Abstract

Non-local reaction-diffusion equations arise naturally to account for diffusions involving jumps rather than local diffusions related to Brownian motion. In ecology, long distance dispersal requires such frameworks. In this work we study a one-dimensional non-local reaction-diffusion equation with bistable and monostable type reactions. The heterogeneity here comes from the presence of a barrier outside of which the equation is a classical homogeneous reaction-diffusion equation and is motivated by applications in ecology, sociology, and biology. In some finite interval, which we refer to as the barrier zone, there is decay. For bistable equations we first establish the existence of a generalized traveling front that approaches a traveling wave solution as $t \to -\infty$, propagating in a heterogeneous environment. We then study the problem of obstructing the propagation of the generalized traveling front. As in the local diffusion case, we prove that obstruction is possible for bistable equations but not for monostable equations. An interesting difference between the local dispersal and the non-local dispersal is that in the latter the obstructing steady states are discontinuous. We characterize these jump discontinuities and discuss the scaling between the range of the dispersal and the critical length of the barrier. We further explore other differences between the local and the non-local dispersal cases and state some open problems.

1 Introduction

This article is devoted to the study the non-local reaction-diffusion equation

\begin{align}
    u_t &= J * u - u + f(x,u) \\
    u(x,0) &= u_0(x),
\end{align}

where $u(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, for example, can be a population density and the dispersal kernel $J : \mathbb{R} \to \mathbb{R}_\geq$ is a symmetric function with unit mass. More specifically,

\begin{equation}
    J \in C^1(\mathbb{R}), \quad J \geq 0, \quad J(x) = J(-x), \quad \text{and} \quad \int_{\mathbb{R}} J \, dx = 1.
\end{equation}

Equation (1) has gained popularity in modeling phenomena where jump processes come into play. In ecology, (1) is a more suitable model for species, such as bees, that jump from one location to another; in social sciences it is natural to consider non-local interactions or movements of agents...
in signal propagation in neural networks it has been suggested that the interaction between neurons is also non-local \[12\]. Another interesting application of (1) is for the propagation of pests, see \[18\].

Non-local dispersal is modeled with the use of a dispersive kernel \( J(x - y) \), which gives the probability that a particle or agent at location \( y \) will jump to location \( x \). Numerous reaction terms have been used to model different phenomena in both homogeneous and heterogeneous environments. In this work we are interested in the particular heterogeneity appearing in the reaction term:

\[
 f_{L}(x, u) = \begin{cases} 
 g(u) & x \notin I_{L} \\
 -\alpha u & x \in I_{L}, 
\end{cases}
\]  

with \( I_{L} = (0, L) \) for \( L \geq 0 \) and \( \alpha \) is a positive fixed constant. We refer to \( I_{L} \) as the barrier zone (\( L = 0 \) corresponds to a homogeneous environment). In particular, we consider the cases when \( g(u) \) is either bistable:

\[
 g(0) = g(a) = g(1) = 0, \quad 0 < a < 1, \quad g < 0 \text{ in } (0, a) \text{ and } g > 0 \text{ in } (a, 1),
\]  

or monostable:

\[
 g(0) = g(1) = 0, \quad g'(0) > 0, \quad g(u) > 0, \quad \text{for } u \in (0, 1).
\]  

The bistable case arises in models with an Allee effect. This refers to the effect that \( u(x, t) \) grows if it is above a certain threshold, \( a \), and decays otherwise. A prototypical example of such reaction term is \( g(u) = u(u-1)(a-u) \) with \( a \in (0, 1) \). The monostable case is the logistic growth nonlinearity that is non-negative up to a limiting carrying capacity. Such equations play a fundamental role in population ecology \[17\].

The interest in (3) arises in multiple applications where most of the environment promotes propagation if the density is above a certain threshold. The invasion to which it leads could possibly be obstructed by the barrier zone, \( I_{L} \), where decay prevails. This has applications in pest control \[18\], crime wave obstruction \[4\], and flame propagation control in which one looks for a barrier to halt the progression of the wave. However, there is also the reverse application, where the barrier zone is interfering with the propagation of a signal, such as in nerve-pulse propagation, or in the propagation of a benign species. In these situations one would like to understand how large the barrier zone can be and still allow propagation. We remark that this problem has been studied in the case of local diffusion and is commonly known as the gap problem in the excitable media literature (see for example \[16\] and references within).

In the homogeneous problem the invasion of a steady state (which can represent a crime wave spreading in space or the maximum density of species) corresponds to the existence of a traveling wave solution. Traveling wave solutions to (1) satisfy

\[
 \mathcal{J} * U(z) - U(z) + cU'(z) + g(U(z)) = 0 \tag{6a}
\]

\[
 \lim_{x \to -\infty} U(z) = 1 \quad \text{and} \quad \lim_{x \to +\infty} U(z) = 0, \tag{6b}
\]

where \( z = x - ct \). These solutions are known to exists for homogeneous reaction term \( g(u) \) of bistable, monostable, and ignition type, \[3\] \[9\] \[11\] \[12\]. In particular, \( c > 0 \) when \( g(u) \) is monostable or when \( g(u) \) is bistable and satisfies

\[
 \int_{0}^{1} g(s) \, ds > 0.
\]

See for example \[3\] and references therein. The speed of the traveling wave solutions, and consequently the speed of propagation, depends on the dispersal potential \( \mathcal{J}(x) \). This stresses the
importance of understanding the role that the range of dispersal, as determined by $\mathcal{J}(x)$, plays on the speed propagation. Motivated by this we analyze the effect of the range of dispersal by studying kernels with the scaling:

(H1) $\mathcal{J}_\lambda(z) = \lambda j(z/\lambda)$ for $\lambda > 0$, 

(H2) $j \in C^1(\mathbb{R}), j(z) = j(-z) \geq 0$ for all $z \in \mathbb{R}$, 

(H3) $\int j(z) \, dz = 1, \int z j(z) \, dz = 0, j' \in L^1(\mathbb{R}), D = \int |j(z)|^2 \, dz < \infty$.

Note that the scaling in (H1) preserves mass and that $\lambda$ measures the scale of the range of dispersal, affording us an explicit way to measure the effect that this has on propagation. The first observation we make when $\mathcal{J}_\lambda(x)$ satisfies (H1)-(H3) is that the transition layer of a traveling wave solution is of order $\lambda$. Furthermore, if $U_\lambda(z)$ is the solution to the homogeneous equation (6) with speed $c_\lambda$ and $g(u)$ a bistable-type reaction term, then by a result of Bates et al. [3] we know that:

$$c_\lambda = \frac{\int_0^1 g(u) \, du}{\int_{\mathbb{R}} |U_\lambda'(z)|^2 \, dz},$$

(7)

At the same time, in [13] Fife and Wang show that the interface layer is at most $O(\sqrt{\lambda} |\ln \lambda|)$, which then gives that $\int_{\mathbb{R}} |U_\lambda'(z)|^2 \, dz$ is at most $O\left(\frac{1}{\sqrt{\lambda} |\ln \lambda|}\right)$. Combining this and (7) we observe that the speed $c_\lambda$ is at least $O(\sqrt{\lambda} |\ln \lambda|)$, which means that the speed of the propagation increases as the range of dispersal increases.

Our first result gives meaning to the notion of propagation of an invading solution in a heterogeneous environment. Indeed, in a heterogeneous environment traveling waves, as defined in (6), do not exist. However, the idea of similar solutions, referred to as a generalized traveling fronts, was introduced by Berestycki and Hamel in [5, 6] for reaction-diffusion equations with local diffusion. In this work we prove the existence of a solution of this type, which will be defined below in section 2, for the case when $g(u)$ is of bistable-type. The generalized traveling fronts represent, for example, crime waves, invasions of pests, propagation of a signal in a neural network in a heterogeneous environment/media.

Having proved the existence of solutions which are invasive it is natural to consider the problem of obstructing such solutions. Our next results deal with the existence of blocking solutions, which we define to be steady-state solutions to (1) that monotonically decrease to zero as $x \to \infty$. More precisely, the solutions satisfy

$$\mathcal{J} * u - u + f_L(x, u) = 0, \quad (8a)$$

$$\lim_{x \to -\infty} u(x) = 1 \text{ and } \lim_{x \to +\infty} u(x) = 0. \quad (8b)$$

In the case when $\mathcal{J}(x)$ is a general dispersal potential that satisfy [2] we prove the existence solutions [8] for sufficiently large, but finite, $L$. This result holds for the case when the reaction term $g(u)$ is bistable and such solutions do not exist when $g(u)$ is of monostable-type, as a consequence of our results - see below. We remark that the proof of existence of solutions to (8a) is very different than the proof of local diffusion - see our work in [4]. One difficulty we encounter here is the lack of an explicit solution in the barrier zone, that is we do not have an explicit solution to

$$\mathcal{J} * u = -\alpha u,$$

for a general potential. This was something that we exploited in [4] and without this we are unable to prove that there is a critical length needed to prevent the propagation of invasive solutions. Our
proof of existence relies on the use of wisely chosen stationary waves, whose existence and regularity is proved in [3], which are super and subsolutions. Since the operator $J^* u - u$ generates a jump process it maintains a maximum principle, which consequently enables a comparison principle, see for example [13]. Therefore, can use a barrier method to build a solution to (8). Moreover, we prove that the blocking solutions, unlike the solutions from our work in the local diffusion case [4], are not continuous and we further characterize the jump-discontinuity.

Next, we study dispersal potentials which satisfy (H1)-(H3). From numerical simulations we observe that for any positive $\lambda$, there exists a sufficiently large $L_\lambda$ that allows the obstruction of invading solutions. Moreover, we see that if $\lambda_1 < \lambda_2$ then $L_{\lambda_1} < L_{\lambda_2}$, which is intuitive if one recalls that $J_\lambda(y - x)$ gives the probability that a particle at location $x$ jumps to a location $y$. Indeed, as $\lambda$ increases so does the probability of particles making long jumps; thus, one expects that a larger barrier zone is necessary to prevent the propagation of a traveling front as the range of dispersal increases.

In our next result we prove the existence of a critical length, $L_\lambda$, for any $\lambda > 0$, which is the minimum length of the barrier zone that will obstruct propagation. Thus, on one hand when $L > L_\lambda$ there exists a solution to (8) with $J(x) = J_\lambda(x)$. On the other hand, no such solutions exist for $L < L_\lambda$. The proof relies on using the fact that if $u_1(x)$ is a blocking solution for $J(x) = J_1(x)$, then $u(\lambda x)$ is a blocking solution for $J(x) = J_\lambda(x)$, which reduces the analysis to understanding the behavior of the equation for a fixed $\lambda$. We choose to study the case when $\lambda << 1$ because we can then approximate the solution of the non-local equation by the solution to a local reaction-diffusion problem that we understand, owing to our previous work with L. Ryzhik in [4]. Indeed, this approximation allows us to prove that a sufficiently small $L$ will allow an invading solution to propagate. Furthermore, if $L_\lambda$ is the critical length for the problem with $J(x) = J_\lambda(x)$, then $L_\lambda$ is linear in $\lambda$.

Remark 1. An implications of this, on the one hand, is that for applications where we seek to obstruct propagation (such as criminal activity and pest propagation) large range of dispersal have a negative effect. On the other hand, for models in which one seeks to understand conditions of propagation, then large range of dispersal have a positive effect.

The previous results deals with $g(u)$ of bistable-type. When we consider $g(u)$ of monostable-type the only steady-state solution to (1) satisfies:

$$\lim_{x \to \pm \infty} u(x) = 1,$$

which implies propagation. This result is proved in [13] but we state it below.

In many situations it is more realistic that one can apply resources in a region of fixed size and instead increase the effect of the decay. Therefore, for our last result we consider a fixed length of the barrier zone, $L \geq 0$, and consider the heterogeneous reaction term

$$f_{L,\alpha}(x, u) = \begin{cases} g(u) & x \notin I_L \\ -\alpha u & x \in I_L, \end{cases}$$

where we instead vary $\alpha > 0$. We are interested in comparing solutions of the non-local reaction-diffusion equations,

$$u_t = J_\lambda * u - u + f_{L,\alpha}(x, u),$$

and the local reaction-diffusion equation

$$u_t = \Delta u + f_{L,\alpha}(x, u).$$
We prove that for any bistable \( g(u) \) there exists \( \alpha > 0 \), which is sufficiently large, so that there exists a blocking solution \( u_\alpha(x) \) to (12) satisfying (5b). On the other hand, for the non-local problem (11): given any \( \alpha > 0 \) there exists an \( \lambda > 0 \) for which there are no blocking solutions to (11). Thus, no matter how strong is the obstruction to propagation in the barrier zone, a sufficiently long range dispersal will always manage to skip it and propagate.

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### 1.1 Notation, definitions, and background

In this section we introduce notation, definitions and some preliminary theory that will be used throughout the paper. The existence of traveling wave solutions for the homogeneous problem (1) has been the focus of many works, see for example [3, 9, 11]. In particular, the first existence result for the bistable case was given by [3], relying in part on the work of [12]. For the local diffusion case we refer readers to [14, 15, 1, 2]. The following theorem summarizes the existence and uniqueness of a solution to (6) in the homogeneous case.

**Theorem 1** ([3, 11]). Let \( J(z) \) satisfy (2). If \( g(u) \) is bistable then assume in addition that

\[
\int_\mathbb{R} J(z) |z| \, dz < \infty. \tag{13}
\]

Then, there exists a unique solution \((U, c^*)\) (modulo translations) to (6). Furthermore, the sign of \( c^* \) depends on the sign of

\[
\int_0^1 g(u) \, du.
\]

If \( g(u) \) is monostable assume in addition that for all \( \mu > 0 \)

\[
\int_\mathbb{R} J(z) e^{\mu z} \, dz < \infty.
\]

Then there exists a minimum speed \( c^* > 0 \) such that (6) admits solutions, \((U, c)\) for all \( c \geq c^* \). Furthermore, it does not admit solutions for \( c < c^* \).

In the bistable case if the reaction term satisfies:

\[
\int_0^1 g(u) \, du = 0, \tag{14}
\]

then \( c = 0 \) and the solution is a stationary wave. With regards to regularity of solutions an interesting difference, pointed out in the work of Bates et al. in [3], between the stationary waves of the local problem (in which the Laplacian replaces the non-local operator) and the stationary waves solutions of (1) is that the former are continuous while the latter can have a finite number of
jump discontinuities depending on the reaction term. The reaction terms leading to discontinuous stationary waves were characterized in [3].

In what follows we denote by $L^2(\Omega)$ the space of square integrable functions in the domain $\Omega \subset \mathbb{R}$ and use the short hand notation $f(a^\pm)$ to denote
\[ \lim_{x \to \pm a} f(x). \]

1.2 Main results

The first theorem deals with the heterogenous problem (1). We first establish the existence of a unique entire solution that is born from a homogeneous traveling wave solution as $t \to -\infty$. Such solution can be thought of as a generalization of a traveling wave and gives meaning to the notion of invasion. Let
\[ -\beta := \min_{u \in [0,1]} g'(u), \]
for the proof of the following theorem we make the technical assumption that
\[ \beta < \alpha. \]

Furthermore, assume that
\[ g(u) \in C^r(\mathbb{R}) \quad \text{for} \quad r > 1. \]

**Theorem 2** (Existence and uniqueness of an entire solution). Let $L > 0$, $g(u)$, as defined in (3), be of bistable-type satisfying (17), and $J(x)$ satisfy (2) and (13). There exists a unique entire solution (up to a time shift) $\tilde{u}(x,t)$ to (1) defined on $\mathbb{R} \times \mathbb{R}$, such that $0 < \tilde{u}(x,t) < 1$ and $\tilde{u}_t(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$. Furthermore, if $U(x-ct)$ is the unique solutions from Theorem 1 then
\[ \tilde{u}(x,t) \to U(x-ct), \]
as $t \to -\infty$ uniformly in $x \in \mathbb{R}$. Moreover, $\tilde{u} \in C^r((-\infty,\infty),C^{1+r}(\mathbb{R}\setminus\{0,L\}))$.

The proof of Theorem 2 follows the approach developed in the work of Berestycki, Hamel, and Matano in [7] (see also [3]). Next, we are interested in characterizing the possible obstruction of the propagation of the entire solution $\tilde{u}(x,t)$ from Theorem 2. The following result shows that there is a finite $\tilde{L}$, such that for all $L > \tilde{L}$, there exists a blocking solution. Finally, we prove that this solution has to be discontinuous at the gap endpoints.

**Theorem 3** (Blocking solutions for large $L$). Let $J(x)$ satisfy (2). There exists a positive and finite $\tilde{L}$ such that for all $L \geq \tilde{L}$ there exist a blocking solution, $u_L(x)$, to (5). Moreover, $u_L(x)$ has two jump-discontinuities, one at $x = 0$ and the other at $x = L$, which satisfy:
\[ u_L(0^-) - g(u_L(0^-)) = (1 + \alpha)u_L(0^+) \]
\[ (1 + \alpha)u_L(0^-) = u_L(0^+) - g(u_L(0^+)). \]

**Remark 1.** A dynamic consequence of Theorem 3 is that if $u(x,t)$ is the solution to (1) with initial data $u_0(x) \leq u_L(x)$, then $u(x,t)$ will never propagate the barrier zone. That is,
\[ u(x,t) \leq u_L(x) \quad \text{for all} \quad t > 0. \]
Figure 1: Figure 1 displays the functions $h(u)$ defined in (20) and $h_k(u)$ defined in (21).

Note that (19a) yields $u(0^+)$ given $u(0^-)$ (the same holds for the discontinuity at $x = L$). This, in principle, leaves an infinite number of possibilities. However, we provide a stronger characterization of the values of the solution and the jump of the discontinuities. For this purpose, let

$$h(u) = u - g(u). \quad (20)$$

Given $k > 0$ define the function

$$h_k(u) = \begin{cases} 2u & \text{for } u \in [0, \frac{k}{2}] \\ k & \text{for } u \in [\frac{k}{2}, h^{-1}(k)] \\ h(u) & \text{for } u \in [h^{-1}(k), 1]. \end{cases} \quad (21)$$

We refer to Figure 1 for an illustration of $h(u)$ and $h_k(u)$.

**Proposition 1** (Characterization of jump discontinuities for $L = \infty$). Let $u_L(x)$ be the solution from Theorem 3 with $L = \infty$. Let $k$ be the value such that

$$\int_0^1 h_k \, dx = \frac{1}{2}, \quad (22)$$

then,

$$u_L(0^-) = h^{-1}(k) \quad \text{and} \quad u_L(0^+) = \frac{k}{2}.$$ 

From Figure 1 we observe that $\int h(u) \, du \leq \frac{1}{2}$ and that there exists a $k$ such that (22) holds. The case when $L < \infty$ is more technical and not as clear to understand due to the fact that there are two discontinuities. This case is the object of the following result.

**Proposition 2** (Characterization of discontinuities for $L < \infty$). For $L < \infty$, the values $u(0^-)$, $u(0^+)$, $u(L^-)$, and $u(L^+)$ must satisfy

$$u_L(0^-) = h^{-1}(k_1) \quad \text{and} \quad u_L(0^+) = \frac{k_1}{2} \quad u_L(L^-) = \frac{k_2}{2} \quad \text{and} \quad u_L(L^+) = h^{-1}(k_2).$$

for some positive $k_1, k_2$ so that the following holds

$$\int_{u(0^-)}^1 h(u) \, du + \int_0^{u(0^+)} h(u) \, du + \int_{u(L^-)}^{u(0^+)} (1 + \alpha)u \, du + k_1(u(0^-) - u(0^+)) + k_2(u(L^-) - u(L^+)) = \frac{1}{2}.$$
Figure 2: Results of numerical simulations for the barrier zone problem with dispersal kernels satisfying (H1) with $\lambda = 3$ and $\lambda = 5$. Figure 2a displays a sufficiently large barrier zone that obstructs the propagation of the invading solution for $\lambda = 3$, but allows propagation for the case when $\lambda = 5$. On the other hand, in Figure 2b the length of the barrier zone is sufficiently large to prevent the propagation in both cases.

Up to now the results did not use the scaling parameter $\lambda$ that appears in condition (H1). The next results establishes the dependence of the critical length on the range of dispersal $\lambda$. In fact, we take advantage of the scaling condition (H1) and obtain a much stronger result for this subclass of potentials. We state this result next.

**Theorem 4** (Critical length dependence on range of dispersal). Let $\bar{u}(x,t)$ be the solution from Theorem 2 with $J(x)$ satisfying (H1)-(H3) for $\lambda > 0$. There exists a critical length, $L_\lambda$, such that

(i) if $L \geq L_\lambda$ then there exists a blocking solution, $u_\lambda(x)$, that satisfies (8),

(ii) if $L < L_\lambda$ then $\bar{u}(x,t)$ will propagate. That is, for any $\epsilon > 0$ there exists an $x_\epsilon > L$ and $t_\epsilon > 0$ such that

$$\bar{u}(x,t) > 1 - \epsilon,$$

for all $x > x_\epsilon$ and $t > t_\epsilon$.

Furthermore, if $x > L$ then $u_\lambda(x)$ is monotonically decreasing and $L_\lambda$ is a linear increasing function of $\lambda$.

Remark 2. Theorem 4 can be extended to solutions that are invasive. In particular, any solution with initial data that is close to an homogeneous traveling front far off to the left, will be an invasive solution.

In the monostable case we are not able to construct an entire solutions. However, any invasive solution will eventually propagate (see Remark 2).

**Theorem 5** (Passing the barrier zone). Let $J(x)$ satisfy (2) and the conditions from Theorem 2 and let $g(u)$ be of monostable-type. For any $L \geq 0$ any invading solution $u(x,t)$ will propagate: for any $\epsilon > 0$ there exists a $t_\epsilon > 0$ and $x_\epsilon > 0$ such that

$$u(x,t) \geq 1 - \epsilon \quad \text{for all} \quad x > x_\epsilon, t > t_\epsilon.$$
Our last results marks a significant difference between the case of local diffusion and the case of non-local diffusion for the bistable case. In particular, we prove that when the barrier zone is of fixed size then local diffusion allows obstruction when the decay rate in the barrier zone is sufficiently large. On the other hand, if the diffusion is non-local then for any decay rate in the barrier zone there exists a range of dispersal that allows propagation.

**Theorem 6** (Intense decay rate in barrier zone of fixed length). *Let* $g(u)$ *be a bistable function and let* $L \geq 0$ *be fixed.*

**(i) (Local dispersal case)** *In the case of local dispersal there exists a sufficiently large* $\alpha > 0$ *so that* \([12]\) *has a blocking solution, $u_\alpha(x)$.* *In particular,* $u_\alpha(x)$ *satisfies*

\[
\begin{align*}
  u_{xx} + f_{L,\alpha}(x,u) &= 0 \\
  u(-\infty) &= 1 \quad \text{and} \quad u(\infty) = 0.
\end{align*}
\]

**(ii) (Non-local dispersal case)** *For any* $\alpha > 0$ *there exists a* $\lambda_\alpha > 0$ *so that the non-local problem \([1]\) *with* $J_{\lambda_\alpha}$ *satisfying (H1) has no blocking solution. That is, \([8]\) *has not solutions.*

Since the proof of (i) in Theorem 6 is very similar to previous work with L. Ryzhik in [4] we omit the proof.

**Outline:** *In §2 we prove the existence of generalized traveling fronts. Section 3 is devoted to the proof of obstruction of traveling fronts for general kernels for sufficiently large* $L$. *In §4 we explore the effect that the range of dispersal has on the propagation and obstruction of traveling fronts. In §5 we consider the problem of fixed barrier length while the decay term in the barrier zone varies. We summarize our findings and examine some of their consequences in §6.*

### 2 Existence and uniqueness a generalized transition wave

In this section we prove the existence of a solution, $\bar{u}(x,t)$, to \([1]\) defined for $t \in (-\infty, \infty)$ such that

\[
\lim_{t \to -\infty} \bar{u}(x,t) = U(x - ct),
\]

uniformly in $x \in \mathbb{R}$. Note that because $g(u)$ is of bistable-type then $U(z)$ is unique, modulo translations, and so without loss of generality we assume that

\[
U(0) = \frac{a}{2},
\]

where $a$ is defined in [4]. The proof will follow the construction used in [7], see also [4]. For the proof of the existence we need to characterize the asymptotic behavior of the traveling wave solution and its derivative, $(U,U')$.

**Lemma 1.** *(Asymptotic estimates)* *Let* $U(z)$ *be the solution to* \([6]\) *when* $g(u)$ *is of bistable-type. There exist positive constants* $\kappa, \mu, \gamma, \delta, \tilde{\gamma}$, *and* $\tilde{\delta}$ *such that the following bounds hold,*

\[
\begin{align*}
  \gamma e^{-(\kappa+\epsilon)z} &\leq U(z) \leq \delta e^{-(\kappa-\epsilon)z} &\quad \text{for} \quad z \geq 0 \\
  \gamma e^{-(\mu+\epsilon)z} &\leq 1 - U(z) \leq \delta e^{-(\mu-\epsilon)z} &\quad \text{for} \quad z < 0 \\
  -\tilde{\gamma} e^{-(\kappa+\epsilon)z} &\leq U'(z) \leq -\tilde{\delta} e^{-(\kappa-\epsilon)z} &\quad \text{for} \quad z \geq 0 \\
  -\tilde{\gamma} e^{(\mu-\epsilon)z} &\leq U'(z) \leq -\tilde{\delta} e^{-(\mu-\epsilon)z} &\quad \text{for} \quad z < 0,
\end{align*}
\]

*for any* $\epsilon > 0$. 

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Proof. The linearize equations for \((U, U') := (U, V)\) about \((0, 0)\) are
\[
\mathcal{J} \ast U(z) - U(z) + cU'(z) + g'(0)U = 0,
\]
\[
\mathcal{J} \ast V(z) - V(z) + cV'(z) + g'(0)V = 0,
\]
where \(c\) is the speed of the wave. Note that for \(w(x)\) to be a supersolutions we need \(Mw \leq 0\) where we define \(M\)
\[
M(w) := \mathcal{J} \ast w(x) - w(x) + cw'(z) + g'(0).
\]
Let \(w(x) = e^{-\kappa x}\) for \(\kappa > 0\) and first note that
\[
\int_{\mathbb{R}} \mathcal{J}(x-y)e^{-\kappa y} \, dy = e^{-\kappa x} \int_{\mathbb{R}} \mathcal{J}(z)e^{\kappa z} \, dz,
\]
thus substituting \(w(x)\) into (27) yields
\[
M(w) = e^{-\kappa x} \left( \int_{\mathbb{R}} \mathcal{J}(z)e^{\kappa z} \, dz - 1 - c\kappa - |g'(0)| \right),
\]
for \(x\) large. Now, recall that \(\mathcal{J}(x)\) has mass one and since
\[
\int_{\mathbb{R}} \mathcal{J}(z)e^{\kappa z} \, dz
\]
is a continuous function of \(\kappa\), there exists a \(\kappa_2\) such that
\[
\int_{\mathbb{R}} \mathcal{J}(z)e^{\kappa_2 z} \, dz \leq 1 + c_\lambda \kappa_2 + |g'(0)|.
\]
At the same time we observe that there exists a sufficiently large \(\kappa_1 > \kappa_2\)
\[
\int_{\mathbb{R}} \mathcal{J}(z)e^{\kappa_1 z} \, dz > 1 + c_\lambda \kappa_1 + |g'(0)|,
\]
which changes the sign of (28). Note that we can choose \(\kappa_1\) and \(\kappa_2\) arbitrarily close, if fact by the intermediate value theorem there is a \(\kappa\) such that
\[
\int_{\mathbb{R}} \mathcal{J}(z)e^{\kappa z} \, dz - 1 - c_\lambda \kappa - |g'(0)| = 0.
\]
Hence, we may choose any \(\kappa_1 = \kappa + \epsilon\) and \(\kappa_2 = \kappa - \epsilon\) for any \(\epsilon > 0\) where \(\kappa\) satisfies (30). Therefore, we can find \(\delta, \gamma\) such that (26a) holds. A similar argument works for the remaining estimates.

\(\square\)

Remark 3. From the proof of Lemma [3] we know that \(\mu > \kappa\) implies that \(-g'(0) < -g'(1)\).

Remark 4. Any \(\mathcal{J}_\lambda(x)\) for \(\lambda > 0\) that satisfies (H1)-(H3) also yield similar asymptotic decay at \(\pm\infty\). In fact, \(\kappa\) and \(\mu\) are decreasing functions of \(\lambda\).

For simplicity we keep the notation
\[
\kappa_2 = \kappa - \epsilon, \kappa_1 = \kappa + \epsilon, \mu_2 = \mu - \epsilon, \mu_1 = \kappa + \epsilon.
\]
To prove existence it is useful to define, following [7], the auxiliary function

\[ \xi(t) = \frac{1}{\kappa_2} \log \frac{1}{1 - c^{-1}K e^{\kappa_2 c t}}, \]  

(32)

where \( c \) is the speed of the traveling wave, \( \kappa_2 \) is as in [31], and \( K \) a positive constant to be chosen later. Note that \( \xi(t) \) is well-defined on \( t \in (-\infty, -T) \) with \( T := \frac{1}{\kappa_2 c} \ln(\frac{c}{K + c}) \). Furthermore, \( \xi(-\infty) = 0 \) and

\[ \dot{\xi}(t) = Ke^{\kappa_2 (ct + \xi(t))}. \]  

(33)

For uniqueness we consider the spatial region where the front of the entire solution is located at a given time. To make this precise, for a given \( \eta \in [0, \frac{1}{2}) \) we define

\[ \mathcal{F}_\eta(t) := \{ x < 0 : \eta \leq \bar{u}(x, t) \leq 1 - \eta \}. \]

Since the waves are moving to the right there exists a time, \( T_\eta \in \mathbb{R} \), such that \( F_\eta(t) \subset \{ x \leq -1 \} \) for \( t \in (-\infty, T_\eta) \). The following lemma states that for \( x \in F_\eta(t) \) the time derivative of the solution is bounded below by a positive constant.

**Lemma 2.** For any \( \eta \in [0, 1/2) \), there exists a \( \delta_\eta > 0 \) such that for \( \bar{u}(x, t) \), the entire solution to (1), satisfies

\[ \bar{u}_t \geq \delta_\eta \quad \text{for } x \in \mathcal{F}_\eta(t), \quad t \in (-\infty, T_\eta). \]  

(34)

The proof of Lemma 2 follows similarly as that of Lemma 3.1 in [7] so we omit the proof.

**Proof.** (Theorem 2) Let \( J(x) \) satisfy (2). We first prove existence of an entire solution by constructing a suitable supersolution and subsolution.

**Existence:** Let \( U(z) \) satisfy (6) and (25) and let \( \xi(t) \) be be defined as in (32). Define

\[ w_+(x, t) = \begin{cases} U(x - ct - \xi(t)) + U(-x - ct - \xi(t)) & x < 0 \\ 2U(-ct - \xi(t)) & x \geq 0, \end{cases} \]  

(35)

and

\[ w_-(x, t) = \begin{cases} U(x - ct + \xi(t)) - U(-x - ct + \xi(t)) & x \leq 0 \\ 0 & x > 0. \end{cases} \]  

(36)

We first show that \( w_+(x, t) \) and \( w_-(x, t) \) are respectively super and subsolutions for \( t \in (-\infty, -T) \), for \( T \) sufficiently large. For this purpose, it is convenient to define the operator \( \mathcal{N} \) as follows

\[ \mathcal{N} u := u_t - J \ast u + u - f_L(x, u). \]  

(37)

**Step 1 (supersolution):** We first prove that \( \mathcal{N} w_+ \geq 0 \) in a suitable time range and note that

\[ w_+(0^+, t) = w_+(0^-, t), \]

for all \( t \) and thus is suffices to check that \( w_+(x, t) \) is a supersolution for \( x < 0 \) and then for \( x \geq 0 \). We compute \( \mathcal{N} w_+ \) when \( x < 0 \), with the additional assumption that \( \kappa > \mu \) (we consider the opposite case later). Let \( z_+ := x - ct - \xi(t) \) and \( z_- := -x - ct - \xi(t) \) and apply \( \mathcal{N} \) to \( w_+ \),

\[ \mathcal{N} w_+ = -(c + \dot{\xi}(t))(U'(z_+) + U'(z_-)) - J \ast (U(z_+ + U(z_-)) + U(z_+) + U(z_-) - g(U(z_+) + U(z_-))) \]

\[ = -\dot{\xi}(t)(U'(z_+) + U'(z_-)) + g(U(z_+)) + g(U(z_-)) - g(U(z_+) + U(z_-)). \]
For the reaction terms above we make use of the inequality
\[
|g(a) + g(b) - g(a + b)| \leq \bar{L}a,
\] (38)
for any \(0 \leq a, b \leq 1\) and some constant \(\bar{L} > 0\). Using (38) we obtain that
\[
\mathcal{N}w_+ \geq -\xi(t)(U'(z_+) + U'(z_-)) - \bar{L}U(z_+)(z_-).
\]
Consider the case when \(z_+ \leq 0\), this implies that \(|x| \geq -ct - \xi(t)\) which in turn implies that \(z_- \geq 0\). In this case, invoking Lemma 1 we obtain
\[
\mathcal{N}w_+ \geq K\delta e^{\kappa_2(ct+\xi(t))}e^{\mu_2(x-ct-\xi(t))} - L\delta e^{-\kappa_2(-x-ct-\xi(t))}
\]
\[
= e^{\kappa_2(ct+\xi(t))}(K\delta e^{\mu_2(x-ct-\xi(t))} - L\delta e^{\kappa_2x})
\]
\[
\geq 0,
\]
because \(\kappa > \mu\) provided we choose \(K\) and \(T\) to satisfy
\[
\delta K > \bar{L}\delta \quad \text{and} \quad -ct - \xi(t) \geq 0 \quad \forall t \in (-\infty, -T).
\] (39)
On the other hand, if \(z_+ > 0\) then \(z_- > 0\) as well and we obtain the following lower bound
\[
\mathcal{N}w_+ \geq K\delta e^{\kappa_2(ct+\xi(t))}e^{\mu_2(x-ct-\xi(t))} - \bar{L}\delta^2 e^{-\kappa_2(-x-ct-\xi(t))}e^{-\kappa_2(x-ct-\xi(t))}
\]
\[
= e^{2\kappa_2(ct+\xi(t))}(K\delta e^{\mu_2x} - \bar{L}\delta^2)
\]
\[
\geq 0,
\]
provided
\[
K\delta > \bar{L}\delta^2.
\] (40)

Now, consider the case when \(\mu > \kappa\). The case when \(z_+ \geq 0\) (thus \(z_- \geq 0\)) follows by the previous argument as we never used the assumption that \(\kappa > \mu\) and so we are left to consider the case when \(z_+ \leq 0\) and \(z_- \geq 0\). Note that \(\kappa < \mu\) implies, from the proof of Lemma 1 that \(-g'(0) < -g'(1)\), see Remark 3 and thus for \(u \sim 0\) and \(v \sim 1\) we obtain
\[
g(u) + g(v) - g(u + v) = (g'(0) - g'(1))u + \mathcal{O}(u(1-v)) + \mathcal{O}(u^2),
\]
as \(\mathcal{O}((v-1)^2)\) is much smaller. If \(z_+ << -1\) and \(z_- >> 1\) we have that
\[
g(U(z_+)) + g(U(z_-)) - g(U(z_+ + U(z_-))) \geq 0.
\]
Thus, there exists an \(L_1 >> -1\) such that if \(x \in (-\infty, -L_1)\) then \(\mathcal{N}w_+ \geq 0\). Finally, when \(x \in (-L_1, 0)\) we have again
\[
\mathcal{N}w_+ \geq K\delta e^{\kappa_2(ct+\xi(t))}e^{\mu_2(x-ct-\xi(t))} - \bar{L}\delta e^{-\kappa_2(-x-ct-\xi(t))}
\]
\[
= e^{\kappa_2(ct+\xi(t))}(K\delta e^{\mu_2(x-ct-\xi(t))} - \bar{L}\delta e^{\kappa_2x})
\]
\[
\geq e^{\kappa_2(ct+\xi(t))}(K\delta e^{\mu_2L_1} - \bar{L}\delta e^{\kappa_2x})
\]
\[
\geq 0,
\]
provided we choose \(K\) and \(T\) such that
\[
K\delta \geq \bar{L}\delta e^{\mu_2L_1} \quad \text{and} \quad -ct - \xi(t) \geq 0 \quad \forall t \in (-\infty, -T).
\] (41)
Now, consider the case when \( x \geq 0 \) then
\[
Nw_+ = -2(c + \dot{\xi})U'(-ct - \xi(t)) - 2J * U(-ct - \xi(t)) + 2U(-ct - \xi(t)) - f_L(x, 2U(-ct - \xi(t)))
\]
\[
= -2(c + \dot{\xi})U'(-ct - \xi(t)) - f_L(x, 2U(-ct - \xi(t)))
\]
\[
\geq 0,
\]
provided \( T \) is chosen as in (41) since this implies by (25), which allows us to conclude that that \( f_L(x, 2U(-ct - \xi(t))) \leq 0 \) outside of the gap (recall that \( f_L(x, u) < 0 \) for \( x \in I_L \)).

In summary, if we choose \( K \) and \( T \) to satisfy (39), (40), and (41) then \( w_+(x, t) \) is a supersolution to (1) for all \( t \in (-\infty, -T) \).

**Step 2 (Subsolution):** The next step is to find a suitable time range for which \( w_-(x, t) \) is a subsolution. Note that
\[
w_-(0^-) < w_-(0^+) = 0,
\]
and since the case when \( x > 0 \) is trivial we only have to treat \( x \leq 0 \). Let \( y_+ := x - ct + \xi(t) \) and \( y_- := -x - ct + \xi(t) \) and apply \( N \) to \( w_- \),
\[
Nw_- = (\xi'(t) - c)(U'(y_+) - U'(y_-)) - J * (U(y_+) - U(y_-)) + U(y_+) - U(y_-) - g(U(y_+) - U(y_-))
\]
\[
= \xi'(t)(U'(y_+) - U'(y_-)) + g(U(y_+)) - g(U(y_-)) - g(U(y_+) - U(y_-)).
\]
Observe that \( y_+ \leq y_- \) and that \( y_- \geq 0 \) always. Let us first consider the case when \( y_+ \leq 0 \) then
\[
Nw_- \leq Ke^{\kappa_2(x+ct+\xi(t))}(-\tilde{\delta}e^{\mu_1 y_+} + \tilde{\gamma}e^{-\kappa_2 y_-}) + \tilde{L}U(y_+) - U(y_-)
\]
\[
\leq Ke^{\kappa_2(x+ct+\xi(t))}(-\tilde{\delta}e^{\mu_1 (x - ct + \xi(t))} + \tilde{\gamma}e^{-\kappa_2 (x - ct + \xi(t))}) + \tilde{L}e^{-\kappa_2 (x - ct + \xi(t))}
\]
\[
= Ke^{\kappa_2(x+ct+\xi(t))}(-\tilde{\delta}e^{\mu_1 (\kappa_2 - ct) + \xi(t)} + \tilde{\gamma}e^{-\kappa_2 (ct + \xi(t))} + \tilde{L}e^{-\kappa_2 \xi(t)}).
\]
If \( \kappa > \mu \) then
\[
Nw_- \leq Ke^{\kappa_2(x+ct+\xi(t))}(-\tilde{\delta}e^{\mu_1 (ct + \xi(t))} + \tilde{\gamma}e^{-\kappa_2 (ct + \xi(t))} + \tilde{L}e^{-\kappa_2 \xi(t)}) \leq 0,
\]
promised \( T \) is chosen such that
\[
-\tilde{\delta}e^{\mu_1 (ct + \xi(t))} + \tilde{\gamma}e^{-\kappa_2 (ct + \xi(t))} + \frac{\tilde{L}e^{-\kappa_2 \xi(t)}}{K} \leq 0 \quad \forall \ t \in (-\infty, -T).
\]

On the other hand, if \( \mu > \kappa \) then once again \(-g'(0) \leq -g'(1) \) (see Remark 3) which has two consequences for \( y_+ \ll -1 \) and \( y_- \gg 1 \). The first is that \( U'(y_+) - U'(y_-) < 0 \) and the second is that
\[
Nw_- \leq (g'(1) - g'(0))U(y_-) + O(U(y_-)^2) + O(U(y_-)(1 - U(y_+))
\]
which is negative. As a consequence, there exists \( L_2 > 0 \) such that if \( x < -L_2 \) then the above inequality is negative. On the other hand, when \( x \in (-L_2, 0] \) the following is true
\[
Nw_- \leq Ke^{\kappa_2(x+ct+\xi(t))}(-\tilde{\delta}e^{-\mu_1 L_2 e^{\mu_1 (ct + \xi(t))}} + \tilde{\gamma}e^{-\kappa_2 (ct + \xi(t))} + \frac{\tilde{L}e^{-\kappa_2 \xi(t)}}{K} \leq 0
\]
provided provided $T$ is chosen such that
\[-\delta e^{-\mu_1 T} e^{\mu_1 (\xi(t))} + \gamma e^{-\kappa_2 (\xi(t))} + \frac{\tilde{L} \delta}{K} e^{-2\kappa_2 \xi(t)} \leq 0 \quad \forall \ t \in (-\infty, -T).\] (43)

If $y_+ \geq 0$ then we consider two cases: $|x| > L_3$ and $|x| \leq L_3$ for some $L_3$ to be specified later. First note that $y_+ - y_- = 2|x|$. Let us consider the case when $|x| \geq L_3$, in which case either $y_+ \leq L_3$ and $y_- > L_3$ or $y_+, y_- \geq L_3$. In the former case, $y_+ \leq L_3$ and $y_- > L_3$, there exists an $e(L_3) > 0$ such that

$$U'(y_+) - U'(y_-) < -e(L_3).$$

Thus, we obtain
\[
\mathcal{N} w_- \leq -K e^{\kappa_2 (\xi(t))} e(L_3) + \tilde{L} U(y_-) (U(y_+) - U(y_-)) \\
\leq -K e^{\kappa_2 (\xi(t))} e(L_3) + \tilde{L} \delta e^{\kappa_2 (\xi(t))} e^{-\kappa_2 (\xi(t))} e^{\kappa_2 (\xi(t))} \\
= K e^{\kappa_2 (\xi(t))} \left(-e(L_3) e^{\kappa_2 x} + \frac{\tilde{L} \delta}{K} e^{-2\kappa_2 \xi(t)}\right). \tag{44}
\]

Since $x < 0$ is negative if $K$ is chosen to be sufficiently large such that

$$K > \frac{\tilde{L} \delta}{e(L_3)}.$$ (45)

In the latter case when $y_+, y_- > L_3$ then we can choose $L_3$ sufficiently large such that both $U(y_+)$ and $U(y_-)$ are close to zero and

$$2U(y_-) > U(y_+).$$ (46)

In this case we have,

$$g(U(y_+)) - g(U(y_-)) - g(U(y_+) - U(y_-)) \leq 0.$$ \hspace{1cm} \text{Indeed, if we linearize about $U(y_+), U(y_-) = 0$ we have that}

$$g(U(y_+)) - g(U(y_-)) - g(U(y_+) - U(y_-)) = -g''(0) (U(y_+))^2 + \frac{1}{2} g''(0) U(y_+) U(y_-)$$

$$+ \mathcal{O}\left(\sum_{k=1}^{3} U(y_+)^k U(y_-)^{3-k}\right).$$

However, recall for $u \in (0, a)$ it can be verified that $g''(0) > 0$ and so by condition (46) we have that $g(U(y_+)) - g(U(y_-)) - g(U(y_+) - U(y_-)) < 0$. Moreover, we also know that

$$U'(y_+) - U'(y_-) < 0.$$ \hspace{1cm} \text{The final case to consider is $|x| < L_3$ in which case we can choose $T$ to be sufficiently large so that}

for $t \in (-\infty, -T)$ so that once again both $U(y_+)$ and $U(y_-)$ are close to zero and (46) is satisfied. In summary, if $K$ and $T$ are chosen to also satisfy (42), (43), and (44) then $w_-(x, t)$ is a subsolution to (1) for $t \in (-\infty, -T)$.

\textbf{Step 3: (Barrier Method)} From steps one and two we know that there exists a $T > 0$ such that $w_-(x, t)$ and $w_-(x, t)$ are a sub and super solutions, respectively, to (1) for all $(x, t) \in \mathbb{R} \times (-\infty, -T)$. 

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We now construct a solution starting from the subsolution and using the supersolution as a barrier. Let us denote $u_n(x,t)$ to be the solution to $(1)$ with $x \in \mathbb{R}$ and $t \in (-n, -\infty)$, for $n \in \mathbb{N}$ sufficiently large, in particular $-n \leq -T$ (a solution can be found using a fixed-point method, see for example [10]). Additionally, we have the bounds
\[
w_-(x, -n) \leq u_n(x, -n) \leq w_+(x, -n),
\]
and as a consequence of the comparison principle
\[
w_-(x, t) \leq u_n(x, -n) \leq w_+(x, t) \quad \text{for} \quad t \in (-n, -T).
\] (47)
In particular, (47) holds for $t = -n + 1$. Now, we denote $u_{n-1}$ to be the solution to $(1)$ for $t \in (-n+1, \infty)$ and continue this process, which yields an increasing sequence of bounded solutions \( \{u(x,t)\}_{n \in \mathbb{N}} \). Thus, we can take the limit as $n \to \infty$, uniformly in $x$, to obtain a solution $\overline{u}(x,t)$ to $(1)$ on $(x,t) \in \mathbb{R} \times \mathbb{R}$. In addition, an application of the comparison principle gives the bound
\[
w_-(x,t) \leq \overline{u}(x,t) \leq w_+(x,t), \quad \text{for} \quad (x,t) \in \mathbb{R} \times (-\infty, -T),
\] (48)
and the definitions of $w_+(x,t), w_-(x,t)$ then give (18). We are left to prove that
\[\overline{u}_t(x,t) > 0.\] (49)
For this purpose, we take the time derivative of $w_-$ when $x < 0$
\[\partial_t w_-(t) = (-c + \xi'(t))(U'(y_+) - U'(y_-)).\]
Take $t$ sufficiently negative so that $-c + \xi'(t) < 0$ and arguing as before we know that $(U'(y_+) - U'(y_-)) < 0$. On one hand, if $y_+ \geq 0$ this inequality is clear since $0 < y_+ < y_-$ implies that
\[U'(y_+) < U'(y_-).\] (50)
On the other hand, if $y_+ \leq 0$ then
\[|x| > c|t| + \xi(t).\]
Note that $|y_-| = 2|x| + |y_+|$, and so there exists a $T$ sufficiently negative so that (50) is satisfied. Thus, $\partial_t w_+ > 0$ for $t$ sufficiently negative, which by (48) implies that $\partial_t u_n(x,t) > 0$ if $n$ is sufficiently large as $u_n(x, -n) = w_-(x, -n)$. Applying the maximum principle gives that
\[\partial_t u_n(x,t) > 0 \quad \text{for} \quad t \in (-n, \infty).\]
Taking the limit as $n \to \infty$ and using the fact that $\partial_t \overline{u}(x,-\infty) > 0$ proves (49).

**Uniqueness:** Assume that $u(x,t)$ and $v(x,t)$ are two entire solutions that satisfy (18). Then for any $\epsilon > 0$ there exists a $t_\epsilon < 0$ such that
\[\|u(\cdot,t) - v(\cdot,t)\|_{L_(\infty(\mathbb{R}))} < \epsilon \quad \forall \ t \in (-\infty, t_\epsilon).\] (51)
Let $\beta$ satisfy (15) and define
\[U^+(x,t) := u(x,t + t_0 + \epsilon \xi(t)) + \epsilon e^{-\beta t} \quad U^-(x,t) := u(x,t + t_0 - \epsilon \xi(t)) - \epsilon e^{-\beta t},\]
where $\xi(t)$ is a positive, increasing, and bounded function to be specified later. Our first objective it to show that $U^+(x,t)$ and $U^-(x,t)$ are, respectively, super and subsolutions for $t \in [0, T_\eta - t_0 - \epsilon)$ with $T_\eta$ defined in (34). Let $\tau := t + t_0 + \epsilon \xi(t)$ and apply $\mathcal{N}$, as defined in (37), to $U^+(x,t)$
\[\mathcal{N}U^+ = (1 + \epsilon \xi'(t))u_t(x,\tau) - \epsilon \beta e^{-\beta t} - \mathcal{J} \ast u(x,\tau) + u(x,\tau) - f_L(x,u(x,\tau) - \epsilon e^{-\beta t})
\]
\[= \epsilon \xi'(t)u_t(x,\tau) - \epsilon \beta e^{-\beta t} + f_L(x,u(x,\tau)) - f_L(x,u(x,\tau) + \epsilon e^{-\beta t}).\]
If \( x \in \mathbb{R} \setminus I_L \) then
\[
f_L(x, u(x, \tau)) - f_L(x, u(x, \tau) + \varepsilon e^{-\beta t}) = g(u(x, \tau)) - g(u(x, \tau) + \varepsilon e^{-\beta t}).
\]
For the remaining of the proof we need the natural extension \( g(u) \) to \((-\infty, \infty)\) defined as follows
\[
g(u) = g'(0)u \quad \text{for } u \leq 0 \quad \text{and} \quad g(u) = g'(1)(u - 1) \quad \text{for } u \leq 1.
\]
On one hand, if \( x \notin \mathcal{F}_\eta(t + t_0 + \varepsilon \xi(t)) \) then for \( \eta \) sufficiently close to one so that \( u + \varepsilon e^{-\beta t} \in [-\delta, \delta] \cup [1 - \delta, 1 + \delta] \) for some small delta such that
\[
g(u(x, \tau) + \varepsilon e^{-\beta t} - g(u(x, \tau) + \varepsilon e^{-\beta t}) \geq \varepsilon \beta e^{-\beta t},
\]
in which \( NU^+ \geq 0 \) holds since \( \xi'(t) \geq 0 \) and \( u_t > 0 \). On the other hand, if \( x \in \mathcal{F}_\eta(t + t_0 + \varepsilon \xi(t)) \) then by Lemma \( 2 \) \( u_t(x, t) \geq \delta \eta \) and thus,
\[
NU^+ \geq \varepsilon \delta \eta \xi'(t) - \varepsilon \beta e^{-\beta t} + g(u(x, \tau)) - g(u(x, \tau) + \varepsilon e^{-\beta t})
\]
\[
\geq \varepsilon \delta \eta \xi'(t) - \varepsilon \beta e^{-\beta t} - r \varepsilon e^{-\beta t},
\]
where \( r = \max_{u \in [0, 1]} g'(u) \). Let \( \xi(t) \) satisfy
\[
\xi'(t) = \frac{r + \beta}{\delta \eta} e^{-\beta t} \quad \text{with} \quad \xi(0) = 0.
\]
Note that \( \xi(t) \) is increasing but approaches a finite limit which we denote by \( K \). Now, with such \( \xi(t) \) we see that \( NU^+ \geq 0 \) in this case. Note that \( x \in \mathcal{F}_\eta(t + t_0 + \varepsilon \xi(t)) \) for \( t < T_\eta - t_0 - K \varepsilon \). The last case to consider is \( x \in [0, L] \), in which case
\[
f_L(x, u(x, \tau)) - f_L(x, u(x, \tau) + \varepsilon e^{-\beta t}) = \varepsilon \alpha e^{-\beta t},
\]
and so by \( 16 \) we also have that \( NU^+ \geq 0 \) in this case.

Similarly, one can show that \( U^- \) is a subsolution. Now, from \( 51 \) we see, using the fact that \( \xi(0) = 0 \), that
\[
U^-(x, 0) \leq v(x, t_0) \leq U^+(x, 0).
\]
A comparison principle then implies that
\[
U^-(x, t) \leq v(x, t + t_0) \leq U^+(x, t),
\]
for all \((x, t) \in \mathbb{R} \times [0, t - t_0 - K \varepsilon] \). Now, we perform a change of variables, \( \hat{t} = t + t_0 \), and obtain by substituting in the definitions of \( U^+ \) and \( U^- \) that
\[
u(x, \hat{t} - \varepsilon \xi(\hat{t} - t_0)) - \varepsilon e^{-\beta(\hat{t} - t_0)} \leq u(x, \hat{t} - \varepsilon \xi(\hat{t} - t_0)) + \varepsilon e^{-\beta(\hat{t} - t_0)},
\]
which holds for \( t \in [t_0, T_\eta - K \varepsilon] \) and \( t_0 \in (-\infty, T_\eta - K \varepsilon] \). Taking the limit as \( t_0 \to -\infty \) we obtain
\[
u(x, \hat{t} - \varepsilon K) \leq u(x, \hat{t}) \leq u(x, \hat{t} + \varepsilon K),
\]
and as these estimates where independent of \( \varepsilon \) we can take the limit \( \varepsilon \to \infty \) and obtain
\[
u(x, t) \equiv v(x, t).
\]
This concludes the proof of uniqueness.
Regularity: Consider the evolution of the $L^2$-norm of $u'(x,t) = \partial_x u(x,t)$:

$$\frac{d}{dt} \int_{\mathbb{R}} (u'(\cdot,t))^2 \, dx = \int_{\mathbb{R}} (\mathcal{J} * u'(\cdot,t) u'(\cdot,t) - (u(\cdot,t))^2 + f'(x,u)(u'(\cdot,t))^2) \, dx.$$ 

An application of Hölder and then Young’s inequality the gives

$$\frac{d}{dt} \|u'(\cdot,t)\|^2 \leq (\|\mathcal{J}\|_1 - 1 + \|f'(x,u)\|_{\infty}) \|u'(\cdot,t)\|^2.$$ 

Then, by an application of Grönwal’s inequality and a bootstrap argument $u(x,t)$ inherits the smoothness of the initial data $U(x - ct)$ and $f'(x,u)$. If $g(u) \in C^r$ for $r \geq 1$ then by Theorem 3.1 in [3] $U(x - ct) \in C^{r+1}$ for $c \neq 0$ and similarly for the time derivative.

3 General dispersal kernels

In the proof of Theorem 3 we use a simple variation of a Lemma proved in [11], which will enable us to construct a solution from an appropriate supersolution and subsolution.

Lemma 3 ([11]). Let $f \in C^0(\mathbb{R} \setminus \{0, L\}) \cap L^2(\mathbb{R})$, then there exists a unique solution $v \in C^0(\mathbb{R} \setminus \{0, L\}) \cap L^2(\mathbb{R})$ to

$$\begin{align*}
\{ \mathcal{J} * u - u &= f, \\
u(\pm \infty) &= 0.
\end{align*}$$

(52)

Proof. We rewrite (52) as $u = \mathcal{J} * u - f,$ and since $\mathcal{J} * u - f$ is a strict contraction in $C^0(\mathbb{R} \setminus \{0, L\})$ then (52) admits a unique fixed point. Moreover, if $v$ is this unique solution then

$$\|v\|_{L^2(\mathbb{R})} = \|\mathcal{J} * u - f\|_{L^2(\mathbb{R})} < \infty,$$

and this concludes the proof.

We are now ready to prove Theorem 3.

Proof. (Theorem 3) Let $g_1(u)$ be a bistable function satisfying (14) and

$$g_1(0) = g_1(1) = 0, \ g(u) \geq g_1(u), \ g'(0) < -1,$$

with only another zero in $(0, 1)$. Such a function exists due to the definition of $g(u)$. Let $U(z)$ be the stationary traveling wave that satisfies

$$\begin{align*}
\{ \mathcal{J} * U(z) - U(z) + g_1(U(z)) &= 0, \\
U(-\infty) &= 1 \quad \text{and} \quad U(\infty) = 0.
\end{align*}$$

In the same spirit, we define $g_2(u)$ also to be a bistable function satisfying (14) and such that

$$g_2(0) = g_2(1) = 0, \ g(u) = g_2(u), \ u \in [0, \gamma] \cup [1 - \gamma, 1],$$

with only another zero in $(0, 1)$. Such a function exists due to the definition of $g(u)$. Let $V(z)$ be the stationary traveling wave that satisfies

$$\begin{align*}
\{ \mathcal{J} * V(z) - V(z) + g_2(V(z)) &= 0, \\
V(-\infty) &= 1 \quad \text{and} \quad V(\infty) = 0.
\end{align*}$$
for some $\gamma > 0$. Now, we let $V(z)$ be the stationary traveling wave that satisfies
\[
\begin{align*}
\mathcal{J} * V(z) - V(z) + g_2(V(z)) &= 0 \\
V(-\infty) &= 1 \quad \text{and} \quad V(\infty) = 0.
\end{align*}
\]
Note that $U(z)$ and $V(z)$ exist and are stationary wave due to [14], see [3] for more details.

**Step 1** (Supersolution and subsolutions): The first step is to construct a supersolution and a subsolution. By our choice of $g_1(u)$ there exists a $\delta > 0$ such that $-\alpha u \geq g_1(u)$ for $u \in [0, \delta]$. We choose a translate of the stationary wave $U$, $u_-(x) := U(x + x_0)$ such that $u_-(0) = \delta$. Note that by construction $u_-(x)$ is a subsolution of (1) for any $L \geq 0$.

Now, we choose a translate of $V(z)$, $u_+ := V(x + x_1)$ such that $u_+(0) = 1 - \gamma$. In this case, since $g(u) > -\alpha u$ for $u \in (\gamma, 1]$, if $L$ is sufficiently large so that $u_+(L) < \gamma$ we see that $u_+$ is a supersolution.

**Step 2** (Sequence of solutions): We now construct a solution starting of with $u_-(x)$ and using $u_+(x)$ as a barrier. Indeed, let $u_0(x) = u_-(x)$ and let $u_n(x)$ be the solution to
\[
\begin{align*}
\mathcal{J} * u_n - u_n + \kappa u_n &= -f_L(x, u_{n-1}) + \kappa u_{n-1} \\
u_n(-\infty) &= 1 \quad u_n(\infty) = 0,
\end{align*}
\]
for some $\kappa > 0$ to be specified later. We first show that given $u_{n-1}$ such $u_n(x)$ exists. Following [11] we choose a $\phi \in C^\infty_c$ satisfying $\|\phi\|_{L^1(\Omega)} = 1$ and define
\[
\varphi(x) = \int_x^\infty \phi(s) \, ds.
\]
Then, $v_n = u_n - \varphi$ satisfies
\[
\begin{align*}
\mathcal{J} * v_n - v_n + \kappa v_n &= -\mathcal{J} * \varphi + \varphi - f_L(x, v_{n-1} + \varphi) + \kappa v_{n-1} \\
v_n(\pm \infty) &= 0.
\end{align*}
\]
(53)
Note, that by our choice of $u_0$ we know that $v_0 \in C^0 \cap L^2(\mathbb{R})$ as $\varphi \equiv 0$ for $x$ sufficiently large and positive, and $\varphi \equiv 1$ for $x$ sufficiently negative. Note also that the righthand side of (53) belongs to $C^0(\mathbb{R} \setminus \{0, L\})$. Now, for $x < 0$ we have, using the mean value theorem and $g(1) = 0$, that
\[
|f_L(x, v_0 + \varphi)| < \|g\|_\infty |v_0(x) + \varphi(x) - 1| = \|g\|_\infty |u_-(x) - 1| \in L^2(\mathbb{R}^-)
\]
and for $x > L$ we have that $g(0) = 0$
\[
|f_L(x, v_0 + \varphi)| < |v_0 + \varphi| = \|g\|_\infty |u_-| \in L^2(\mathbb{R}^+).
\]
Thus, $f_L(x, v_0 + \varphi) \in L^2(\mathbb{R})$. Finally, the fact that $\mathcal{J} * \varphi - \varphi \in L^2(\mathbb{R})$ follows from Lemma 3.1 in [11]. An application of Lemma 3 gives a solution to (53). Repeating this process iteratively gives us a sequence of solutions, $\{u_n\}_{n \in \mathbb{N}}$.

**Step 3**: (Passing to the limit as $n \to \infty$) From induction and the maximum principle we know that
\[
u_-(x) \leq u_n(x) \leq u_+(x),
\]
for all $n \in \mathbb{N}$. Furthermore, if we define $w_n = u_n(x - \tau) - u_n(x)$ for some $\tau > 0$ then $w_n$ solves
\[
\mathcal{J} * w_n - w_n + \kappa w_n = -f_L(x, u_{n-1}(x - \tau)) + f_L(x, u_{n-1}(x)) - \kappa(u_{n-1}(x - \tau) - u_{n-1}(x)).
\]
Choose \( \kappa > \|f'\|_{L(\infty)} \) so that \(-f_L + \kappa \) is non-decreasing and positive, making the right hand side of the above equality non-negative for \( u_1 \) since \( u_- \) is monotone decreasing. An application of the maximum principle implies that \( w_1 \) is always non-negative. Iterating this process gives that \( u_n(x) \) is non-increasing in \( x \). Invoking Helly’s lemma we obtain that a subsequence converges to a non-increasing solution \( u(x) \) that also satisfies \( u-(x) \leq u(x) \leq u+(x) \). Moreover, the dominated convergences implies

\[
\mathcal{J} \ast u_n - u_n \rightarrow \mathcal{J} \ast u - u.
\]

Since \( \mathcal{J} \ast u_n \in C^\infty(\mathbb{R}) \) and \( \mathcal{J} \ast u_n = u_n - f_L(x, u_n) \) then \( u_n - f_L(x, u_n) \in C^\infty(\mathbb{R}) \). This gives uniform convergence in \( u_n - f_L(x, u_n) \rightarrow u - f_L(x, u) \). This concludes the existence and we are left to show that the solution must be discontinuous.

**Step 4: (Discontinuity)** The solution \( u(x) \) satisfies

\[
\mathcal{J} \ast u = u + f_L(x, u),
\]

and since \( \mathcal{J} \ast u \) is smooth then \( u + f_L(x, u) \) must be smooth implying that

\[
\begin{align*}
u(0^-) - g(u(0^-)) &= (1 + \alpha)u(0^+) \\
(1 + \alpha)u(L^-) &= u(L^+) - g(u(L^+)).
\end{align*}
\]

If \( u \) is continuous at 0 then there exist an \( u \) that solves

\[
g(u) = -\alpha u,
\]

which holds only when \( u = 0 \) due to (16). Thus, \( u \) is discontinuous at \( x = 0 \). The discontinuity is proved similarly for \( x = L \).

We now prove Propositions [1] and [2] using techniques from [3].

**Proof.** (Proposition 1) Let \( L = \infty \) and so the reaction term is given by:

\[
f_\infty(x, u) = \begin{cases} g(u) & x < 0 \\ -\alpha u & x \geq 0. \end{cases}
\]

Recall that any steady-state solution to (1) with \( L = \infty \) satisfies

\[
\mathcal{J} \ast u - u + f_\infty(x, u) = 0. \tag{54}
\]

Let \( \mathcal{J} \ast u(0) = k \) with \( k > 0 \), then

\[
u(0^-) - g(u(0^-)) = (1 + \alpha)u(0^-) = k.
\]

Multiply (54) by \( u' \) and integrate over \( \mathbb{R} \) to obtain

\[
\int_{-\infty}^{0} \mathcal{J} \ast u u' \, dx + \int_{0}^{\infty} \mathcal{J} \ast u u' \, dx + \int_{u(0^-)}^{1} h(u) \, dx + \int_{0}^{u(0^+)} (1 + \alpha)u \, dx = 0,
\]

where \( h(u) \) is defined by (20). Let \( S(x) \) be the step function

\[
S(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}
\]

and define \( v(x) = u(x) - S(x) \). Note that the following hold:
(i) \( v(\pm \infty) = 0, u'(x) = v'(x) \) for \( x \neq 0 \),
(ii) \( S'(x) = -\delta(x), \mathcal{J}' * S(x) = -\mathcal{J}(x) \),
(iii) \( \int_{-\infty}^{\infty} \mathcal{J}' * vv \, dx = 0 \),
(iv) \( \int_{\mathbb{R}} \mathcal{J}(x) S(x) = \frac{1}{2} \).

Note that (iii) holds by Fubini because \( \mathcal{J}'(z) \) is odd and (iv) holds as \( \mathcal{J} \) is even and with mass one. We first look at the integral of the kernel in the negative reals,

\[
\int_{-\infty}^{0} \mathcal{J} * uu' \, dx = \int_{-\infty}^{0} \mathcal{J} * (v + S)v' \, dx
\]

\[
= \mathcal{J} * u(0)v(0^-) - \int_{-\infty}^{0} \mathcal{J}' * vv \, dx - \int_{-\infty}^{0} \mathcal{J}' * Sv \, dx
\]

\[
= \mathcal{J} * u(0)(u(0^-) - 1) - \int_{-\infty}^{0} \mathcal{J}' * vv \, dx + \int_{0}^{0} \mathcal{J}(x)v(x) \, dx,
\]

Similarly for the positive reals,

\[
\int_{0}^{\infty} \mathcal{J} * uu' \, dx = \int_{0}^{\infty} \mathcal{J} * (v + S)v' \, dx
\]

\[
= -\mathcal{J} * u(0)v(0^+) - \int_{0}^{\infty} \mathcal{J}' * vv \, dx - \int_{0}^{\infty} \mathcal{J}' * sv \, dx
\]

\[
= -\mathcal{J} * u(0)v(0^+) - \int_{0}^{\infty} \mathcal{J}' * vv \, dx + \int_{0}^{\infty} \mathcal{J}(x)v(x) \, dx.
\]

Adding (55) and (56) we obtain

\[
\int_{-\infty}^{0} \mathcal{J} * uu' \, dx + \int_{0}^{\infty} \mathcal{J} * uu' \, dx = \mathcal{J} * u(0)(u(0^-) - u(0^+)) - \mathcal{J} * u(0) + \int_{-\infty}^{\infty} \mathcal{J}' * vv \, dx
\]

\[
+ \mathcal{J} * u(0) - \mathcal{J} * s(0).
\]

\[
= \mathcal{J} * u(0)(u(0^-) - u(0^+)) - \frac{1}{2}.
\]

Thus,

\[
\int_{0}^{1} h_k(u) \, du = \int_{0}^{1} h(u) \, du + \int_{u(0^-)}^{u(0^+)} (1 + \alpha)u \, dx + k(u(0^-) - u(0^+)) = \frac{1}{2},
\]

and we conclude.

\[\square\]

**Proof.** (Proposition 2) Let \( L < \infty \) and \( \mathcal{J} * u(0) = k_1 \) and \( \mathcal{J} * u(L) = k_2 \). Following the same steps as in the previous case we obtain

\[
\int_{-\infty}^{0} \mathcal{J} * uu' \, dx + \int_{0}^{L} \mathcal{J} * uu' \, dx + \int_{L}^{\infty} \mathcal{J} * uu' \, dx + \int_{u(0^-)}^{1} h(u) \, du + \int_{0}^{u(L^+)} h(u) \, du + \int_{u(L^-)}^{u(L^+)} (1 + \alpha)u \, du = 0.
\]

In this case,

\[
\int_{-\infty}^{0} \mathcal{J} * uu' \, dx = \mathcal{J} * u(0)(u(0^-) - 1) - \int_{-\infty}^{0} \mathcal{J}' * vv \, dx + \int_{-\infty}^{0} \mathcal{J}(x)v(x) \, dx,
\]

20
\[
\int_{L}^{\infty} J * uu' \, dx = -J * u(L)u(L^+) - \int_{L}^{\infty} J' * vv \, dx + \int_{L}^{\infty} J(x)v(x) \, dx,
\]
and
\[
\int_{0}^{L} J * uu' \, dx = J * u(L)u(L^-) - J * u(0)u(0^+) - \int_{0}^{L} J' * vv \, dx + \int_{0}^{L} J(x)v(x) \, dx.
\]
Adding the three terms gives
\[
\int_{-\infty}^{0} J * uu' \, dx + \int_{0}^{L} J * uu' \, dx + \int_{L}^{\infty} J * uu' \, dx = k_1(u(0^-) - u(0^+)) + k_2(u(L^-) - u(L^+)).
\]
Thus,
\[
\int_{u(0^-)}^{1} h(u) \, du + \int_{0}^{u(L^+)} h(u) \, du + \int_{u(L^-)}^{u(0^+)} (1 + \alpha)u \, du + k_1(u(0^-) - u(0^+)) + k_2(u(L^-) - u(L^+)) = \frac{1}{2}
\]
and from this we conclude. \(\square\)

## 4 The effect of the range of dispersal

In this section we prove Theorem 4. The proof consists of three steps. We first prove that there exists a blocking solution when \(L\) is sufficiently large. The second step it to show that the invading solution will propagate for \(L, \lambda\) sufficiently small. The final step is to generalize the result for any \(\lambda > 0\).

**Proof.** (Theorem 4)

**Step 1** (\(L\) sufficiently large): Let \(J_\lambda(x)\) satisfy (H1) – (H3) and \(\bar{u}(x,t)\) be the entire solution from Theorem 2. By Theorem 3 for any \(\lambda > 0\) there exists a sufficiently large \(L_b > 0\) for which there exists a blocking solution, \(u_b(x)\), to (8). Moreover, since \(\bar{u}(x,t)\) is a supersolution to (8) with \(L > L_b\) we can also conclude that \(\bar{u}(x,t)\) is also obstructed for \(L > L_b\).

**Step 2** (\(L << 1\) and \(\lambda << 1\)): Next we show that for a sufficiently small \(L\) the solution \(\bar{u}(x,t)\) will propagate. First consider \(\lambda << 1\). Since \(\bar{u}(x,t) \in C^{1+\gamma}(\mathbb{R}, C^{2+\gamma}(\mathbb{R} \setminus \{0, L\}))\) we can approximate the non-local operator as follows
\[
J_\lambda * \bar{u}(x,t) - \bar{u}(x,t) = \lambda^2 \partial_{xx} \bar{u}(x,t) + O(\lambda^{2+\gamma}),
\]
for \(x \in \mathbb{R} \setminus \{0, L\}\), see for example [8]. Motivated by this we introduce the auxiliary system

\[
v_t = \lambda^2 \partial_{xx} v - C \lambda^{2+\gamma} + f_L(x,v), \tag{57a}
\]
\[
\lim_{t \to -\infty} v(x,t) = V(x-ct), \tag{57b}
\]
for some $C > 0$ and initial condition $V(x - ct)$ both to be specified later. Let $h(u) = g(u) - C\lambda^{2+\gamma}$ and take $\lambda$ sufficiently small so that $h(u)$ also has exactly three zeros, in particular,

$$h(-\delta_1(\lambda)) = h(1 - \delta_2(\lambda)) = 0.$$  

Note that $\delta_i(\lambda)$ decrease with $\lambda$ and $\lim_{\lambda \to 0} \delta_i(\lambda) = 0$ for $i = 1, 2$. The initial condition, $V(z)$, where $z = x - ct$ in (57a) is the traveling wave solution for (57a) and satisfies:

$$\lambda^2 V''(z) + cV'(z) + h(z) = 0$$

$$\lim_{x \to -\infty} V(z) = 1 - \delta_2(\lambda), \lim_{x \to \infty} V(z) = -\delta_1(\lambda).$$

In fact, there exists a unique entire solution, $v(x, t)$, to (57a) exists and is unique, see for example [4, 7]. Moreover, for $\delta$ sufficiently small $v(x, t)$ satisfies

$$v_t \leq \mathcal{J}_\lambda * v - v + f_L(x, t),$$

for $x \in \mathbb{R} \setminus \{0, L\}$. Define

$$w(x, t) := \max \{v(x, t), 0\},$$

which is a subsolution of (1) as the max of two subsolutions is a subsolution. In addition, we have that

$$V(x - ct) \leq U(x - ct).$$

Therefore,

$$\lim_{t \to -\infty} w(x, t) \leq \lim_{t \to -\infty} \bar{u}(x, -\infty),$$

which consequently by the comparison principle implies that

$$w(x, t) \leq \bar{u}(x, t) \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Hence, if $w(x, t)$ propagates then so does $\bar{u}(x, t)$. However, for $L$ sufficiently small the only steady-state solutions to (57a) approach $1 - \delta_2(\lambda)$ at both $\pm \infty$. Indeed, in [4] we prove that for $L < L_c$, where $L_c$ is the critical length, the only steady state solution, $u_p(x)$, satisfy the property that for any $\epsilon_1 > 0$ there exists an $x_\lambda > 0$ such that

$$u_p(x) > 1 - \epsilon_1 - \delta_2(\lambda) \text{ for } x > x_\epsilon_1.$$  

From this we can conclude that there exists a sufficiently small $L$ so that for any $\epsilon_1$ there exists an $x_\epsilon_1, t_\epsilon_1$ such that

$$\bar{u}(x, t) > 1 - \delta_2(\lambda) - \epsilon_1 \text{ for all } x > x_\epsilon_1, t > t_\epsilon_1.$$  

In fact,

$$\lim_{x \to \pm 1} \bar{u}(x, t) = 1,$$

which can be proved by showing that $g(u) \to 0$ as $x \to \infty$, see the proof of Theorem 6 in [18]. This proves the result for $\lambda << 1$.

**Step 3 (general $L > 0$):** Observe that if $u(x, t)$ is a solution to $u_t = \mathcal{J}_1 * u - u + f_L(x, u)$ then $u(\lambda x, t)$ is a solution $u_t = \mathcal{J}_\lambda * u - u + g_L(x, u)$, where $g_L(x, u)$ is a rescaled version of $f_L(x, u)$. Indeed, we have that $u(\lambda x, t)$ satisfies

$$u_t(\lambda x, t) = \int \mathcal{J}_1(x - y)u(\lambda y, t) \, dy + f_L(\lambda x, u(\lambda x, t)).$$

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By a change of variables, $z = \lambda y$ and $\lambda^{-1} dz = dy$, gives

$$u_t(\lambda x, t) = \frac{1}{\lambda} \int J_1 \left( x - \frac{z}{\lambda} \right) u(\lambda x, t) \, dz + f_L(\lambda x, u(\lambda x, t)).$$

Another change of variables, $w = \lambda x$, gives

$$u_t(w, t) = \frac{1}{\lambda} \int J_1 \left( \frac{w - z}{\lambda} \right) u(w, t) \, dw + f_L(w, u(w, t)) = J_{\lambda} * u(w, t) + f_L(w, u(w, t)) \, dw.$$

Thus, if $u_{L^*}(x, t)$ is a blocking solution to $u_t = J_1 * u - u + f_{L^*}(x, u)$ then $u_{L^*}(\lambda x, t)$ is a blocking solution to

$$u_t = J_{\lambda} * u - u + f_{\lambda L^*}(x, u).$$

Thus, the critical length is rescaled by $\lambda$ linearly and we conclude.

$$\square$$

5 Intense decay in barrier zone of fixed length

This section is devoted to the proof of Theorem 6.

Proof. (Theorem 6) Let $g(u)$ and $L$ be as in the hypothesis of the theorem. Since the proof of (i) is very similar to our work in [4], we skip it and refer the reader to Section 3.4 of [4]. The proof of (ii) is in the same spirit as the previous section. Indeed, for $L, \alpha > 0$ fixed consider the problem

$$u_t = J_{\lambda} * u - u + f_{L, \alpha}(x, u),$$

for $\lambda$ fixed and $f_{L, \alpha}(x, u)$ defined in (10). As we have seen before, we can rescale (11) and obtain that it is equivalent to

$$u_t = J_1 * u - u + f_{L, \alpha}(x, u). \quad (58)$$

From the proof of Theorem 4 we know that if $L/\lambda$ is sufficiently small then (58) does not have a blocking solutions. Thus, if $\lambda$ is sufficiently large any invading solution will propagate.

$$\square$$

6 Conclusion

For many phenomena in ecology, physiology, and the social sciences non-local dispersal is a more fitting description than that of local dispersal. In this work we have explored the effect that the non-local range of dispersal has on the propagation and obstruction of invading solutions. We were able to prove the existence of generalized traveling fronts in the case when reaction term is bistable and fairly general class of dispersal kernel. The proof of existence relied on the construction of super and subsolutions, which heavily took advantage of the fact that the reaction term was bistable. It would be interesting to construct such solutions for the monostable case and for a larger set of dispersal kernels.

For the bistable case we proved that for any range of dispersal there is a critical length required for the obstruction of invading solutions. For $L$ less than this critical length we prove that any invading solution will propagate. This proof is not direct, as was the case for the local diffusion.
system in [4], because the non-local diffusion causes the construction used in [4] to fail. Indeed, in this work we really took advantage of the results for the local diffusion case.

We also consider the case when the barrier zone is of fixed size and the strength of the decay term varies, which is a more realistic method of implementing resources. An interesting consequence of this work is that if the range of diffusion is too large then invading solutions will not be obstructed, even in the bistable case. On one hand, this implies that a large range of diffusion is beneficial in applications to propagation neural network and for persistence of species in ecology. On the other hand, a large range of diffusion is detrimental in the obstruction of criminal activity and in pest-control for example.

References


