A few fewnomials

1. Prove that for \( x \geq 0 \), we have
\[
x^7 + x^4 + x^3 + 1 \geq 2x^6 + 2x.
\]

2. Prove that for \( x \geq 0 \), we have
\[
x^{\sqrt{2}} + 2\sqrt{2} \geq 2^{\frac{3-\sqrt{2}}{2}}x + 2.
\]

3. Prove that for all \( x \) we have
\[
3x^2 e^x + 4e^2 \geq 8xe^x.
\]

4. Prove that for \( x \geq 0 \), we have
\[
x^{22} + x^{11} + x^9 + 1 \geq x^{21} + x^{15} + x^4 + x^2.
\]

5. Prove that for \( x > 0 \), we have
\[
x^{\sqrt{2}} + 2 + \frac{1}{x\sqrt{2}} \geq 2x + \frac{2}{x}.
\]

6. Prove that for \( x \geq 0 \), we have
\[
x^9 + 281x^3 + 100 \geq 22x^6 + 360x^2.
\]

7. For \( x, y > 0 \), define their logarithmic mean to be
\[
\text{LM}(x, y) = \frac{x - y}{\ln(x) - \ln(y)},
\]
where \( \ln(x) \) is the natural logarithm of \( x \). Prove that
\[
\frac{x + y}{2} \geq \text{LM}(x, y) \geq \sqrt{xy}.
\]

8. Prove that for any \( x > 0 \) we have
\[
x^{66} + x^{29} + x^{26} + \frac{1}{x^{26}} + \frac{1}{x^{29}} + \frac{1}{x^{66}} \geq x^{62} + x^{45} + x^2 + \frac{1}{x^2} + \frac{1}{x^{45}} + \frac{1}{x^{62}}.
\]

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9. Prove that if \( f \) is differentiable and \( f' \) is convex, then we have
\[
\begin{align*}
  f(3) + 3f(1) & \geq 3f(2) + f(0), \\
  f(6) + f(2) + f(1) & \geq f(5) + f(4) + f(0).
\end{align*}
\]

10. (Vasc) Prove that if \( f \) is differentiable and \( f' \) is convex, then for any \( x \geq y \geq z \) we have
\[
\begin{align*}
  f(2x + y) + f(2y + z) + f(2z + x) & \geq f(2x + z) + f(2z + y) + f(2y + x).
\end{align*}
\]

11. Popoviciu defines the divided differences of a polynomial inductively, as follows:
\[
\begin{align*}
  [a; f] &= f(a), \\
  [a, b; f] &= \frac{f(b) - f(a)}{b - a}, \\
  [a_0, ..., a_n; f] &= \frac{[a_1, ..., a_n; f] - [a_0, ..., a_{n-1}; f]}{a_n - a_0}.
\end{align*}
\]

Prove, by induction on \( n \), that if the \( n \)th derivative of \( f \) exists and is nonnegative, that for any \( a_0, ..., a_n \) we have
\[
[a_0, ..., a_n; f] \geq 0.
\]

12. Show that \([a_0, ..., a_n; f]\) is a symmetric function of \( a_0, ..., a_n \).

13. Let \( n \) be an integer which is at least 3. Suppose \( f \) is a function such that for every \( a_0, ..., a_n \) we have \([a_0, ..., a_n; f] \geq 0\). Show that \( f \) is differentiable and that for any \( b_0, ..., b_{n-1} \) we have
\[
[b_0, ..., b_{n-1}; f'] \geq 0.
\]

14. Suppose that \( f \) is a function such that for every integer \( n \) and every \( a_0, ..., a_n \) we have \([a_0, ..., a_n; f] \geq 0\). Prove that
\[
f(2) + 4f(0) \geq 4f(1).
\]

15. Prove that if \( a_1, ..., a_9 \) are real numbers satisfying \( \sum_{i=1}^{9} a_i = \sum_{i=1}^{9} a_i^3 = 0 \) and \( \sum_{i=1}^{9} a_i^2 = 8 \), then for any \( x > 0 \) we have
\[
x^2 + 7 + \frac{1}{x^2} \geq \sum_{i=1}^{9} x^{a_i} \geq 4x + 1 + \frac{4}{x}.
\]

16. Prove that for any \( x, y, z > 0 \) we have
\[
\sum_{\text{sym}} \frac{x^2}{y^2} + \sum_{\text{sym}} \frac{x\sqrt{2}}{y\sqrt{2}} \geq \sum_{\text{sym}} \frac{x^2}{yz} + \sum_{\text{sym}} \frac{xy}{z^2}.
\]