where we build our solutions.

Proof of Theorem 1. Define

$$\begin{aligned} \mathcal{Y}_\mathcal{T} = \Big\{ & u(x,t) \in L^{p'}(0,T;L^2(\Omega)) \mid \nabla u \\ & \in L^2(0,T;L^2(\Omega)),\ \partial_\nu u|_\Omega = 0\Big\} \end{aligned}$$

where p' is defined by (19) and the norm defined by

$$\|u\|_{\mathcal{Y}_{\mathcal{T}}} = \left(\int_0^T \|u(t)\|_2^{p'}\ dt\right)^{1/p'}.$$

Let $\tilde{\rho} \in \mathcal{Y}_{\mathcal{T}}$ be given and consider the linear problem

$$A_t = \eta\Delta A - A + \tilde{\rho} + \overline{A}(x) \tag{21a}$$

$$\rho_t = \nabla\cdot(\nabla\rho - \rho\nabla\chi(A)) - A\rho + \overline{B}(x). \tag{21b}$$

Let $\tilde{A}$ be the solution to (21a), by classical theory we know that such a solution exists and its formulation is given by (11b). Furthermore, (21b) is a uniformly parabolic linear problem. Classical theory for linear uniformly parabolic systems provides us with a solution to (21b) (see [24, Chapter III]). Indeed, one only needs the following inclusions

$$\left|\nabla\chi(\tilde{A})\right|^2 \in L^r(0,T;L^q) \quad \text{for } 1/r + n/(2q) \le 1$$

$$\frac{\partial\chi(\tilde{A})}{\partial\nu} \in L^{r'}(0,T;L^{q'}) \quad \text{for } 1/r' + (n-1)/(2q') \le 1/2,$$

where n is the dimension. Note that the a priori estimates provided by Lemma 3 guarantee that these conditions are satisfied. Now, we define the map $\mathcal{S} : \mathcal{Y}_{\mathcal{T}} \to \mathcal{Y}_{\mathcal{T}}$ by $\mathcal{S}(\tilde{\rho}) = \rho$, where ρ is the solution to (21b) where, $A(x,t)$ is given by Duhamel's formula (11b). Also, let $\mathcal{X}_R^T = \left\{v \in \mathcal{Y}_{\mathcal{T}} : \|v\|_{\mathcal{Y}} \le R, \|\nabla v\|_{L^2} \le M\right\}$. We prove that the hypothesis of Theorem 5 are satisfied in three steps. We first show that $\mathcal{X}_R^T$ is invariant under $\mathcal{S}$ for small enough T and R. Next, we show that $\mathcal{S}$ is continuous on $L^{p'}(0,T;L^2(\Omega))$. Finally, we prove that $\mathcal{S}$ is a compact map.

Step 1: *Invariance of* $\mathcal{X}_R^T$ *under* $\mathcal{S}$. Let $\tilde{A}$ be defined by (11b) and note that the weak solutions satisfy

$$\frac{1}{2}\int_\Omega \rho^2 + \int_0^t\int_\Omega |\nabla\rho|^2 + \int_0^t\int_\Omega \tilde{A}\rho^2 = \frac{1}{2}\int_\Omega \rho_0^2 + \int_0^t\int_\Omega \rho\nabla\chi(\tilde{A})\cdot\nabla\rho + \int_0^t\int_\Omega \overline{B}\rho.$$

This equality can be formally written in the differential form

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 + \int_\Omega |\nabla\rho|^2 + \int_\Omega \tilde{A}\rho^2 \le \int_\Omega \rho\nabla\chi(\tilde{A})\cdot\nabla\rho + \left\|\overline{B}(x)\right\|_\infty M_\rho(t). \tag{22}$$

To bound the first term on the right hand side of (22) we make use of the following Gagliardo–Nirenberg type inequality

$$\|u\|_3 \lesssim C_\Omega \|u\|_{H^1}^{1/3}\|u\|_2^{2/3}. \tag{23}$$

This estimate can be obtained directly from Theorem 6, which can be found in Appendix A. Now, general Hölder's inequality gives

$$\begin{aligned}\left\|\rho\nabla\rho\cdot\nabla\chi(\tilde{A})\right\|_1 &\le \|\nabla\rho\|_2\|\rho\|_3\left\|\nabla\chi(\tilde{A})\right\|_6\\ &\overset{(23)}{\le} C_\Omega\|\nabla\rho\|_2\|\rho\|_{H^1}^{1/3}\|\rho\|_2^{2/3}\left\|\nabla\chi(\tilde{A})\right\|_6\\ &\le C_\Omega\|\rho\|_{H^1}^{4/3}\|\rho\|_2^{2/3}\left\|\nabla\chi(\tilde{A})\right\|_6\\ &\le \frac{2}{3}\|\rho\|_{H^1}^2 + C\|\rho\|_2^2\left\|\nabla\chi(\tilde{A})\right\|_6^3.\end{aligned}$$

Combining the above estimate with (22) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 + \frac{1}{3}\int_\Omega |\nabla\rho|^2 + \int_\Omega A\rho^2 \le \alpha(t)\int \rho^2 + \gamma(t), \tag{24}$$

where $\alpha(t) = C\left\|\nabla\log\tilde{A}(t)\right\|_6^3 + 2/3$ and $\gamma(t) = \left\|\overline{B}(x)\right\|_{L^\infty} M_\rho(t)$. Note that from Lemma 3 we see that both $\alpha(t)$ and $\gamma(t)$ are bounded over $[0,T]$; hence, we can apply Grönwall's inequality to obtain

$$\|\rho(t)\|_2^2 \le Ce^{\int_0^T\alpha(s)ds}\left(\|\rho_0\|_2^2 + \int_0^T \gamma(s)ds\right) = C_1(T).$$

Raising to the exponent $p'/2$ and integrating over time gives

$$\|\rho\|_{L^{p'}(0,T;L^2(\Omega))} \le T^{1/p'}C_1(T)^{1/2}. \tag{25}$$

Using the a priori bounds from Lemma 3 the set is invariant for large enough R and small enough T.

Remark 1. Note that the differential inequality (24) provides the additional bounds on $\|\nabla\rho\|_{L^2(0,T;L^2(\Omega))}$.

Step 2: *Continuity of* $\mathcal{S}$. Next, we show that the map $\mathcal{S}$ is continuous in $\mathcal{Y}_{\mathcal{T}}$. Given $\rho_1,\rho_2 \in \mathcal{Y}_{\mathcal{T}}$ we define $u_i = \mathcal{S}(\rho_i)$ for $i = 1,2$. Also, let A_i be defined through (11b). Consider the evolution equation of $\|u_1 - u_2\|_2$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(u_1-u_2)^2dx &= \int(u_1-u_2)\partial_t(u_1-u_2)dx\\ &= \int(u_1-u_2)\Delta(u_1-u_2)\\ &\quad - \int(u_1-u_2)\nabla\cdot(u_1\nabla\chi(A_1) - u_2\nabla\chi(A_2))\\ &\quad - \int(u_1-u_1)(A_1u_1 - A_2u_2)\\ &:= I_1 + I_2 + I_3.\end{aligned}$$

The first term is bounded easily using integration by parts. Indeed,

$$I_1 = -\int|\nabla(u_1-u_2)|^2\,dx.$$

The third term is also easily bounded

$$\begin{aligned}I_3 &\le -\int A_1(u_1-u_2)^2dx + \int u_2(A_1-A_2)(u_1-u_2)dx\\ &\le -\int A_1(u_1-u_2)^2dx + \frac{1}{2}\int(u_1-u_2)^2dx\\ &\quad + \frac{1}{2}\|A_1-A_2\|_\infty\int u_2^2\,dx.\end{aligned}$$

Similar to the proof of (iii) in Lemma 3 we get that

$$\|A_1(\cdot,t) - A_2(\cdot,t)\|_\infty \le \mathcal{P}(t)\,\|\rho_1-\rho_2\|_{L^{p'}(0,t,L^2)}.$$

Unfortunately, the second term is not as easy to bound.

$$\begin{aligned}I_2 &= \int\nabla(u_1-u_2)\cdot(u_1\nabla\chi(A_1) - u_2\nabla\chi(A_2))\\ &= \int(u_1-u_2)\nabla(u_1-u_2)\cdot\nabla\chi(A_1)\\ &\quad + \int u_2\nabla(u_1-u_2)\cdot\nabla(\chi(A_1)-\chi(A_2))\\ &:= R_1 + R_2.\end{aligned}$$

We can bound R_1 similarly to (24)

$$R_1 \le \beta_1(t)\int(u_1-u_2)^2 + \frac{2}{3}\int|\nabla(u_1-u_2)|^2, \tag{26}$$

where $\beta_1(t) = c\|\nabla\chi(A_1)\|_6^3$, which is integrable by (iv) in Lemma 3. Furthermore,

$$\begin{aligned} R_2 \le\ & \frac{1}{4}\int |\nabla(u_1-u_2)|^2\,dx \\ & + C\,\|u_2\|_3^2 \left\|\nabla\big(\chi(A_1)-\chi(A_2)\big)\right\|_6^2 \\ \overset{(23)}{\le}\ & \frac{1}{4}\|\nabla(u_1-u_2)\|_2^2 \\ & + C\,\|u_2\|_{H^1}^{2/3}\|u_2\|_2^{4/3}\left\|\nabla\big(\chi(A_1)-\chi(A_2)\big)\right\|_6^2. \end{aligned}$$

Now, condition (4b) allows us to conclude, following the proof of Lemma 3 that

$$\begin{aligned} R_2 \le\ & \frac{1}{4}\|\nabla(u_1-u_2)\|_2^2 \\ & + \alpha_1(t)\,\|\nabla(A_1-A_2)\|_6 + \alpha_2(t)\,\|(A_1-A_2)\|_\infty \\ \le\ & \frac{1}{4}\|\nabla(u_1-u_2)\|_2^2 + \alpha_3(t)\,\|\rho_1-\rho_2\|_{L^{p'}(0,t;L^2)} \end{aligned} \tag{27}$$

where $\alpha_1(t) = \frac{C}{A_{\min}^2(t)}\|u_2\|_{H^1}^{1/2}\|u_2\|_2^{1/2}\|A_2\|_\infty$, $\alpha_2(t) = \frac{C}{A_{\min}^2}\|u_2\|_{H^1}^{1/2}\|u_2\|_2^{1/2}\|\nabla A_1\|_6$, and $\alpha_3(t) = C(\alpha_1(t)+\alpha_2(t))\mathcal{P}(t)$. Combining (26) and (27) gives a final bound for I_2

$$\begin{aligned} I_2 \le\ & \beta_1(t)\int (u_1-u_2)^2dx + \frac{3}{4}\int |\nabla(u_1-u_1)|^2\,dx \\ & + \alpha_3(t)\,\|\rho_1-\rho_2\|_{L^{p'}(0,t;L^2)}. \end{aligned} \tag{28}$$

Finally,

$$\begin{aligned} I_3 \le\ & -\int A_1(u_1-u_2)^2dx \\ & + \frac{1}{2}\int u_2^2dx\,\|A_1-A_2\|_\infty^2 + \frac{1}{2}\int (u_1-u_2)^2dx \\ \le\ & \frac{1}{2}\int (u_1-u_2)^2dx + C\,\|u_2\|_2^2\,\mathcal{P}(t)\,\|\rho_1-\rho_2\|_{L^{p'}(0,t;L^2)}. \end{aligned} \tag{29}$$

We combine (28) and (29) and using the fact that $I_1 = -\|\nabla(u_1-u_2)\|_2^2$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_1-u_2\|_2^2 \le\ & \beta_2(t)\,\|u_1-u_2\|_2^2 \\ & + \beta_3(t)\,\|\rho_1-\rho_2\|_{L^{p'}(0,t;L^2)}. \end{aligned} \tag{30}$$

Another application of Grönwall's inequality we have, since $\beta_2(t)$ and $\beta_3(t)$ are integrable on $[0,T]$ then $\|u_1-u_2\|_2$ depends continuously on $\|\rho_1-\rho_2\|_{L^{p'}(0,T;L^2)}$.

Step 3: *Compactness of* $\mathcal{S}$

Consider $u = \mathcal{S}(\rho)$ for $\rho \in \mathcal{Y}_\mathcal{T}$ and recall from Remark 1 the bound on $\|\nabla u\|_{L^2(0,T;L^2(\Omega))}$. Furthermore, $u_t \in L^2(0,T;H^{-1}(\Omega))$. A variant of Aubin–Lions compactness lemma gives the pre-compactness of $\mathcal{S}$ in $L^{p'}(0,T;L^2(\Omega))$. Since $u \in L^\infty(0,T;L^2)$ then S is pre-compact in $\mathcal{Y}_\mathcal{T}$ as well, and we obtain a local in time solution to (3). Applying Theorem 5 gives a solution $\rho \in \mathcal{Y}_\mathcal{T}$ such that $\rho = \mathcal{S}(\rho)$. □

4. Global existence for logarithmic sensitivity function

In the previous section we proved the existence of a local weak solution to the system (3). As a result of the proof of Theorem 1 we obtain as a corollary that $\rho(x,t)$ can be continued in time as long as its L^2-norm remains bounded. A key ingredient missing in the analysis of the system (1), studied in [11,18], was an energy functional that could be controlled for all time. An advantage (3) has $\mathcal{F}_{\log}(t)$ defined by (6) is bounded above for all time for any $A(x,t)$, $\rho(x,t)$ solutions of (3). Note that $\mathcal{F}_{\log}(t)$ is not necessarily dissipated; however, an upper bound proves to be sufficient for the purpose of our analysis. We take advantage of the specific form of the sensitivity function (3), *i.e.* $\nabla\log A$, to obtain such control. This bound is stated more precisely in the following proposition.

Proof of Proposition 1. We prove the proposition formally but note that the computations can be made rigorous using standard regularization techniques, which we leave out for clarity. Consider the evolution equation of $\mathcal{F}_{\log}(t)$, we drop the domain of integration for notational simplicity,

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{\log}(t) =\ & \int (\log\rho - \gamma\log A) \\ & \times\Big(\nabla\cdot(\nabla\rho - \rho A^{-1}\nabla A) - A\rho + \overline{B}\Big) \\ & + \int \rho_t - \gamma\int \rho A^{-1}A_t \\ =\ & -\int \frac{1}{\rho}|\nabla\rho|^2 + (1+\gamma)\int A^{-1}\nabla\rho\cdot\nabla A \\ & -\gamma\int \rho\,|\nabla A|^2 A^{-2} + \int \overline{B} - \int A\rho \\ & + \underbrace{\int (\overline{B}-A\rho)(\log\rho-\gamma\log A)}_{\mathcal{R}_1} \\ & \underbrace{-\gamma\int \rho A^{-1}(\eta\Delta A - A + \rho + \overline{A})}_{\mathcal{R}_2}. \end{aligned}$$

We can simplify the last term of the above inequality by integrating by parts

$$\begin{aligned} -\mathcal{R}_2 =\ & \gamma\eta\int A^{-1}\nabla\rho\cdot\nabla A - \gamma\eta\int \rho A^{-2}|\nabla A|^2 \\ & + \gamma\int\rho - \gamma\int \rho^2A^{-1} - \gamma\int \overline{A}(x)\rho A^{-1} \\ \le\ & \gamma\eta\int A^{-1}\nabla\rho\cdot\nabla A - \gamma\eta\int \rho A^{-2}|\nabla A|^2 + \gamma M_\rho(t). \end{aligned}$$

Substituting this back into the evolution equation gives

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{\log}(t) + \int A\rho \le\ & -\int\frac{1}{\rho}|\nabla\rho|^2 + (1+\gamma+\gamma\eta) \\ & \times\int A^{-1}\nabla\rho\cdot\nabla A + \mathcal{R}_2 + \mathcal{R}_1 \\ \le\ & \left[\frac{(1+\gamma+\gamma\eta)^2}{4} - \gamma(1+\eta)\right] \\ & \times\int \rho\,|\nabla A|^2A^{-2} + M_{\overline{B}} + \gamma M_\rho(t) + \mathcal{R}_1 \\ \overset{(13)}{\le}\ & M_{\overline{B}} + \gamma M_\rho(t) + \mathcal{R}_1. \end{aligned}$$

The only term left to control is $\mathcal{R}_1$. First, let us discuss the term

$$D(t) := -\int A\rho(\log\rho - \gamma\log A).$$

This term comes from the decay in the criminal equations; hence, we expect it to help in preventing blow up. Nevertheless, it requires some care. Note that for t such that $\mathcal{F}_{\log}(t) > 0$ then $D(t) \le -A_{\min}(t)\mathcal{F}_{\log}(t)$, and so in this case it can be ignored. On the other hand, if $\mathcal{F}_{\log}(t) < 0$ then $D(t)$ will push for $\mathcal{F}_{\log}(t)$ to approach zero. Hence, either way this term is helping the energy remain bounded in absolute value. Given that we are trying to prove an upper bound we consider the case when $\mathcal{F}_{\log}(t) > 0$. Therefore,

$$\mathcal{R}_1 = \int \overline{B}(\log\rho - \gamma\log A) - \int A\rho(\log\rho - \gamma\log A)$$
$$\le \left\|\overline{B}\right\|_\infty M_\rho(t) + \gamma\,|\log A_{\min}(t)|\,M_{\overline{B}} - A_{\min}(t)\mathcal{F}_{\log}(t).$$

Since the initial energy is positive then the second term in the above equality will contribute to the decay of the energy. Combining everything gives

$$\frac{d}{dt}\mathcal{F}_{\log}(t) + \int A\rho \le \Gamma(t) - A_{\min}(t)\mathcal{F}_{\log}(t),$$

where $\Gamma(t) = M_{\overline{B}}\left(1+\frac{1}{2}|\log A_{\min}(t)|\right) + M_\rho(t)\left(\frac{1}{2}+\left|\overline{B}\right|_\infty\right)$. The final step is to integrate this in time and obtain

$$\mathcal{F}_{\log}(t) + \int_0^t\int A\rho \le e^{-\int_0^t A_{\min}(s)\,ds} \times\left(\mathcal{F}_{\log}(0) - \frac{1}{A_{\min}(t)}\int_0^t \Gamma(s)ds\right) + \frac{1}{A_{\min}(t)}\int_0^t \Gamma(s)ds.$$

Since, by definition, $\Gamma(t)$ is integrable on $[0, T]$ the above integral inequality proves (14). □

Remark 2. We do not know how general the condition (13) is. Two pairs of γ and η that satisfy the inequality are $\gamma = 1/2$ and $\eta = 1$ and for $\eta = 0$ and $c = 1$.

Remark 3. The proof above gives us bounds on $\int A^2dx$ and $\int_0^t\int |\nabla A|^2\,dx$ on any finite time interval. Furthermore, this is a naive estimate as we do not make use of all the terms with a useful sign, for example the negative term $\int A\rho$.

Unfortunately, the upper bound on the energy, $\mathcal{F}_{\log}(t)$, proved above does not guarantee that the solutions will exist globally in time. Indeed, we need control of $\int|\rho\log\rho|\,dx$ to be able to continue our solution; hence, the importance of the lower bound. Below, we state a simple corollary to Proposition 1.

Lemma 4 (*Lower Bound of Entropy*)**.** *Let $A(x,t)$ and $\rho(x,t)$ be the weak solutions to (3) obtained in Theorem 1 then $\int\rho\,|\log\rho|\ \ dx \le \mathcal{P}(t) < \infty$, for all $t > 0$.*

Proof. Fix $t^* \ge 0$ and for notational simplicity we suppress the time dependence and define $M_\rho = M_\rho(t^*)$ and the measure $\frac{\rho}{M_\rho(t*)}$. Note that $\frac{1}{M_\rho}\int\rho(x,t^*)dx = 1$ and we can apply Jensen's inequality.

$$\int\rho(\log\rho - \log A)dx = -M_\rho\int\frac{\rho}{M_\rho}\left(\log\frac{A}{\rho}\right)dx$$
$$\ge -M_\rho\log\left(\int\frac{A}{M_\rho}dx\right)$$
$$= M_\rho\log M_\rho - M_\rho\log M_A.$$

In particular this implies that

$$-\frac{1}{2}\int\rho\log\rho \le -\frac{1}{2}\int\rho\log A - \gamma M_\rho\log M_\rho + \frac{1}{2}M_\rho\log M_A.$$

This gives us a bound on the entropy $\int\rho\log\rho\ dx$. Indeed,

$$\frac{1}{2}\int\rho\log\rho = \int\rho\log\rho - \frac{1}{2}\rho\log\rho$$
$$\le \int\rho\left(\log\rho - \frac{1}{2}\log A\right)dx + \mathcal{P}_2(t)$$
$$= \mathcal{F}_{\log}(t) + \mathcal{P}_2(t)$$

where $\mathcal{P}_2(t) := \frac{1}{2}M_\rho(t)\left(\log M_A(t) - \log M_\rho(t)\right)$. □

Lemma 4 gives us a bound on the entropy for any finite time interval, which turns out to be key in extending the solution to global ones.

Remark 4. We can easily see the importance of the logarithmic sensitivity function via the Young type inequality

$$\int|uv|\ \ dx \le \int u\log u\ dx + \int e^{v-1}dx.$$

One can obtain a stronger bound that provided by the lemma above. Indeed, for the logarithmic case we have $\int|\rho\log A| \le \int\rho\log\rho dx + e^{-1}\int A$. Then

$$\frac{1}{2}\int\rho\log\rho \le \int\rho\log\rho - \frac{1}{2}\rho\log\rho \le \mathcal{F}_{\log}(t) + \frac{e^{-1}}{2}M_A(t).$$

The importance of the bound on the entropy will begin to become apparent after we state a sequence of lemmas. The first is a logarithmic Sobolev embedding, which is a variant of a well-known result (see for example [4]).

Lemma 5 (*Logarithmic Sobolev Embedding*)**.** *Let $\epsilon(t) > 0$ be given and let w satisfy the hypothesis of Theorem 6 then the following logarithmic Sobolev embedding holds*

$$\|w\|_4^4 \le \epsilon(t)\,\|w\|_{H^1}^3\,\|w\log w\|_1 + C_\epsilon(t)\,\|w\|_1^4. \tag{31}$$

Proof. Let $N(t) > 1$ such that $\epsilon(t) \ge (\log N(t))^{-1}$, for example $N(t) = e^{\frac{1}{\epsilon(t)}}$, and define

$$\phi_{N(t)}(k) = \begin{cases} 0 & |k| \le N(t)\\ 2\,(|k| - N(t)) & N(t) < |k| \le 2N(t)\\ |k| & |k| > 2N(t).\end{cases}$$

Note that $\|w\|_4^4 \lesssim \left\||w| - \phi_{N(t)}(w)\right\|_4^4 + \left\|\phi_{N(t)}(w)\right\|_4^4$ and

$$\left\||w| - \phi_{N(t)}(w)\right\|_4^4 \le \int_{|w|\le 2N(t)}|w|^4\,dx \le (2N(t))^3\int|w|\,dx.$$

Once more, we make use of the extended Sobolev inequality for $n = 2$ (see [25] or Theorem 6 in Appendix A) $\left\|\phi_{N(t)}(w)\right\|_4^4 \lesssim \left\|\phi_{N(t)}(w)\right\|_{H^1}^3\left\|\phi_{N(t)}(w)\right\|_1$. The L^1-norm of $\phi_{N(t)}$ can be bounded as follows

$$\int\phi_{N(t)}(w)dx$$
$$\le \frac{1}{\log N(t)}\int_{|w|\ge N(t)} w\log w dx \le \epsilon(t)\,\|w\log|w|\|_1.$$

Furthermore, let $v' = \partial_w v$ then the H^1-norm can be bounded by

$$\left\|\phi_{N(t)}(w)\right\|_{H^1}^3 = \left(\left\|\phi'_{N(t)}\nabla w\right\|_2^2 + \left\|\phi_{N(t)}(w)\right\|_2^2\right)^{3/2}$$
$$\lesssim \|\nabla w\|_2^3 + 2\,\|w\|_2^3$$
$$\lesssim \|w\|_{H^1}^3.$$

Above we used that $\phi'_{N(t)}$ and $\phi_{N(t)}$ can be bounded. Letting $C_\epsilon(t) \ge (N(t))^2$ concludes the proof. □

The second lemma states that control of the entropy is enough to control $\|\nabla A\|_2$. Also, (31) will enable us to use the Sobolev embedding, which is critical in the two-dimensional case.

Lemma 6 (*Global Bound on the L^2-norm of Gradient of Attractiveness*)**.** *Let $A(x,t)$ be a solution to (3a) obtained from Theorem 1 then $\|\nabla A\|_2$ remains bounded for all $t \in [0, T]$ for any $T < \infty$.*

Control of the entropy suffices to control $\|\nabla A\|_2$. Heuristically, we obtain this by studying the integral representation of $A(x,t)$ by passing the gradient to the heat kernel and using the bound on the entropy. Note that in higher dimension the entropy does not help us control this quantity and so it is critical that we work in $\mathbb{R}^2$.

Proof of Theorem 2. From the proof of local existence (Theorem 1) we see that we can continue our solution provided we maintain control of the $\|\rho(t)\|_2$. Now, recall that the weak solutions satisfy

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 + \int_\Omega |\nabla\rho|^2 &= \int_\Omega \rho\nabla\log A\cdot\nabla\rho + \int_\Omega \bar{B}\rho \\ &\le \|\nabla\rho\|_2\,\|\nabla\log A\|_4\,\|\rho\|_4 + \mathcal{P}(t) \\ &\le \frac{1}{2}\|\nabla\rho\|_2^2 + \frac{1}{2}\|\nabla\log A\|_4^2\,\|\rho\|_4^2 + \mathcal{P}(t) \\ &\le \frac{1}{2}\|\nabla\rho\|_2^2 + C(t)\,\|\rho\|_4^{8/3} + \frac{1}{4C(t)}\|\nabla\log A\|_4^8 + \mathcal{P}(t).\end{aligned}$$

The inequality (31) gives that $\|\rho\|_4^{8/3} \le \epsilon(t)\,\|\rho\|_{H^1}^2\,\|\rho\log\rho\|_1 + C_\epsilon(t)M_\rho(t)$. Let $\epsilon(t) = \frac{1}{4C(t)}$, then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 + \frac{1}{2}\int_\Omega|\nabla\rho|^2 \le \frac{1}{4}\|\rho\|_{H^1}^2 + \frac{1}{4C(t)}\|\nabla\log A\|_4^8 + \mathcal{P}(t).$$

Note that the integral above is taken over the bounded domain Ω. However, A is defined over $\mathbb{R}^2$ and $\|\nabla A\|_{L^4(\Omega)} \le \|\nabla A\|_{L^4(\mathbb{R}^d)}$, which we can bound using a homogeneous Gagliardo–Nirenberg inequality (see [26])

$$\begin{aligned}\|\nabla A\|_4 &\lesssim \left\|D^3A\right\|_2^{1/4}\|\nabla A\|_2^{3/4} \\ &\lesssim \left\|D^3A\right\|_2^{1/4}.\end{aligned}$$

From the lower bound of $A(x,t)$ we have that

$$\|\nabla\log A\|_4^8 \le \left\|\frac{1}{A_{\min}(t)}\right\|_\infty^2\|\nabla A\|_4^8 := C_A(t)\,\|\nabla A\|_4^8.$$

Before choosing $C(t)$ note that we have that $\int_0^t\int\left|D^3A\right|^2dxds \le C\int_0^t\int|\nabla\rho|^2dxds$. Indeed,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\Delta A|^2dx &\le -\eta\int\left|D^3A\right|^2dx - \int|\Delta A|^2 \\ &\quad - \int\nabla\rho\cdot\nabla(\Delta A) \\ &\le -c\int\left|D^3A\right|^2dx - \int|\Delta A|^2 + C\int|\nabla\rho|^2.\end{aligned}$$

Hence, if we chose $C(t)$ such that $C_A(t) \le 4C(t)$ we have the bound

$$\begin{aligned}\frac{1}{4C(t)}\|\nabla\log A\|_4^8 &\le \frac{C_A(t)}{4C(t)}\|\nabla A\|_4^8 \\ &\le \left\|D^3A\right\|_2^2\|\nabla A\|_2^6 \\ &\le C\left\|D^3A\right\|_2^2.\end{aligned}$$

Combining all of these estimates gives

$$\begin{aligned}\|\rho(t)\|_2^2 + \frac{1}{4}\int_0^t\|\nabla\rho\|_2^2 &\le c\int_0^t\|\rho\|_2^2 \\ &\quad + C\int_0^t\left\|D^2A\right\|_2^2 + \int_0^t\mathcal{P}(s) \\ &\le C\int_0^t\|\rho\|_2^2 + \int_0^t\mathcal{P}(s).\end{aligned}$$

An application of Grönwalls inequality for integral form then concludes the proof. □

5. Linear sensitivity function

In this section we study (3) with linear sensitivity function, $\chi(A) = A$. Recall that this system has a controlled energy given by (7), that is this energy remains bounded along trajectories of the dynamical system associated with the system. This is stated more precisely in the next proposition.

Proposition 2 (*Bounds on Energy for the Linear Sensitivity Function*). *Let $A(x,t),\ \rho(x,t)$ be solutions (3) with $\chi(A) = A$ and initial data, $(A_0(x),\ \rho_0(x)) \ge (\bar{A},0)$, such that $\mathcal{F}_L(0)$ is positive and finite then $\mathcal{F}_L(t)$ is bounded from above $t > 0$.*

Proof.

$$\begin{aligned}\frac{d}{dt}\mathcal{S}_L(t) &= \int(\log\rho - A)\left(\nabla\cdot(\nabla\rho - \rho\nabla A) - A\rho + \bar{B}(x)\right)dx \\ &\quad + \int\rho_t - \int A_t\rho\,dx \\ &= -\int\rho\,|\nabla(\log\rho - A)|^2\,dx \\ &\quad + \int(\log\rho - A)(\bar{B} - A\rho)dx \\ &\quad + \int\bar{B}dx - \int A\rho dx - \int A_t\rho\,dx.\end{aligned}$$

Now, multiplying (3a) by A_t and integrating gives

$$\begin{aligned}\int|A_t|^2\,dx &= -\frac{\eta}{2}\frac{d}{dt}\int|\nabla A|^2\,dx - \frac{1}{2}\frac{d}{dt}\int A^2dx \\ &\quad + \int\rho A_t + \int\bar{A}A_tdx.\end{aligned}$$

From this we see that the time evolution of $\mathcal{F}_L(t)$ is bounded as follows

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_L(t) &\le -\int\rho\,|\nabla(\log\rho - A)|^2\,dx - \int|A_t|^2\,dx - \int A\rho dx \\ &\quad + M_{\bar{B}} + \int(\log\rho - (A))(\bar{B} - A\rho)dx.\end{aligned}$$

The last term in the above inequality can be bounded as was done in Section 4 and we conclude. □

As in the case with logarithmic sensitivity function we have an upper bound on $\mathcal{F}_L(t)$. However, recall that control of $\int\rho\,|\log\rho|\,dx$ is ultimately what provided the global existence. To obtain this we need a lower bound on $\mathcal{F}_L(t)$ as well. To obtain this bound we will have to use more machinery as will be shown next.

Proof of Theorem 3. We make use of Lemma 1 to obtain a bound for the entropy provided the mass of the initial data is small enough. The objective now is to try to obtain a lower bound for $\mathcal{S}_L$, following the proof for the lower bound of $\mathcal{F}_{\log}(t)$ we get that for a fixed t^*, which we drop for notational simplicity,

$$\begin{aligned}\mathcal{S}_L(t^*) &= -M_\rho\int\frac{\rho}{M_\rho}\log\left(\frac{e^A}{\rho}\right)dx \\ &\ge -M_\rho\log\left(\frac{1}{M_\rho}\int e^Adx\right) \\ &\ge M_\rho\log M_\rho - M_\rho\log\int e^Adx.\end{aligned}$$

Immediately we see that this lower bound is not as useful as the one obtained from the logarithmic sensitivity function. We use Lemma 1 to bound the last term in the above inequality

$$\log\left(\int e^{A}dx\right) \leq \log\left\{C\exp\left(\frac{1}{|\Omega|}\left|\int_{\Omega}A\right| + \frac{1}{8\theta}\int|\nabla A|_2^2\right)\right\}$$
$$\leq \log C + \frac{1}{|\Omega|}M_A + \frac{1}{8\theta}\int|\nabla A|_2^2.$$

We conclude that

$$0 \leq \mathcal{S}_L(t) + M_\rho(t)C + \frac{1}{|\Omega|}M_A(t)M_\rho(t) + \frac{M_\rho(t)}{8\theta}|\nabla A|_2^2 - M_\rho(t)\log M_\rho(t)$$
$$= \mathcal{F}_L + \left(\frac{M_\rho(t^*)}{8\theta} - \frac{\eta}{2}\right)\int|\nabla A|^2\,dx - \frac{1}{2}\int|A|^2\,dx + \mathcal{P}(t).$$

Therefore, if $\rho_0(x)$ satisfies (15) then (18a) and $\overline{B}(x) \equiv 0$ implies that

$$M_\rho(t) < 4\theta\eta, \tag{32}$$

for all positive time. Hence, the energy $\mathcal{F}_L$ controls $|\nabla A|_2$, which then provides controls of $\int|\rho\log\rho|\,dx$. The remaining of the proof follows that of Theorem 2. □

6. Discussion

In the above sections we studied a system based on the assumption that the attractiveness value of a house increased proportionally to the criminal density. In Section 4 we saw that the logarithmic sensitivity function is sufficient to prevent finite time blow-up. Recall that in that section we did not capitalize on the fact that criminals were removed when they committed a crime and that the result was true for all of the initial mass. However, in Section 5 we proved that starting out with a small enough initial mass was sufficient to prevent blow-up even with a linear sensitivity function (with additional assumptions on the system). From numerical observations we believe that the result should hold for all initial mass; hence, the complications are simply technical. It is reasonable to believe that this should be the case due to the fact that we never took advantage of the decay terms in the criminal density quantity. One can claim that the system proposed by Short and collaborators, (2), has two mechanisms to prevent finite time blow-up, that is the logarithmic dependence of the velocity field on the sensitivity function and the removal of criminals when a crime is committed. This might seem excessive; however, recall that this model assumed the 'repeat and near-repeat victimization effect'. This leads to the nonlinear growth in the attractiveness value equations. We conjecture that both of the blow-up prevention mechanisms discussed here are essential for the global well-posedness of the original system; however, this will be the addressed in more detail in further work.

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Appendix A. Auxiliary theorem

In this section we state some classical results that are used in this paper. We begin with the fixed point theorem [22].

Theorem 5 (*Fixed Point Theorem*)**.** *Let $\mathcal{S}$ be a closed, convex subset of a Banach space $\mathcal{B}$ and let $J : S \to S$ be a continuous and compact map, e.g $J[B]$ is pre-compact for any bounded set $B \subset \mathcal{B}$. Then J has a fixed point.*

We will also make use of the Aubin–Lions lemma, or a variant of it.

Lemma 7 (*Aubin–Lions*)**.** *Let $B,\ B_0,\ B_1$ be Banach spaces with $B_0 \subset B \subset B_1$ and $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$ and $B_0,\ B_1$ be reflexive and define*

$$W \equiv \left\{u \in L^p(0,T;B_0) : u' \in L^q(0,T;B_1)\right\},$$

then the inclusion $W \hookrightarrow L^p(0,T;B)$ is compact.

Theorem 6 (*Extended Sobolev Inequalities in Bounded Domains*)**.** *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega) \cap L^p(\Omega)$, $1 \leq r, q \leq \infty$. For any integer j, $0 \leq j \leq m$, and for any number a in the interval $j/m \leq a \leq 1$, set*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

If $m - j - n/r$ is a nonnegative integer, then

$$\left\|D^j u\right\|_{L^p} \leq C\,\|u\|_{W^{m,r}}^{a}\,\|u\|_{L^q}^{(1-a)}. \tag{A.1}$$

If $m - j - n/r$ is a nonnegative integer, then (A.1) holds for $a = j/m$. The constant C depends only on Ω, r, q, m, j, a.

Appendix B. Local existence for the parabolic–elliptic system

Recall we are now studying

$$-\eta\Delta A = -A + \rho + \overline{A}(x) \tag{B.1a}$$
$$\rho_t = \Delta\rho - \nabla\cdot(\rho\nabla A) - A\rho. \tag{B.1b}$$

where (A, ρ) are defined on $\mathbb{R}^2 \times \Omega$ for $\Omega \subset \mathbb{R}^2$. Note that $A = \mathcal{B}_\eta(x) * (\rho + \overline{A}(x))$, where $\mathcal{B}_\eta(x)$ is the Bessel potential,

$$\mathcal{B}_\eta(x) = \frac{1}{4\eta\pi}\int_0^\infty \frac{1}{t}e^{-\frac{|x|^2}{4\eta t} - t}dt,$$

is well-defined. As the proof Theorem 4 is very standard we only provide an outline.

Proof of Theorem 4. The local existence proof of this theorem follows the same steps as the proof of the local existence theorem for the parabolic–parabolic case, Theorem 1. Hence, we only state any major differences. Of course, the goal, once more, is to use the fixed point theorem. First, define $\mathcal{O} = \left\{u \in H^s(\Omega) : \|u\|_{H^2} \leq M\right\}$ for fixed and finite scalar M. Given $\rho_1 \in \mathcal{O}$ we have that

$$A(x) = \mathcal{B}_\eta(x) * (\rho_1 + \overline{A}(x)). \tag{B.2}$$

Now, we are interested in solving the criminal density equation with $A(x)$ given by (B.2). Let $\mathcal{J}(\rho_1) = \rho$ be the solution of the criminal density equation. As before, we need to show that J is a continuous map and that $\mathcal{O}$ is invariant under J.

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{H_s} = \frac{1}{2}\sum_{|\alpha|\leq s}\frac{d}{dt}\int(D^\alpha\rho)^2\,dx$$
$$= \sum_{|\alpha|\leq s}D^\alpha\rho D^\alpha\rho_t\,dx$$
$$= \sum_{|\alpha|\leq s}\int D^\alpha\rho D^\alpha\left(\Delta\rho - \nabla\cdot(\rho\nabla A) - A\rho + \overline{B}\right)\,dx$$

$$= \sum_{|\alpha|\le s}\left(-\int |D^\alpha\nabla\rho|^2\ dx + \int D^\alpha\nabla\rho\cdot D^\alpha\rho\nabla A - \int D^\alpha\nabla\rho\cdot D^\alpha\rho A + \int D^\alpha\nabla\rho\cdot D^\alpha\overline{B}\right)$$

$$= -\|\nabla\rho\|_{H^s}^2 + \epsilon\|\nabla\rho\|_{H^s}^2 + \frac{1}{\epsilon}\|\rho\nabla A\|_{H^s} + \|\rho\|_{H^s}^2 + \frac{1}{2}\|A\rho\|_{H^s}^2 + \frac{1}{2}\left\|\overline{B}\right\|_{H^s}^2.$$

Note that we can bound the terms with A in terms of ρ_1, that is

$$\|\rho\nabla A\|_{H^s}^2 \lesssim \|\nabla A\|_\infty\|\rho\|_{H^s} + \|\rho\|_\infty\|\nabla A\|_{H^s} \lesssim \|\rho\|_{H^s}\left(\left\|\nabla\mathscr{B}_\eta(x)\right\|_{L^1}\|\rho_1\|_\infty + \left\|\nabla\mathscr{B}_\eta(x)\right\|_{L^1}\|\rho_1\|_\infty\left\|\overline{A}\right\|_\infty\right) + \|\rho\|_\infty\left\|\nabla\mathscr{B}_\eta(x)\right\|_{L^1}\left(\|\rho_1\|_{H^s} + \left\|\overline{A}(x)\right\|_{H^s}\right).$$

Furthermore, we also have that

$$\|\rho A\|_{H^s}^2 \lesssim \|\rho\|_\infty^2\|A\|_{H^s}^2 + \|A\|_\infty^2\|\rho\|_{H^s}^2.$$

Let $y = \|\rho\|_{H^s}^2$ then combining all of the estimates gives a differential inequality of the form

$$\frac{dy}{dt} = c_1 y + c_2$$

and solving the differential inequality gives that for M large enough and T small enough we have that $\rho \in \mathcal{O}$. Continuity follows similarly, considering $\frac{1}{2}\frac{d}{dt}\|\rho_1-\rho_2\|_{H_s}$ for $\rho_1, \rho_2 \in \mathcal{O}$. The rest follows the proof of Theorem 1. Note also, that from the above estimates we can see that the controlling quantity is $\|\rho\|_\infty$. This gives (16). □

Appendix C. Additional proof

C.1. Proof of Lemma 2

Proof. Consider the change of variables $z = \sqrt{(p/4\eta t)}x$ ($dx = (4\eta t/p)^{n/2}dz$), then

$$\left\|K_\eta(x,t)\right\|_p = \frac{1}{(4\eta\pi t)^{n/2}}\left(\int e^{\frac{-p|x|^2}{4\eta t}}dx\right)^{1/p} = \frac{(4\eta t)^{n/2p}}{p^{n/(2p)}(4\pi\eta t)^{n/2}}\left(\int e^{-|z|^2}dz\right)^{1/p} = C_{p,\eta}\ t^{n(1-p)/(2p)}.$$

This gives the estimate (17a). Now, we take the gradient of K_η and get

$$\nabla K_\eta = \frac{-2}{(4\eta t)^{n/2+1}\pi^{n/2}}\int xe^{\frac{-|x|^2}{4\eta t}}dx.$$

Hence, making the same change of variables

$$\left\|\nabla K_\eta\right\|_p = \frac{2}{(4\eta t)^{n/2+1}\pi^{n/2}}\left(\int |x|^p e^{\frac{-p|x|^2}{4\eta t}}dx\right)^{1/p} = C_p(4\eta t)^{n(1-p)/2p-1/2}\left(\int z^p e^{-|z|^2}dz\right)^{1/p}.$$

This concludes the proof. □

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