EXAMPLES, COUNTEREXAMPLES, AND STRUCTURE IN BOUNDED WIDTH ALGEBRAS

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Abstract. We study bounded width algebras which are minimal in the sense that every proper reduct does not have bounded width. We show that minimal bounded width algebras can be arranged into a pseudovariety with one basic ternary operation. We classify minimal bounded width algebras which have size at most three, and prove a structure theorem for minimal bounded width algebras which have no majority subalgebra, which form a pseudovariety with a commutative binary operation. We provide a counterexample to a recent conjecture about three variable Mal’cev conditions for bounded width algebras, and we show that any two variable Mal’cev condition which holds in the two element semilattice and majority algebras holds in every bounded width algebra.

1. Introduction

In the past several years, a number of beautiful characterizations of bounded width algebras have been found. We summarize a few of them in the next proposition.

Proposition 1. Let \( \mathbb{A} \) be a finite idempotent algebra. The following are equivalent:

1. \( \mathbb{A} \) has bounded relational width.
2. \( \mathbb{A} \) has relational width at most \((2,3)\).
3. Every “Prague instance” of \( \text{CSP}(\mathbb{A}) \) has a solution.
4. \( \text{CSP}(\mathbb{A}) \) has a “robust” satisfiability algorithm.
5. \( \text{CSP}(\mathbb{A}) \) is solved by the canonical SDP relaxation.
6. \( \mathbb{A} \) generates a congruence meet-semidistributive variety.
7. \( \mathbb{A} \) generates a variety which omits types 1 and 2.
8. \( \text{CSP}(\mathbb{A}) \) does not have “the ability to count”.
9. No nontrivial quotient of any subalgebra of \( \mathbb{A} \) is affine.
10. \( \mathbb{A} \) has weak near-unanimity terms \( g(x,y,z), h(x,y,z,w) \) satisfying

\[
g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx h(x,x,x,y) \approx h(x,x,y,x) \approx h(x,y,x,x) \approx h(y,x,x,x).
\]

11. For every sufficiently large \( k \), \( \mathbb{A} \) has a \( k \)-ary weak near-unanimity term.

Proof. For the equivalence of (1), (2), and (3), see [2] (or [7], for a slightly weaker result). For the equivalence of (1), (4), and (5), see [4]. For the equivalence of (6) and (7), see Theorem 9.10 of [10]. For the equivalence of (7), (8), and (9), see [16]. For the equivalence of (1) and (7), see [5]. For the equivalence of (7) and (10), see [15]. For the equivalence of (7) and (11), see [17]. \( \square \)

It is curious that there are only a few explicit examples of bounded width algebras in the literature. The two basic examples of bounded width algebras are the majority algebras and semilattice algebras. A majority algebra is an algebra having a ternary operation \( g \), called a majority operation, which satisfies the identity

\[
g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx x.
\]
The simplest example of a majority algebra is the dual discriminator algebra, with the basic operation $d$ which is given by

$$d(x, y, z) = \begin{cases} x & \text{if } y \neq z, \\ y & \text{if } y = z. \end{cases}$$

Another fundamental example of a majority algebra is the median algebra on an ordered set, whose basic operation returns the median of its three inputs. Generalizing majority algebras, there are the algebras of bounded strict width, which (by the Baker-Pixley Theorem [1]) are the algebras which have a near-unanimity term of some arity. A near-unanimity term $t$ of arity $n$ is defined to be a term which satisfies the identity

$$t(x, x, \ldots, y, x) \approx \cdots \approx t(y, x, \ldots, x, x) \approx t(y, x, \ldots, x, x) \approx x.$$

A binary operation $s$ is called a semilattice operation if it is associative, commutative, and idempotent. A standard example is the operation $\lor$ of any lattice, and in general every semilattice corresponds to a poset in which every pair of elements have a unique least upper bound. Generalizing semilattices are the so-called 2-semilattice operations, in which associativity is replaced by the weaker identity

$$s(x, s(x, y)) \approx s(x, y).$$

Unlike semilattices, a 2-semilattice need not be associated to a consistent partial ordering.

The goal of this paper is to develop a better understanding of bounded width algebras. In particular we will study several Malcev conditions described in [11]. In that paper, they prove that in every locally finite variety of bounded width, there exists an idempotent term $t$ satisfying the identity

(SM 3) \quad t_3(x, x, x, y) \approx t_3(x, x, y, x) \approx t_3(x, y, x, x) \approx t_3(y, x, x, x)

using a difficult Ramsey theoretic construction and the fact that every bounded width algebra has width $(2, 3)$. In this paper we’ll show that there is a much more direct argument for the existence of such a term (and, in fact, for many more terms) using Barto’s refinement in terms of “Prague instances” [2]. Two other conjectured terms from [11] are idempotent terms satisfying either the identity

(SM 5) \quad t_5(x, y, z, y) \approx t_5(y, x, z, z) \approx t_5(z, x, x, y)

or the identity

(SM 6) \quad t_6(x, y, z, y) \approx t_6(y, x, x, z) \approx t_6(z, y, x, x).

We give a three element bounded width algebra which does not have terms satisfying either identity (SM 5) or (SM 6).

Another unsolved conjecture about bounded width algebras is that the constraint satisfaction problems derived from them may be solved with the singleton arc consistency algorithm described in [9]. (TODO: reference recent solution. [14])

Since an algebra with fewer term operations corresponds to a relational clones containing more constraints, it is natural to study bounded width algebras such that their clone of operations is minimal among bounded width clones. The main result of this paper is a (nonconstructive) proof of the existence of an underlying structure for bounded width algebras which are minimal in this sense. This structure result rests crucially on the following coherence theorem, the proof of which is inspired by a clever argument of Bulatov from [8].

**Theorem 7.** Let $\mathcal{V}$ be a locally finite idempotent bounded width variety such that every algebra in $\mathcal{V}$ is connected through two element semilattice and two element majority subalgebras, let $\mathbb{A} \in \mathcal{V}$, and suppose there is a term $m$ and a subset $S \subseteq \mathbb{A}$ which is closed under $m$, such that $(S, m)$ is a
bounded width algebra. Let \( V' \) be the reduct of \( V \) consisting of all terms \( t \) of \( V \) such that \( S \) is closed under \( t \) and such that \( t|_S \in \text{Clo}(m|_S) \). Then \( V' \) also has bounded width.

Immediate consequences of this result are that every subalgebra and every quotient of a minimal bounded width algebra is also a minimal bounded width algebra. Using these, we show that the collection of minimal bounded width algebras can be arranged into a pseudovariety \( V_{mbw} \) (see Theorem \ref{thm:mbw} for details) - this is what is meant by the claim that minimal bounded width algebras have an underlying structure.

We note in passing that there is an analogue of Theorem \ref{thm:mbw} for idempotent varieties having \( p \)-ary cyclic terms. To see this, suppose that \( c \) is a \( p \)-ary cyclic term for \( V \), that \( A \in V \) and that there is a \( p \)-ary term \( m \) and a subset \( S \subseteq A \) which is closed under \( m \) and such that \( m|_S \) is cyclic. Then if we define a \( p \)-ary term \( t \) by

\[
t(x_1, \ldots, x_p) = c(m(x_1, x_2, \ldots, x_p), m(x_2, x_3, \ldots, x_1), \ldots, m(x_p, x_1, \ldots, x_{p-1})),
\]

we see that \( t \) is cyclic and that \( t|_S = m|_S \) (by idempotence of \( c \)). By one of the main results of \cite{bulatov1998infinite}, we can conclude that there is also an analogue of Theorem \ref{thm:mbw} for locally finite idempotent varieties which omit type 1.

Using this structure result, we classify minimal bounded width algebras of size 3 (see Theorem \ref{thm:mbw3} and Figure \ref{fig:mbw3}) as well as minimal bounded width algebras of size 4 which are generated by two elements (see Theorem \ref{thm:mbw4} and Figure \ref{fig:mbw4}).

2. Partial Semilattice Operations

In this section we collect some results which generalize results about 2-semilattices, and which apply in all idempotent varieties (even those varieties which do not omit type 1). Most of these results were found by Bulatov in \cite{bulatov2001minimal} and \cite{bulatov2001infinite}, but the presentation here is different. Lemma \ref{lem:iteration} seems to be an entirely new result: it shows that one may modify the basic operations of an idempotent algebra (by “preparing” their inputs) in order to make any “potential” semilattice subalgebra into an an actual semilattice subalgebra without affecting any of the two-variable linear identities satisfied by these operations.

**Definition 1.** We say that a binary operation \( s \) is a **partial semilattice** if it satisfies the identities

\[
s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y), \quad s(x, x) \approx x.
\]

Note that the definition of a partial semilattice implies that for all \( x, y \), the set \( \{ x, s(x, y) \} \) is closed under \( s \), and the restriction of \( s \) to this set is a semilattice operation. Partial semilattice operations are the binary analogue of unary operations \( u \) which satisfy \( u(u(x)) \approx u(x) \), in that they can be produced using an iteration argument.

**Lemma 1** (Semilattice Iteration Lemma). Let \( t \) be a binary idempotent term of a finite algebra. Then there exists a partial semilattice \( s \in \text{Clo}(t) \) which is built out of \( t \) in a nontrivial way (i.e. both \( x \) and \( y \) show up in the definition of \( s(x, y) \) in terms of \( t \)). In particular, for any \( a, b \) such that \( t(a, b) = t(b, a) = b \), we have \( s(a, b) = s(b, a) = b \).

**Proof.** Define \( t^0(x, y) = y \), \( t^1(x, y) = t(x, y) \), and

\[
t^{i+1}(x, y) = t(x, t^i(x, y)),
\]

and define \( t^\infty(x, y) \) by

\[
t^\infty(x, y) = \lim_{n \to \infty} t^n(x, y) \approx t^{|A|}(x, y).
\]

Note that we have

\[
t^\infty(x, t^\infty(x, y)) \approx t^\infty(x, y).
\]
Now define \( u(x, y) \) by
\[
    u(x, y) = t^\infty(x, t^\infty(y, x)),
\]
and define \( u^i(x, y), u^\infty(x, y) \) in analogy with \( t^i(x, y), t^\infty(x, y) \). We will show that \( s = u^\infty \) is a partial semilattice.

First, note that \( t^\infty(x, u^\infty(x, y)) \approx u^\infty(x, y) \), since we have
\[
\begin{align*}
    t^\infty(x, u^{i+1}(x, y)) &= t^\infty(x, u(x, u^i(x, y))) \\
    &= t^\infty(x, t^\infty(x, t^\infty(u^i(x, y), x))) \\
    &\approx t^\infty(x, t^\infty(u^i(x, y), x)) \\
    &= u(x, u^i(x, y)) = u^{i+1}(x, y)
\end{align*}
\]
for any \( i \geq 0 \). Thus, for any \( i \geq 0 \) we have
\[
\begin{align*}
    u^{i+1}(u^\infty(x, y), x) &= u^i(u^\infty(x, y), u(u^\infty(x, y), x)) \\
    &= u^i(u^\infty(x, y), t^\infty(u^\infty(x, y), t^\infty(x, u^\infty(x, y)))) \\
    &\approx u^i(u^\infty(x, y), t^\infty(u^\infty(x, y), u^\infty(x, y))) \\
    &\approx u^\infty(x, y),
\end{align*}
\]
where the last step follows from idempotence. From this we see that
\[
    u^\infty(u^\infty(x, y), x) \approx u^\infty(x, y),
\]
while
\[
    u^\infty(x, u^\infty(x, y)) \approx u^\infty(x, y)
\]
follows directly from the definition of \( u^\infty \). \qed

Remark 1. Essentially the same result is proved in Proposition 10 of [8], with a slightly different construction. The construction given here has the nice additional property that for any \( a, b \in A \), the value of \( s(a, b) \) may be computed from \( t|_{\text{SG}_A\{a, b\}} \) in time polynomial in the size of \( \text{SG}_A\{a, b\} \).

From any partial semilattice operation \( s \), we can define certain higher-arity terms \( s_n \) which behave nicely when the number of distinct values among their inputs is at most two.

**Proposition 2.** If \( s \) is a partial semilattice operation, then for all \( n \geq 1 \) there are terms \( s_n \in \text{Cl}_n(s) \) of arity \( n \) such that if \( \{ x_1, x_2, \ldots, x_n \} = \{ x, y \} \), then
\[
    s_n(x_1, x_2, \ldots, x_n) \approx s(x, y).
\]

**Proof.** Define \( n \)-ary functions \( s_n(x_1, \ldots, x_n) \) by \( s_1(x) = x, s_2(x, y) = s(x, y) \) and
\[
    s_n(x_1, \ldots, x_n) = s(s_{n-1}(x_1, \ldots, x_{n-1}), s(x_1, x_n)).
\]
We will show by induction on \( n \) that if \( \{ x, x_2, \ldots, x_n \} = \{ x, y \} \), then
\[
    s_n(x, x_2, \ldots, x_n) \approx s(x, y).
\]
There are three cases. If \( x_2 = \cdots = x_{n-1} = x \), then we must have \( x_n = y \), so
\[
    s_n(x, x_2, \ldots, x_n) = s(s_{n-1}(x, \ldots, x), s(x, y)) \approx s(x, s(x, y)) \approx s(x, y).
\]
If \( \{ x, x_2, \ldots, x_{n-1} \} = \{ x, y \} \) and \( x_n = x \), then by the inductive hypothesis
\[
    s_n(x, x_2, \ldots, x_n) = s(s_{n-1}(x, x_2, \ldots, x_{n-1}), s(x, x)) \approx s(s(x, y), x) \approx s(x, y).
\]
Finally, if \( \{ x, x_2, \ldots, x_{n-1} \} = \{ x, y \} \) and \( x_n = y \), then by the inductive hypothesis
\[
    s_n(x, x_2, \ldots, x_n) = s(s_{n-1}(x, x_2, \ldots, x_{n-1}), s(x, y)) \approx s(s(x, y), s(x, y)) \approx s(x, y). \qed
\]
Recall that an identity is \textit{linear} if it does not involve any nesting of operations. Remarkably, the class of bounded width algebras and the class of algebras which omit type 1 can both be characterized by Mal’cev conditions consisting of finitely many two-variable linear identities which involve both variables on each side (see [11] and [12]), making them well-suited to the next lemma.

\textbf{Lemma 2} (Semilattice Preparation Lemma). Let $\mathbb{A} = (A, (f_i)_{i \in I})$ be a finite idempotent algebra, and let $\Sigma$ be the set of all two-variable linear identities which involve both variables on each side and are satisfied in $\mathbb{A}$. Then $\mathbb{A}$ has terms $(f_i')_{i \in I}$ which satisfy the identities in $\Sigma$, and such that for every pair of subalgebras $C \leq B$ of $(A, (f_i')_{i \in I})$ such that there exists a term $t \in \text{Clo}((f_i')_{i \in I})$ with $t(b, c), t(c, b) \in C$ whenever $b \in B, c \in C$, we in fact have $f_i'(b_1, ..., b_m) \in C$ whenever $b_1, ..., b_m \in B$ such that at least one of $b_1, ..., b_m$ is an element of $C$. In particular, if there is a $t \in \text{Clo}((f_i')_{i \in I})$ with $t(b, c) = t(c, b) = c$, then $\{b, c\}$ is a semilattice subalgebra of $(A, (f_i')_{i \in I})$.

\textbf{Proof.} Suppose that $(f_i')_{i \in I}$ are chosen to satisfy the identities in $\Sigma$, such that the number of pairs of subalgebras $C \leq B$ of $(A, (f_i')_{i \in I})$ such that for each $m$-ary term $f_i'$ and any $b_1, ..., b_m \in B$ with at least one of the $b_i$s in $C$ we have $f_i'(b_1, ..., b_m) \in C$. Suppose that there is a pair of subalgebras $C \leq B$ of $(A, (f_i')_{i \in I})$ and a term $t \in \text{Clo}((f_i')_{i \in I})$ with $t(b, c), t(c, b) \in C$ for all $b \in B, c \in C$, and apply Lemma [1] to produce a nontrivial partial semilattice $s \in \text{Clo}(t)$. By the construction of $s$, we will have $s(b, c), s(c, b) \in C$ whenever $b \in B, c \in C$. We just need to show that we can find $f''_i \in \text{Clo}(f'_i)$ which satisfy the identities in $\Sigma$ and are such that for each $m$-ary term $f''_i$ and any $b_1, ..., b_m \in B$ with at least one of the $b_i$s in $C$ we have $f''_i(b_1, ..., b_m) \in C$, since then by choice of $f''_i$ the $f_i'$ must already have this property.

Define functions $s_n$ in terms of $s$ as in Proposition 2. Now for each $m$-ary $f'_i$, we define $f''_i$ by
\[
f''_i(x_1, ..., x_m) = f'_i(s_m(x_1, ..., x_m), s_m(x_2, ..., x_m, x_1), ..., s_m(x_m, x_1, ..., x_{m-1})).\]
It is clear that each $f''_i$ is such that for each $m$-ary term $f''_i$ and any $b_1, ..., b_m \in B$ with at least one of the $b_i$s in $C$ we have $f''_i(b_1, ..., b_m) \in C$, since each $s_m$ has this property and $C$ is closed under $f'_i$. Now suppose we have an identity
\[
f'_i(a_1, ..., a_m) \approx f'_j(b_1, ..., b_n),\]
with $\{a_1, ..., a_m\} = \{b_1, ..., b_n\} = \{x, y\}$. Define $a'_1, ..., a'_m$ by $a'_k = s(x, y)$ if $a_k = x$ and $a'_k = s(y, x)$ if $a_k = y$, and define $b'_1, ..., b'_n$ similarly. Then for each $k$, we have
\[s_m(a_k, ..., a_m, a_1, ..., a_{k-1}) \approx a'_k,\]
and similarly for the $b_i$s, so
\[f''_i(a_1, ..., a_m) \approx f'_i(a'_1, ..., a'_m) \approx f'_i(b'_1, ..., b'_n) \approx f''_i(b_1, ..., b_n). \qquad \Box\]

\textbf{Definition 2.} We say that an idempotent algebra $\mathbb{A}$ has been \textit{prepared} if for every pair $a, b$ and every binary term $t$ such that $t(a, b) = t(b, a) = a, \{a, b\}$ is a semilattice subalgebra of $\mathbb{A}$. A partial semilattice term $s$ of $\mathbb{A}$ is called \textit{adapted} to $\mathbb{A}$ if it can be built out of the basic operations of $\mathbb{A}$ in a nontrivial way.

\textbf{Proposition 3.} If $\mathbb{A}$ has been prepared as above and $a, b, c \in \mathbb{A}$ have $c \in \text{Sg}\{a, b\}$ with $\{a, c\}$ a two element semilattice subalgebra directed from $a$ to $c$, then $\mathbb{A}$ has a partial semilattice term $s$ with $s(a, b) = c$.

\textbf{Proof.} Let $s'$ be an arbitrary partial semilattice term which is adapted to $\mathbb{A}$, and choose $p$ a binary term of $\mathbb{A}$ with $p(a, b) = c$. Then take $s(x, y) = s'(x, p(x, y))$. We clearly have $s(a, b) = s'(a, p(a, b)) = s'(a, c) = c$, so we just have to check that $s$ is a partial semilattice. If $p$ is second projection then $s = s'$ and we are done. Otherwise, since $\mathbb{A}$ has been prepared, $p$ acts as the semilattice operation on $\{x, s'(x, p(x, y))\} = \{x, s(x, y)\}$. Thus,
\[s(x, s(x, y)) = s'(x, p(x, s'(x, p(x, y)))) \approx s'(x, s'(x, p(x, y))) \approx s'(x, p(x, y)) = s(x, y),\]
of finite algebras

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Lemma 3. If s is a partial semilattice operation and a, b have \( s(a, b) = b \), then we write \( a \to s b \), or just \( a \to b \) if s is understood (or if the algebra has been prepared). We say that b is reachable from a if there is a sequence \( a = a_0, a_1, ..., a_k = b \) such that \( a_i \to a_{i+1} \) for \( i = 0, ..., k - 1 \). We say that a set \( A \) is strongly connected if for every subset \( S \subseteq A \) with \( S \neq \emptyset \), \( A \) there is an \( a \in S \) and a \( b \in A \setminus S \) such that \( a \to b \). We say that a set \( A \) is a maximal strongly connected component of an algebra \( \mathbb{A} \) if \( A \) is a strongly connected subset of \( \mathbb{A} \) such that for any \( a \in A \) and any \( b \in \mathbb{A} \) with \( a \to b \) we have \( b \in A \) (note that every finite algebra has at least one maximal strongly connected component). Finally, we call an element of an algebra \( \mathbb{A} \) maximal if it is contained in any maximal strongly connected component of \( \mathbb{A} \).

Remark 2. If an algebra \( \mathbb{A} \) has been prepared and is strongly connected, then it has no proper absorbing subalgebra in the sense of [3]. However, even in the case of strongly connected algebras the next result is not a consequence of the Absorption Theorem since it applies even in varieties which do not omit type 1.

Definition 4. A relation \( \mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B} \) is linked if for any two elements \( a, a' \in \mathbb{A} \) there exists a sequence \( a = a_0, b_1, a_1, ..., b_k, a_k = a' \) with \( (a_i, b_{i+1}) \in \mathbb{R} \) and \( (a_i, b_i) \in \mathbb{R} \) for all i. In other words, \( \mathbb{R} \) is linked if it is connected when viewed as a bipartite graph on \( \mathbb{A} \cup \mathbb{B} \), or equivalently when \( \ker \pi_1 \vee \ker \pi_2 \) is the total congruence of \( \mathbb{R} \). An element of \( \mathbb{A} \) is called a fork for \( \mathbb{R} \) if there exist \( b \neq b' \) such that \( (a, b), (a, b') \in \mathbb{R} \), and similarly for elements of \( \mathbb{B} \).

Lemma 3. Fix a partial semilattice operation s. Suppose that \( \mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B} \) is a subdirect product of finite algebras \( \mathbb{A}, \mathbb{B} \), and that \( \mathbb{B} \) is simple and \( \mathbb{B} = \text{Sg}(B) \), with \( B \) a maximal strongly connected component of \( \mathbb{B} \). Then:

(a) if \( \mathbb{A} \) is also simple and \( \mathbb{A} = \text{Sg}(A) \) with \( A \) a maximal strongly connected component of \( \mathbb{A} \), and if \( \mathbb{R} \cap (A \times B) \neq \emptyset \), then \( \mathbb{R} \) is either the graph of an isomorphism or \( \mathbb{A} \times \mathbb{B} \), and

(b) if \( \mathbb{A} \) is arbitrary and \( \mathbb{R} \) is not the graph of a homomorphism from \( \mathbb{A} \) to \( \mathbb{B} \), then there is an \( a \in \mathbb{A} \) with \( \{a\} \times \mathbb{B} \subseteq \mathbb{R} \).

Proof. We give a retelling of the proof of Lemma 9 of [7] (with a few modifications and corrections), which was a generalization of the corresponding fact for 2-semilattices proved in [6]. Although we will only apply this result when \( \mathbb{B} \) is strongly connected, it seems necessary to prove this result in the form originally given by Bulatov in [7] in order to make the inductive argument work out.

The strategy is to induct on \( |\mathbb{A}| + |\mathbb{B}| \). First we prove (a). Suppose that \( \mathbb{R} \) is not the graph of an isomorphism, then since \( \mathbb{A} \) and \( \mathbb{B} \) are simple \( \mathbb{R} \) must be linked. Let \( \mathbb{E} \) be a proper subalgebra of either \( \mathbb{A} \) or \( \mathbb{B} \) which contains at least one fork of \( \mathbb{R} \) and such that the number of elements of \( \mathbb{R} \) which meet \( \mathbb{E} \) is maximal - since \( \mathbb{R} \) is linked, some one-element subalgebra of \( \mathbb{A} \) must be a fork, so such \( \mathbb{E} \) exists. Assume without loss of generality that \( \mathbb{E} \) is contained in \( \mathbb{A} \). Since \( \mathbb{R} \) is linked, there must be some pair \( a, a' \in \mathbb{A} \) with \( a \in \mathbb{E}, a' \not\in \mathbb{E} \), and such that some element \( b \in \mathbb{B} \) has \( (a, b), (a', b) \in \mathbb{R} \). Thus the set of elements \( b \in \mathbb{B} \) for which there exists \( a \in \mathbb{E} \) with \( (a, b) \in \mathbb{R} \) is a subalgebra of \( \mathbb{B} \) which contains a fork and meets strictly more edges of \( \mathbb{R} \) than \( \mathbb{E} \) does, so by the choice of \( \mathbb{E} \) this set must be all of \( \mathbb{B} \), that is, \( \mathbb{R} \cap (\mathbb{E} \times \mathbb{B}) \) is a subdirect product of \( \mathbb{E} \) and \( \mathbb{B} \) which is not the graph of a homomorphism from \( \mathbb{E} \) to \( \mathbb{B} \). Then by part (b) applied inductively to \( \mathbb{R} \cap (\mathbb{E} \times \mathbb{B}) \), we see that there is some \( e \in \mathbb{E} \) with \( \{e\} \times \mathbb{B} \subseteq \mathbb{R} \).

For any \( a \in \mathbb{A} \), let \( S_a \) be the set of \( b \in \mathbb{B} \) such that \( (a, b) \in \mathbb{R} \). Since \( \mathbb{R} \cap (A \times B) \neq \emptyset \), there is some \( a \in \mathbb{A} \) with \( S_a \neq \emptyset \). Choose \( a \in \mathbb{A} \) such that \( S_a \) is maximal, and suppose for contradiction that \( S_a \neq B \). Then since \( B \) is strongly connected, we can find \( b \to b' \in B \) with \( b \in S_a \) and \( b' \not\in S_a \). Since \( a \to s(a, e) \) and \( A \) is a maximal strongly strongly connected component of \( \mathbb{A} \) we also have
Now suppose for contradiction that $A$ is strongly connected. Let $(a, b) \in A$ be a maximal congruence of $A$, so $s(a, b) = \{a, b\}$ is a maximal strongly connected component, this implies that $s(b, b' \in B$ and $s(b, b') \subseteq S_{a'} \neq \emptyset$. Thus $S_{a'} \neq \emptyset$. Then since $B$ is strongly connected, there are $b \rightarrow b'$ in $B$ with $b \in S_{a'}$, $b' \notin S_{a'}$. Since $S_{a'} = B$, we get

$$(a', b') = (s(a, a'), s(b, b')) = s((a, b'), (a', b')) \in \mathbb{R},$$

so $b' \in S_{a'}$, a contradiction. Thus we must in fact have $A \times B \subseteq \mathbb{R}$, and so $\mathbb{A} \times \mathbb{B} = Sg(A \times B) \subseteq \mathbb{R}$.

Now we prove (b). Let $(a, b)$ be a maximal element of $A$ which are reachable from $a$ and such that there exist $b' \in B$ with $(a', b') \in \mathbb{R}$. Suppose for a contradiction that there is $a' \rightarrow a''$ with $a' \in A$ and $a'' \notin A$, then by the definition of $A$ and the fact that $\mathbb{R}$ is a subdirect there are $b', b''$ with $(a', b'), (a'', b'') \in \mathbb{R}$ and $b' \in B$, so

$$(a'', s(b', b'')) = (s(a', a''), s(b', b'')) = s((a', b'), (a'', b'')) \in \mathbb{R},$$

and since $b' \rightarrow b''$ we see that $a'' \in A$, a contradiction. Thus by the choice of $(a, b)$, $A$ must be a maximal strongly connected component of $\mathbb{A}$.

Suppose now that $\mathbb{R}$ is not the graph of a homomorphism from $\mathbb{A}$ to $\mathbb{B}$. Then since $\mathbb{B}$ is simple, and the congruence $\pi_2(\ker \pi_1 \lor \ker \pi_2)$ is a nontrivial congruence of $\mathbb{B}$, it must be the total congruence, so $\mathbb{B}$ is in a connected component of $\mathbb{R}$ considered as a bipartite graph on $\mathbb{A} \cup \mathbb{B}$. Choose any $b' \in B$ with $b \rightarrow b'$ and $b \neq b'$ (such $b'$ exists since $B$ is strongly connected), and find a path $b = b_0, a_1, b_1, ..., a_k, b_k = b'$ with $(a_i, b_{i-1}) \in \mathbb{R}$ and $(a_i, b_i) \in \mathbb{R}$ for all $i$. Then from $s(b, b) = b \neq b' = s(b, b')$, there exists some $i$ such that $s(b, b_{i-1}) \neq s(b, b_i)$. Then

$$(s(a, a_i), s(b, b_{i-1})) = s((a, b), (a_i, b_{i-1})) \in \mathbb{R}$$

and

$$(s(a, a_i), s(b, b_i)) = s((a, b), (a_i, b_i)) \in \mathbb{R}$$

have $s(a, a_i) \in A$ and $s(b, b_{i-1}) \neq s(b, b_i) \in B$. Setting $a' = s(a, a_i)$, we have shown that $a' \in A$ and $a'$ is a fork of $\mathbb{R} \cap (A \times B)$.

Let $\mathbb{A}' = Sg(A)$, and let $\theta$ be a maximal congruence of $\mathbb{A}'$. Then $\mathbb{A}'/\theta$ simple and is generated by $A/\theta$, which is a strongly connected component of $\mathbb{A}'/\theta$, and $(a/\theta, b) \in \mathbb{A}'/\theta \times \mathbb{B}$, so by part (a) and the fact that $a'/\theta \in \mathbb{A}'/\theta$ is a fork of $\mathbb{R}/\theta$ we have

$$(\mathbb{R} \cap (\mathbb{A}' \times \mathbb{B}))/\theta = \mathbb{A}'/\theta \times \mathbb{B}.$$

If $\mathbb{A}' = \mathbb{A}$ and $\theta$ is trivial, then we are done. Otherwise, letting $\mathbb{A}''$ be the congruence class of $a'$ modulo $\theta$ in $\mathbb{A}'$, we see that $\mathbb{A}''$ is a proper subalgebra of $\mathbb{A}$ such that $\mathbb{R} \cap (\mathbb{A}'' \times \mathbb{B})$ is a subdirect product of $\mathbb{A}''$ and $\mathbb{B}$ and that $a' \in \mathbb{A}''$ is a fork of $\mathbb{R} \cap (\mathbb{A}'' \times \mathbb{B})$, so by the inductive hypothesis there is some $a'' \in \mathbb{A}''$ with $\{a''\} \times \mathbb{B} \subseteq \mathbb{R}$, and we are done. \qed

**Theorem 1.** Fix a partial semilattice operation $s$. If $R \subseteq A \times B \times C$ is closed under $s$, $A, B, C$ are finite and strongly connected, and $\pi_{1,2}R = A \times B$, $\pi_{1,3}R = A \times C$, $\pi_{2,3}R = B \times C$, then $R = A \times B \times C$. 

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Proof. We give a simplification of the proofs of Lemmas 10, 12, and 13 of [7] which does not rely on the results of [13].

Let \( Q \subseteq C \) be the set of all \( c \in C \) such that \( A \times B \times \{c\} \subseteq R \). First we show that if \( Q \neq \emptyset \) then \( Q = C \). Suppose otherwise. Then by the definition of a strongly connected set, there are \( c, c' \in C \) with \( c \in Q, c' \notin Q \) and \( c \to c' \). Since \( A \times B \) is strongly connected and since \( R \cap (A \times B \times \{c'\}) \neq \emptyset \), there must be \( (a, b), (a', b') \in A \times B \) with \( (a, b) \to (a', b') \) and \( (a, b, c') \in R, (a', b', c') \notin R \). Since \( c \in Q \) we have \( (a', b', c') \in R \), so
\[
(a', b', c') = (s(a, a'), s(b, b'), s(c', c)) = s((a, b, c'), (a', b', c')) \in R,
\]
a contradiction. Thus we must have \( Q = C \), so \( R = A \times B \times C \). A similar argument shows that if there is any \( a \in A \) with \( \{a\} \times B \times C \subseteq R \) or \( b \in B \) with \( A \times \{b\} \times C \subseteq R \) then \( R = A \times B \times C \).

We divide into cases depending on whether any of \( A, B, C \) are simple. Suppose first that two of \( A, B, C \) are simple, say \( A \) and \( B \). For every \( c \in C \) define \( R_c \) to be
\[
R_c = \{(a, b) \in A \times B \ | (a, b, c) \in R\} = \pi_{1,2}(R \cap (A \times B \times \{c\})).
\]
If any \( R_c = A \times B \), then \( A \times B \times \{c\} \subseteq R \) and we are done. Otherwise, by Lemma 3, each \( R_c \) must be the graph of an isomorphism from \( A \) to \( B \). Choose a pair \( a \to a' \) in \( A \) and some \( c \in C \). Since \( R_c \) is an isomorphism, there are elements \( b \to b' \) in \( B \) such that \( (a, b, c), (a', b', c) \in R \). Since \( \pi_{1,2}R = A \times B \), there is some \( c' \in C \) with \( (a, b', c') \in R \). Then
\[
(a', b', c(c, c')) = (s(a, a), s(b, b'), s(c, c')) = s((a, b, c), (a, b', c')) \in R
\]
and
\[
(a, b', s(c, c')) = (s(a', a), s(b', b'), s(c, c')) = s((a', b', c), (a, b', c')) \in R,
\]
so \( R_{s(c,c')} \) is not the graph of an isomorphism from \( A \) to \( B \), a contradiction.

Finally, we suppose that at least one of \( A, B, C \) is not simple, say \( C \), and suppose for a contradiction that we have a minimal counterexample. Let \( \theta \) be a maximal congruence of \( C \). Then since \( C/\theta \) is smaller than \( C \), we can apply the induction hypothesis to see that \( R/\theta = A \times B \times (C/\theta) \). Let \( C' \) be any congruence class of \( \theta \), and let \( C'' \) be a maximal strongly connected component of \( C' \). Define a subset \( P \subseteq A \times B \) by
\[
P = \pi_{1,2}(R \cap (A \times B \times C'')).
\]
Since \( C'' \subset \pi_3R \), \( P \) is not empty. If \( P = A \times B \), then \( A \times B \times C'' \subseteq R \), so \( R = A \times B \times C \). Thus we may assume that \( P \neq A \times B \) as well. Then since \( A \times B \) is strongly connected, there are \( (a, b) \to (a', b') \) with \( (a, b) \in P \) and \( (a', b') \notin P \). Since \( (a, b) \in P \), there is some \( c \in C'' \) with \( (a, b, c) \in R \), and since \( R/\theta = A \times B \times (C/\theta) \) there is some \( c' \in C'' \) with \( (a', b', c') \in R \). Then
\[
(a', b', s(c, c')) = (s(a, a'), s(b, b'), s(c, c')) = s((a, b, c), (a', b', c')) \in R
\]
and since \( C'' \) is a maximal strongly connected component of \( C' \) we must have \( s(c, c') \in C'' \). But then \( (a', b') \in P \), a contradiction. \( \square \)

3. PRAGUE INSTANCES

We will use the homomorphism description of the constraint satisfaction problem. Let \( A, B \) be two relational structures with the same signature \( \sigma \). The constraint satisfaction problem for the pair \( B, A \) asks whether there exists a homomorphism \( B \to A \). It might be helpful to think of \( A \) as a set of possible values together with library of constraint relations indexed by \( \sigma \), and to think of \( B \) as an edge-colored, directed hypergraph on a set of variables, where the colors are the elements of \( \sigma \), and to think of the homomorphism as an assignment of values to the variables satisfying the constraints corresponding to the edges of \( B \).

If we restrict to constraint satisfaction problems with fixed target \( A \), the resulting constraint satisfaction problem is denoted \( \text{CSP}(A) \). The complexity of this problem is known to only depend
on the algebra of polymorphisms of \( A \), which we will denote \( \mathcal{A} \), so we may also sometimes speak
of the problem CSP\( (\mathcal{A}) \), by which we mean CSP\( (\text{Inv}(\mathcal{A})) \), where Inv\( (\mathcal{A}) \) is the relational structure
on the same underlying set having as relations all (underlying sets of) subpowers of the algebra \( \mathcal{A} \).

We now repeat Barto’s definition of a Prague instance of a constraint satisfaction problem, as
well as his main theorem which we will use as a black box.

**Definition 5.** Let \( A, B \) be relational structures with the same signature \( \sigma \). The instance \( (B, A) \) is
called 1-minimal if for any \( R, S \in \sigma \) and any \((v_1, \ldots, v_k) \in R^B \), \((w_1, \ldots, w_l) \in S^B \) and \( i, j \) such that
\( v_i = w_j \), we have \( \pi_i(R^A) = \pi_j(S^A) \). In this case, if \( b \) is any element of \( B \), we write \( A_b \) for \( \pi_i(R^A) \),
where \( R \in \sigma, (v_1, \ldots, v_k) \in R^B, 1 \leq i \leq k, v_i = b \) (note that this is well-defined by 1-minimality), or
\( A_b = A \) if no such \( R, (v_1, \ldots, v_k), i \) exist.

**Definition 6.** Let \( (B, A) \) be a 1-minimal instance. A pattern \( p \) of length \( k - 1 > 0 \) from \( b_1 \) to \( b_k \)
in \( B \) is a tuple

\[
p = (b_1, (R_1, i_1, j_1), b_2, (R_2, i_2, j_2), \ldots, (R_{k-1}, i_{k-1}, j_{k-1}), b_k)
\]
such that each \( b_i \in B \), each \( R_i \in \sigma \) and for each \( l \) there is a tuple \((t_1, \ldots, t_m) \in R_i^B \) such that
t\( t_i = b_1, t_j = b_{l+1} \). We set \( [[p]] = \{b_1, \ldots, b_k\} \), the set of all variables showing up in \( p \). The pattern
\( p \) is closed if \( b_1 = b_k \).

A realization of \( p \) in \( A \) is a tuple \((f_1, \ldots, f_{k-1}) \) such that for each \( l \) we have \( f_l \in R_l^A \) and
\( \pi_{i_l}(f_l) = \pi_{i_{l+1}}(f_{l+1}) \). We say that this realization connects \( \pi_{i_l}(f_l) \) to \( \pi_{i_{l+1}}(f_{l+1}) \), and we say that
\( p \) connects \( x \in A_{b_l} \) to \( y \in A_{b_k} \) if it has a realization connecting \( x \) to \( y \).

If \( S \subseteq B \), \( b \in S \), and \( x, y \in A_b \), we say that \( x, y \) are connected in \( S \) if there exists a pattern \( p \)
connecting \( x \) to \( y \) with \([[p]] \subseteq S \).

If \( p \) is a pattern from \( b_1 \) to \( b_k \) and \( U \subseteq A_{b_1} \), we define a subset \( U + p \) of \( A_{b_k} \) by

\[
U + p = \{y \in A_{b_k} \mid \exists x \in U \ p \text{ connects } x \text{ to } y\}.
\]

If \( p = (b_1, (R_1, i_1, j_1), \ldots, (R_{k-1}, i_{k-1}, j_{k-1}), b_k) \) and \( q = (b_k, (R_k, i_k, j_k), \ldots, (R_{k+l-1}, i_{k+l-1}, j_{k+l-1}), b_{k+l}) \),
we put \( p + q = (b_1, (R_1, i_1, j_1), \ldots, (R_{k+l-1}, i_{k+l-1}, j_{k+l-1}), b_{k+l}) \) (note that we need the last
element of \( p \) to match the first element of \( q \) to make this definition). We also define \(-p = (b_{k+l}, (R_{k-1}, i_{k-1}, j_{k-1}), \ldots, (R_1, i_1, j_1), b_1) \). For a closed pattern \( p \) and \( m \geq 1 \) we put \( m \times p = p + p + \cdots + p \), with \( m \) copies of \( p \) on the right hand side.

**Definition 7.** An instance \( (B, A) \) is a Prague instance if it is 1-minimal and for any \( b \in B \), any
closed pattern \( p \) from \( b \) to \( b \), and any \( x, y \in A_b \), if \( x, y \) are connected in \([[p]] \) then there exists \( m \geq 1 \)
such that \( m \times p \) connects \( x \) to \( y \).

**Theorem 2** (Barto, [2]). If a finite algebra \( \mathcal{A} \) has bounded width, then every Prague instance \( (B, A) \)
has a solution.

Prague instances are preserved by taking the closure with respect to an algebra of polymorphisms:

**Proposition 4.** If \( (B, S) \) is a Prague instance, and \( \mathcal{A} \) is a finite algebra with underlying set containing
the underlying set of \( S \) and we define \( C \) to be the relational structure such that for each
\( R \in \sigma \) of arity \( k \) we have \( R^C = \text{Sg}_{\mathcal{A}^k}(R^S) \) (here \( \text{Sg}_{\mathcal{A}^k}(X) \) is the subalgebra generated by \( X \) in \( \mathcal{A}^k \)),
then \( (B, C) \) is also a Prague instance. In particular, if \( \mathcal{A} \) has bounded width then the instance
\( (B, C) \) has a solution.

*Proof.* This is routine. For a proof of a similar statement about weak Prague instances, see [3]. \( \square \)

4. **Intersecting Families of Sets**

**Definition 8.** Let \( S \) be a set. A family \( \mathcal{F} \subseteq P(S) \) is called an intersecting family of subsets of \( S \)
if \( A, B \in \mathcal{F} \) implies \( A \cap B \neq 0 \).
Proposition 5. An intersecting family of subsets of a set $S$ is maximal (with respect to containment) if and only if for every set $A \subseteq S$ we have either $A \in \mathcal{F}$ or $(S \setminus A) \in \mathcal{F}$. For every $n \geq 1$ there is an obvious bijection between maximal intersecting families $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ and self-dual monotone boolean functions $f : \{0,1\}^n \to \{0,1\}$.

Theorem 3. Let $\mathcal{V}$ be a locally finite idempotent variety of bounded width. Then there is an idempotent term $f$ of arity 2 of $\mathcal{V}$ and a family of idempotent terms $h_\mathcal{F}$ in $\mathcal{V}$, indexed by maximal intersecting families $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ for $n \geq 2$, such that for each maximal intersecting family $\mathcal{F}$ of $\{1, \ldots, n\}$, $h_\mathcal{F}$ has arity $n$, and for every sequence $(a_1, \ldots, a_n)$ such that $\{a_1, \ldots, a_n\} = \{x, y\}$ and $\{i \in \{1, \ldots, n\} | a_i = x\} \in \mathcal{F}$ we have

$$h_\mathcal{F}(a_1, \ldots, a_n) \approx f(x, y).$$

Furthermore, we can choose $f$ such that for any binary term $t$ in the clone generated by the functions $h_\mathcal{F}$ and any pair $a, b$ of distinct elements of some algebra in $\mathcal{V}$, if $t(a, b) = t(b, a) = a$ then $f(a, b) = f(b, a) = a$, and such that

$$f(f(x, y), f(y, x)) \approx f(x, y).$$

Proof. We assume without loss of generality that $\mathcal{V}$ is idempotent. The strategy is to apply Proposition 4 with $\mathcal{S}$ a relational structure on the two element set $\{x, y\}$, with signature $\sigma$ equal to the collection of all maximal intersecting families $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ for all $n \geq 2$ and $\mathcal{A}$ equal to the free algebra on two generators in $\mathcal{V}$, $\mathcal{F}_\mathcal{V}(x, y)$ (which is finite and bounded width by assumption). A solution to the resulting instance $(\mathcal{B}, \mathcal{C})$ will then correspond to a family of terms, one for each relation symbol $R \in \sigma$. In order to show that these terms satisfy the required identities we will think of the solution to this instance as a coloring of the vertices of the directed hypergraph $\mathcal{B}$, and use a pigeonhole argument to show that each $\mathcal{F}_\mathcal{B}$ contains an edge such that all of its vertices have the same color.

First we have to describe how we realize the relations $\mathcal{F}$ in $\mathcal{S}$. If $\mathcal{F}$ is a maximal intersecting family of subsets of $\{1, \ldots, n\}$, first we fix any ordering $A_1, \ldots, A_{2^n−1}$−1 of the elements of $\mathcal{F} \setminus \{\{1, \ldots, n\}\}$ such that $|A_1| \leq |A_2| \leq \cdots \leq |A_{2^n−1}|$. Then $\mathcal{F}_\mathcal{S}$ will be the arity $2^n−1 − 1$ relation with elements $t^1, \ldots, t^n \in \{x, y\}^{2^n−1−1}$ given by

$$t_i^j = x \iff i \in A_j.$$ 

Next we describe $\mathcal{B}$. As the underlying set, we take the set of natural numbers $\mathbb{N}$. For each $\mathcal{F}$ a maximal intersecting family of subsets of $\{1, \ldots, n\}$, we take $\mathcal{F}_\mathcal{B}$ to be the set of all strictly increasing sequences of $\mathbb{N}$ of length $2^n−1 − 1$.

Since no element of any $\mathcal{F} \setminus \{\{1, \ldots, n\}\}$ is either the empty set or the entire set, for each $1 \leq i \leq 2^n−1 − 1$ we have $\pi_i(\mathcal{F}_\mathcal{S}) = \{x, y\}$, so the instance $(\mathcal{B}, \mathcal{S})$ is 1-minimal. In order to check that it is a Prague instance, we note that for any $1 \leq i < j \leq 2^n−1 − 1$ the relation $\pi_{i,j}(\mathcal{F}_\mathcal{S})$ necessarily contains both $(x, x)$ (since $\mathcal{F}$ is an intersecting family) and $(y, x)$ (since $|A_i| \leq |A_j|$), and contains at least one of $(x, y), (y, y)$ (since $A_j \neq \{1, \ldots, n\}$). From this and the fact that any closed pattern $p = (b_1, \ldots, b_k)$ in $\mathcal{B}$ must contain an element $b_j$ with $b_{j−1} < b_j > b_{j+1}$ (indices considered modulo $k$), it is easy to see that $\{x\} + p + p = \{y\} + p + p = \{x, y\}$, which shows that $(\mathcal{B}, \mathcal{S})$ is a Prague instance.

Applying Proposition 4 we find a solution to the instance $(\mathcal{B}, \mathcal{C})$ where $\mathcal{C}$ is the closure of $\mathcal{S}$ in $\mathcal{F}_\mathcal{V}(x, y)$. Applying the infinite pigeonhole principle, we see that there is an infinite subset $X$ of $\mathcal{B}$ which is all assigned the same value $f(x, y) \in \mathcal{F}_\mathcal{V}(x, y)$. Thus since each $\mathcal{F}_\mathcal{B}$ contains a tuple from $X$, there must be an element of $\text{Sg}_{\mathcal{F}_\mathcal{V}(x, y)}^{n−1−1}(\mathcal{F}_\mathcal{S})$ such that each coordinate is equal to $f(x, y)$. Calling the $n$ elements of $\mathcal{F}_\mathcal{S}$ $t^1, \ldots, t^n$ as above, we then have a term $h_\mathcal{F}$ such that each coordinate of $h_\mathcal{F}(t^1, \ldots, t^n)$ is $f(x, y)$, which is what was needed.

In order to guarantee that whenever there is a term $t$ with $t(a, b) = t(b, a) = b$ we have $f(a, b) = f(b, a) = b$, we apply Lemma 2 to the free algebra on $\mathcal{V}$ with two generators.
Now we show how to modify these functions to ensure that \( f(f(x,y), f(y,x)) \approx f(x,y) \). First we define the dictator family \( D_n = \{ A \subseteq \{1, \ldots, n\} \mid 1 \in A \} \), and set \( f_n(a_1, \ldots, a_n) = h_{D_n}(a_1, \ldots, a_n) \). Then if \( \{x, a_2, \ldots, a_n\} = \{x, y\} \), we have

\[
f_n(x, a_2, \ldots, a_n) \approx f(x, y).
\]

Now we define a sequence of functions \( f^i(x,y) \) by \( f^0(x,y) \approx x \), \( f^1(x,y) \approx f(x,y) \)

\[
f^{i+1}(x,y) \approx f(f^i(x,y), f^i(y,x)),
\]

and define a sequence of functions \( h^i_F(a_1, \ldots, a_n) \) by \( h^1(a_1, \ldots, a_n) = h(a_1, \ldots, a_n) \),

\[
h^{i+1}_F(a_1, \ldots, a_n) \approx h^i_F(f_n(a_1, \ldots, a_n), f_n(a_2, \ldots, a_n, a_1), \ldots, f_n(a_n, a_1, \ldots, a_{n-1})).
\]

It’s then easy to show by induction on \( i \geq 1 \) that if \( \{a_1, \ldots, a_n\} = \{x, y\} \) and \( \{i \in \{1, \ldots, n\} \mid a_i = x\} \in F \), then

\[
h^i_F(a_1, \ldots, a_n) \approx f^i(x,y).
\]

Replacing \( f \) by \( f^i \) and \( h_F \) by \( h^i_F \) finishes the proof.

A term \( t_3 \) satisfying (SM 3) is now given by \( h_H \), where

\[
H = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.
\]

Note that the identity

\[
f(f(x,y), f(y,x)) \approx f(x,y)
\]

implies that for any \( a, b \) either have \( f(a,b) = f(b,a) \) or the two element set \( \{f(a,b), f(b,a)\} \) is closed under all of the operations \( h_F \), and that they act on this set as the corresponding self-dual monotone functions. Furthermore, setting \( g = h_M \), where \( M = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \), we see that \( g \) acts as a majority operation on the set \( \{f(a,b), f(b,a)\} \). Since every self-dual monotone function on a two element set is contained in the clone generated by the majority function, it’s easy to show that we can in fact write all the functions \( h_F \) in terms of the \( f_n \)s (corresponding to the dictator families) and \( g \). Curiously, it doesn’t seem easy to find a way to write the functions \( f_n \) for \( n \geq 3 \) directly in terms of \( g \).

**Corollary 1.** A locally finite variety \( V \) has bounded width if and only if it has an idempotent weak majority term \( g \) satisfying the identity

\[
g(g(x,x,y), g(x,x,y), g(x,y,y)) \approx g(x,x,y) \approx g(y,x,y) \approx g(y,x,x).
\]

Furthermore, if \( V \) has bounded width we can choose \( g \) such that there exist \( f, h_F \) as in the previous theorem with \( g = h_M \) where \( M = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \subseteq P(\{1, 2, 3\}) \), such that each \( h_F \in \text{Clo}(g) \), and such that \( g \) satisfies the additional identity

\[
g(g(x,y,z), g(y,z,x), g(z,x,y)) \approx g(x,y,z).
\]

**Proof.** We’ve already shown that if \( V \) has bounded width then such a \( g \) exists. In order to prove the reverse implication, it’s enough to show that such a \( g \) can’t be realized in a nontrivial module over a ring. Suppose for contradiction that \( M \) is a module over a ring \( \mathbb{R} \), and that we have \( g^M(x,y,z) \approx ax + by + cz \) for some fixed \( a, b, c \in \mathbb{R} \). Since \( g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \), we have \( ax \approx bx \approx cx \), and from \( g(x,x,x) \approx x \), we have \( 3ax \approx x \). From

\[
g(g(x,x,y), g(x,x,y), g(x,y,y)) \approx g(x,x,y),
\]

we see that

\[
5a^2 x + 4a^2 y \approx 2ax + ay.
\]
Multiplying both sides by 9 and using $3ax \approx x$, we get

$$5x + 4y \approx 6x + 3y,$$

so $x \approx y$ and the module $\mathbb{M}$ must be trivial.

In particular, for any such $g$ the clone $\text{Clo}(g)$ must be bounded width as well. Choose such a $g$ such that $\text{Clo}(g)$ contains as few terms of arity 3 as possible. By Theorem 3, there are terms $f, h \in \text{Clo}(g)$. Let $g' = h \cdot g$, then by the choice of $g$ we must have $\text{Clo}(g) = \text{Clo}(g')$, so in particular all $h \in \text{Clo}(g')$. Replace $g$ by this $g'$ to get the second to last assertion of the Corollary.

In order to prove the last assertion, consider the map $\gamma : \mathcal{F}(x, y, z) \rightarrow \mathcal{F}(x, y, z)$ defined by

$$\gamma : (a, b, c) \mapsto (g(a, b, c), g(b, c, a), g(c, a, b)).$$

Since $\mathcal{V}$ is locally finite, there exists $i \geq 1$ such that $\gamma^{2i} = \gamma^i$. Note that if any two of $a, b, c$ are equal, then $\gamma^{i}(a, b, c) = (g(a, b, c), g(a, b, c), g(a, b, c))$ for all $i \geq 1$, so we may replace $g$ by the first coordinate of $\gamma^i$ without changing the value of $g(x, y)$.

**Theorem 4.** If $\mathcal{V}$ is a locally finite idempotent bounded width variety with terms $f, g$ as in Corollary 1 such that the clone generated by $g$ is minimal, then there is a sequence of binary terms $p_0, \ldots, p_n \in \text{Clo}(f)$ such that $p_0(x, y) \approx x, p_n(x, y) \approx y$, and for each $0 \leq i < n$, the set $\{p_i(x, y), p_{i+1}(x, y)\} \subseteq \mathcal{F}(x, y)$ is closed under $g$.

**Proof.** Let $f, g$ be terms as in Corollary 1 let $\mathbb{A}$ be an algebra in $\mathcal{V}$, and let $A$ be a subset of $\mathbb{A}$ which is closed under $f$. Set $\mathbb{A}' = (A, f)$, the algebra on the underlying set $A$ with $f$ as its only basic operation. Define a graph $\mathcal{G}(\mathbb{A}') = \mathcal{G}(\mathbb{A}', g)$ on $\mathbb{A}'$ with an edge connecting a pair of elements $a, b \in \mathbb{A}'$ whenever $\{a, b\}$ is closed under $g$. It’s enough to show that we can choose $g$ such that each such graph $\mathcal{G}(\mathbb{A}', g)$ is connected. Let $f_3 = h \cdot g \in \text{Clo}(g)$ be as in the proof of Theorem 3. Suppose that the clone generated by $g$ is minimal and that $g$ is the only basic operation of $\mathcal{V}$, and suppose for contradiction that $\mathbb{A}'$ has minimal size such that $\mathcal{G}(\mathbb{A}')$ is not connected.

For every element $a \in \mathbb{A}'$, let $C(a)$ be the connected component of $a$ in $\mathcal{G}(\mathbb{A}')$. Letting $a, b \in \mathbb{A}'$ be any pair of elements with $C(a) \neq C(b)$, from the minimality of $\mathbb{A}'$ we see that $\text{Sg}_{\mathbb{A}'}\{a, b\} = \mathbb{A}'$. Define a subalgebra $\mathcal{S} = \mathcal{S}_{a, b}$ of $\mathbb{A}'$ to be $\text{Sg}_{\mathbb{A}'}\{(a, b), (b, a)\}$. Since $\{a, b\}$ is not an edge of $\mathcal{G}(\mathbb{A}')$, Theorem 3 and the choice of $g$ implies that $(a, a), (b, b) \notin \mathcal{S}$. Since $\text{Sg}_{\mathbb{A}'}\{a, b\} = \mathbb{A}'$, we see that $\mathcal{S}$ is subdirect in $\mathbb{A}'$.

Suppose now that $(u, t), (v, w) \in \mathcal{S}$ and that $(u, v)$ is an edge of $\mathcal{G}(\mathbb{A}')$. Let $\mathbb{B} = \mathcal{B}_{u, v} \subseteq \mathbb{A}'$ be the subalgebra of $x \in \mathbb{A}'$ such that at least one of $(u, x), (v, x)$ is in $\mathcal{S}$, i.e. $\mathbb{B} = \pi_2\{(u, v) \times \mathbb{A}' \cap \mathcal{S}\}$. Suppose, for contradiction, that $\mathbb{B} \neq \mathbb{A}'$. Then at least one of $C(a)$ or $C(b)$ is not equal to $C(u) = C(v)$, say $C(a)$. From $b \in \mathbb{B}$, we see that at least one of $(u, b), (v, b) \in \mathbb{B}$, say $(u, b) \in \mathbb{B}$, and then from $C(a) \neq C(u)$ we see that $(b, b) \in \mathbb{A}' \times \{b\} = \text{Sg}_{\mathbb{A}'}\{(a, b), (b, a)\} \subseteq \mathcal{S}$, a contradiction. Thus $\mathbb{B}$ must be a proper subalgebra of $\mathbb{A}'$, and thus from the choice of $\mathbb{A}'$ the graph $\mathcal{G}(\mathbb{B})$ must be connected. In particular, since $t, w \in \mathbb{B}$ we see that $C(t) = C(w)$. By a straightforward induction on the length of the path, we now see that for any $(u, t), (v, w) \in \mathcal{S}$ with $C(u) = C(v)$ we must have $C(t) = C(w)$. Thus there must exist an involution $\iota$ on the set $\mathcal{C}$ of connected components of $\mathcal{G}(\mathbb{A}')$ such that

$$\mathcal{S} \subseteq \bigcup_{C \in \mathcal{C}} C \times \iota(C).$$

Suppose, for contradiction, that $C(f(a, b)) = C(a)$. From $(a, b), (b, a) \in \mathcal{S}$ we see that $(f(a, b), f(b, a)) \in \mathcal{S}$, and so $C(f(b, a)) = \iota(C(f(a, b))) = \iota(C(a)) = C(b)$. Since $\{f(a, b), f(b, a)\}$ is a subalgebra of $\mathbb{A}'$, we see that $C(f(a, b)) = C(f(b, a))$ as well, so $C(a) = C(b)$, a contradiction. Thus, for any $a, b$ with $C(a) \neq C(b)$, we have $C(a) \neq C(f(a, b))$ as well. Defining a sequence of binary terms $f^i$ by $f^0(x, y) \approx y, f^1 = f$, and

$$f^{i+1}(x, y) \approx f(f^i(x, y)),$$
we see by induction on $i$ that if $C(a) \neq C(b)$ then $C(a) \neq C(f^i(a, b))$. Letting $N \geq 1$ be such that $f^N = f^{2N}$, we see that $C(a) \neq C(f^N(a, b))$ and $f^N(a, f^N(a, b)) = f^N(a, b)$. Replacing $b$ by $f^N(a, b)$, we may assume without loss of generality that $f^N(a, b) = b$.

Define also a sequence of ternary terms $g^i$ by $g^0(x, y, z) \approx z$, $g^1 = g$, and

$$g^{i+1}(x, y, z) \approx g(x, y, g^i(x, y, z)).$$

We now define functions $f', g', f_3'$ by

$$f'(x, y) \approx f(f(x, y), f^{N-1}(y, f(x, y))),$$
$$g'(x, y, z) \approx g(f^{N-1}(x, g^{N-1}(f(y, z), f(z, y), g(x, y, z))),$$
$$f^{N-1}(y, g^{N-1}(f(z, x), f(x, z), g(y, z, x))), f^{N-1}(z, g^{N-1}(f(x, y), f(y, x), g(z, x, y))),$$
and

$$f_3'(x, y, z) \approx f_3(f^{N-1}(x, f_3(x, y, z),$$
$$f^{N-1}(y, f^{N-1}(x, f(f(x, y), f_3(x, y, z))), f^{N-1}(z, f^{N-1}(x, f(f(x, z), f_3(x, y, z))))).$$

We have the identity

$$g'(x, y) \approx g(f^{N-1}(x, g^{N-1}(f(x, y), f(y, x), f(x, y))),$$
$$f^{N-1}(x, g^{N-1}(f(x, y), f(y, x), f(x, y))), f^{N-1}(y, g^{N-1}(x, x, f(x, y))))$$
$$\approx g(f(x, y), f^{N}(x, y), f^{N-1}(y, f^{N}(x, y)))$$
$$\approx f'(x, y),$$
and similarly $g'(x, y, x) \approx g'(y, x, x) \approx f'(x, y)$. We also have

$$f_3'(x, y) \approx f_3(f^{N-1}(x, f(x, y),$$
$$f^{N-1}(x, f^{N-1}(x, f(f(x, y), f(x, y)))), f^{N-1}(y, f^{N-1}(x, f(f(x, y), f(x, y))))))$$
$$\approx f_3(f^{N}(x, y), f^{2N}(x, y), f^{N-1}(y, f^{N}(x, y)))$$
$$\approx f'(x, y),$$
and similarly $f_3'(x, y, x) \approx f'(x, y)$, as well as

$$f_3'(x, y) \approx f_3(f^{N-1}(x, f(x, y),$$
$$f^{N-1}(y, f^{N-1}(x, f(f(x, y), f(x, y)))), f^{N-1}(y, f^{N-1}(x, f(f(x, y), f(x, y))))))$$
$$\approx f_3(f^{N}(x, y), f^{N-1}(y, f^{N}(x, y)), f^{N-1}(y, f^{N}(x, y)))$$
$$\approx f'(x, y),$$
so the pair of terms $f_3', g'$ satisfies (SM 4).

From $f^N(a, b) = b$, we have

$$f'(a, b) = f(f(N, a, b), f^{N-1}(b, f^{N}(a, b))) = f(b, f^{N-1}(b, b)) = b.$$ Using this we will construct terms $f''$, $g''$ with

$$f''(x, y) \approx g''(x, x, y) \approx g''(y, x, x) \approx f''(x, y, x))$$

and with $C(f''(a, b)) = C(a)$, $C(f''(b, a)) = C(b)$. Then $g''$ generates a bounded width algebra by Corollary [1] so by our choice of $g$ we will have $\text{Clo}(g) = \text{Clo}(g'')$. Thus, since $\{f''(a, b), f''(b, a)\}$ will be closed under $g''$, it will also be closed under $g$, so $\{f''(a, b), f''(b, a)\}$ will be an edge of $G(\mathcal{A}')$ which connects $C(a)$ and $C(b)$, giving us a contradiction.
In order to construct \( f''_i, g''_i \), we define sequences \( f'^i, g'^i \) by \( f'^1 = f', g'^1 = g' \), and
\[
\begin{align*}
  f'^{i+1}(x, y) &= f'^i(f'(x, y), f'(y, x)) \\
  g'^{i+1}(x, y, z) &= g'^i(f'_3(x, y, z), f'_3(y, z, x), f'_3(z, x, y)).
\end{align*}
\]
We then choose an even \( K \geq 1 \) such that \( f''^{iK} = f'^{2K} \) and take \( f'' = f'^K, g'' = g'^K \).

It remains to show that \( C(f''(a, b)) = C(a) \) and \( C(f''(b, a)) = C(b) \). We will prove by induction on \( i \) that, considering \( S \) as an undirected graph on \( \mathcal{A}' \), \( a \) is connected to \( f'^i(a, b) \) by a sequence of \( i \) edges of \( S \) since \( (f'^i(a, b), f'^i(b, a)) \in S \). For \( i = 1 \), this follows from \( (a, b) \in S \) and \( (f'(a, b), f'(b, a)) \in S \). For the induction step, let \( u_0, u_1, \ldots, u_i \) be such that \( u_0 = a, u_1 = b, u_i = f'^i(a, b) \) with \( (u_j, u_{j+1}) \in S \) for \( 0 \leq j < i \), and set \( u_{i+1} = f'^i(b, a) \) so that \( (u_i, u_{i+1}) \in S \) as well. Now for \( 1 \leq j \leq i + 1 \), set \( v_j = f'(u_{j-1}, u_j) \), and set \( v_0 = a \). Then \( (v_0, v_1) = (a, f'(a, b)) = (a, b) \in S \), and
\[
(v_j, v_{j+1}) = f'((u_{j-1}, u_j), (u_j, u_{j+1})) \in S,
\]
and \( v_{i+1} = f'(u_i, u_{i+1}) = f'(u_i, f'^i(a, b), f'^i(b, a)) = f'^{i+1}(a, b) \), so \( v_0, \ldots, v_{i+1} \) is a path through \( i + 1 \) edges of \( S \) connecting \( a \) to \( f'^{i+1}(a, b) \). Thus, we have
\[
C(f'^i(a, b)) = i(C(a)) \quad i \equiv 0 \pmod{2},
\]
\[
C(b) \quad i \equiv 1 \pmod{2},
\]
so \( C(f''(a, b)) = C(f'^K(a, b)) = C(a), C(f''(b, a)) = i(C(f''(a, b))) = C(b) \), and we are done.

**Corollary 2.** Let \( \mathcal{A} \) be a finite bounded width algebra. If \( \text{Aut}(\mathcal{A}) \) is 2-transitive, then \( \mathcal{A} \) has a majority term. If \( \text{Aut}(\mathcal{A}) \) is 3-transitive, then \( \mathcal{A} \) has the dual discriminator as a term.

**Proof.** Since \( \text{Aut}(\mathcal{A}) \) is transitive, every unary operation of \( \mathcal{A} \) must be surjective, so \( \mathcal{A} \) is core and we may replace it with its idempotent reduct. Choose \( f, g \) as in the previous corollary. If there is any pair \( x, y \in \mathcal{A} \) such that \( f(x, y) \neq f(y, x) \), then from \( f(f(x, y), f(y, x)) = f(x, y) \) and 2-transitivity we see that \( f \) is first projection and \( g \) is a majority operation. Now suppose that \( f(x, y) = f(y, x) \) for all \( x, y \in \mathcal{A} \). Choose any \( a \neq b \in \mathcal{A} \), and let \( \sigma \) be an automorphism of \( \mathcal{A} \) such that \( \sigma(a) = f(a, b) \) and \( \sigma(b) = a \). Define a sequence of functions \( g_i \) by \( g_1 = g \) and \( g_{i+1}(x, y, z) = g_i(f(y, z), f(z, x), f(x, y)) \).

Since \( \sigma \) is an automorphism of \( \mathcal{A} \), we have
\[
g_{i+1}(a, a, b) = g_i(f(a, b), f(b, a), a) = g_i(\sigma(a), \sigma(a), \sigma(b)) = \sigma(g_i(a, a, b)) = \sigma^i(a)
\]
by induction on \( i \), and similarly we get \( g_{i+1}(a, b, a) = g_{i+1}(b, a, a) = \sigma^{i+1}(a) \). Choosing \( n \) such that \( \sigma^n \) fixes \( a \), we see that \( g_n(a, a, b) = g_n(a, b, a) = g_n(b, a, a) = a \), and thus by 2-transitivity \( g_n \) is a majority operation.

Now suppose that \( \text{Aut}(\mathcal{A}) \) is 3-transitive. Let \( g \) be a majority operation on \( \mathcal{A} \). By iteration, we may assume that \( g \) also satisfies the identity
\[
g(g(x, y, z), y, z) \approx g(x, y, z).
\]
Let \( a, b, c \) be any three distinct elements of \( \mathcal{A} \). If \( g(a, b, c) \in \{a, b, c\} \), then by 3-transitivity (and possibly permuting the inputs of \( g \)) we are done. Otherwise, letting \( d = g(a, b, c) \) we see that \( d, b, c \) are three distinct elements of \( \mathcal{A} \) with \( g(d, b, c) = d \), and we are done by 3-transitivity.

**5. Construction of special weak near-unanimity terms**

**Theorem 5.** Let \( \mathcal{A} \) be a finite algebra of bounded width. Then \( \mathcal{A} \) has an idempotent term \( t(x, y) \) satisfying the identity
\[
t(x, t(x, y)) \approx t(x, y)
\]
along with an infinite sequence of idempotent weak near-unanimity terms \( w_n \) of every arity \( n > 2\lcm\{1,2,...,|\mathbb{A}|-1\} \) such that for every sequence \( (a_1,...,a_n) \) with \( \{a_1,...,a_n\} = \{x,y\} \) and having strictly less than \( \frac{n}{2\lcm\{1,2,...,|\mathbb{A}|-1\}} \) of the \( a_i \) equal to \( y \) we have

\[
w_n(a_1,...,a_n) \approx t(x,y).
\]

Proof. Let \( f,h_F \) be as in the conclusion to Theorem 3. Define a sequence of functions \( t^i \) by

\[
t^0(x,y) \approx x, \ t^1(x,y) \approx f(x,y),
\]

\[
t^{i+1}(x,y) \approx f(x,t^i(x,y)).
\]

We will inductively construct for each \( i, k \geq 1 \) and each \( n > 2ik \) a sequence of functions \( w^{i,k}_n(a_1,...,a_n) \) such that for every sequence \( (a_1,...,a_n) \) with \( \{a_1,...,a_n\} = \{x,y\} \) and having at most \( k \) of the \( a_j \) equal to \( y \) we have

\[
w^{i,k}_n(a_1,...,a_n) \approx t^i(x,y).
\]

We start with \( w^{1,k}_n = h_F \) for some maximal intersecting family containing the family of all subsets of \( \{1,...,n\} \) of size at least \( n - k \) (which is an intersecting family if \( n > 2k \)). Now assume we have already constructed \( w^{i,k}_n \), and we will construct \( w^{i+1,k+2}_n \). Fix a bijection of \( s : (\{1,...,n+2k\} \rightarrow \{1,...,\binom{n+2k}{n}\} \). Let \( F \) be a maximal intersecting family of subsets of \( \{1,...,\binom{n+2k}{n}\} \) containing the intersecting family of subsets \( \{s(j_1),...,s(j_{n+k}) \mid A \subseteq \{1,...,n+2k\}, |A| \geq n+k \} \) (these are intersecting since if \( A,B \subseteq \{1,...,n+2k\} \) with \( |A|,|B| \geq n+k \), then \( |A \cap B| \geq n \), so there is some \( n \)-element set which is contained in both \( A \) and \( B \)). Then we take

\[
w^{i+1,k+2}_n(a_1,...,a_{n+2k}) \approx h_F(x_1,...,x_{(n+2k)}),
\]

where \( x_s(j_1,...,j_n) = w^i_n(a_{j_1},a_{j_2},...,a_{j_n}) \) for \( 1 \leq j_1 < j_2 < \cdots < j_n \leq n + 2k \).

Now take

\[
w_n(a_1,...,a_n) \approx w^i_n(a_1,...,a_n)
\]

with \( i = \lcm\{1,...,|\mathbb{A}|-1\} \) and \( k = \lfloor \frac{n-1}{2i} \rfloor \), and note that for this \( i \) we necessarily have

\[
t^i(x,t^i(x,y)) \approx t^i(x,y).
\]

6. Counterexample to (SM 5) and (SM 6)

Let \( \mathbb{A} = (\{a,b,c\},f,g) \) be the idempotent algebra given by

\[
f(a,b) = f(b,a) = f(b,c) = g(a,b,c) = b,
\]

\[
f(a,c) = f(c,a) = f(c,b) = g(a,c,b) = c,
\]

\[
f(x,y) \approx g(x,x,y),
\]

\[
g(x,y,z) \approx g(y,z,x).
\]

Note that this algebra satisfies \( f(f(x,y),f(y,x)) \approx f(x,y) \), and thus has bounded width. Also, note that

\[
(b,c) \notin Sg_{\mathbb{A}^2}\{(a,a),(a,b),(b,b),(c,c)\} = \{(a,a),(a,b),(b,b),(c,b),(c,c)\},
\]

\[
(b,b) \notin Sg_{\mathbb{A}^2}\{(a,a),(a,c),(b,c),(c,b)\} = \{(a,a),(a,c),(b,c),(c,b),(c,c)\},
\]

that \( Sg_{\mathbb{A}^n}\{x_1,...,x_k\} \) contains the vector \( (a,a,...,a) \) if and only if one of the \( x_i \) is equal to \( (a,a,...,a) \), and that \( (b,c) \in \text{Aut}(\mathbb{A}) \).

**Theorem 6.** The algebra \( \mathbb{A} \) does not have any 4-ary term \( t \) satisfying

(SM 5) \( t(x,y,z,y) \approx t(y,x,z,z) \approx t(z,x,x,y) \).
and also does not have any 4-ary term \( t \) satisfying

\[(SM\ 6) \quad t(x, y, z, y) \approx t(y, x, x, z) \approx t(z, y, x, x).\]

Proof. If a term satisfying (SM 5) existed, then for every algebra \( \mathbb{F} \) in the variety generated by \( A \) and for every \( x, y, z \in \mathbb{F} \), we would have

\[
S_{g_{\mathbb{F}}} \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} y \\ x \\ z \end{bmatrix}, \begin{bmatrix} z \\ x \\ y \end{bmatrix} \right\} \cap \Delta \neq \emptyset,
\]

where \( \Delta \) is the diagonal of \( \mathbb{F}^3 \). To show that this is not the case, take \( \mathbb{F} = A^2 \), \( x = (a, b) \), \( y = (b, c) \), \( z = (c, a) \). We aim to show that

\[
S_{g_{A^6}} \left\{ \begin{bmatrix} a & b \\ b & c \\ c & a \end{bmatrix}, \begin{bmatrix} b & c \\ a & b \\ c & a \end{bmatrix}, \begin{bmatrix} c & a \\ a & b \\ a & b \end{bmatrix}, \begin{bmatrix} b & c \\ c & a \\ a & b \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} \alpha & \beta \\ \alpha & \beta \end{bmatrix} \right| \alpha, \beta \in A \} = \emptyset.
\]

Suppose for contradiction that \( \begin{bmatrix} \alpha & \beta \\ \alpha & \beta \end{bmatrix} \) was in the intersection. We clearly must have \( \alpha, \beta \in \{b, c\} \), and by

\[
\begin{bmatrix} * & * \\ b & * \\ b & * \end{bmatrix} \notin S_{g_{A^6}} \left\{ \begin{bmatrix} * & * \\ b & * \\ c & * \end{bmatrix}, \begin{bmatrix} * & * \\ a & * \\ a & * \end{bmatrix}, \begin{bmatrix} * & * \\ a & * \\ b & * \end{bmatrix} \right\}
\]

and

\[
\begin{bmatrix} * & b \\ * & * \\ * & * \end{bmatrix} \notin S_{g_{A^6}} \left\{ \begin{bmatrix} * & b \\ * & c \\ * & * \end{bmatrix}, \begin{bmatrix} * & c \\ * & b \\ * & * \end{bmatrix}, \begin{bmatrix} * & a \\ * & * \\ * & * \end{bmatrix} \right\}
\]

we see that \( \alpha = \beta = c \). But this is a contradiction, since

\[
\begin{bmatrix} c & * \\ * & * \\ * & c \end{bmatrix} \notin S_{g_{A^6}} \left\{ \begin{bmatrix} a & * \\ * & * \\ * & * \end{bmatrix}, \begin{bmatrix} b & * \\ * & * \\ * & * \end{bmatrix}, \begin{bmatrix} c & * \\ * & * \\ * & * \end{bmatrix} \right\}.
\]

Similarly, if a term satisfying (SM 6) existed, we would have

\[
S_{g_{\mathbb{F}^3}} \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} y \\ x \\ z \end{bmatrix}, \begin{bmatrix} z \\ x \\ y \end{bmatrix} \right\} \cap \Delta \neq \emptyset,
\]

so it is enough to show that

\[
S_{g_{A^6}} \left\{ \begin{bmatrix} a & b \\ b & c \\ c & a \end{bmatrix}, \begin{bmatrix} b & c \\ a & b \\ a & b \end{bmatrix}, \begin{bmatrix} b & c \\ a & b \\ c & a \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} \alpha & \beta \\ \alpha & \beta \end{bmatrix} \right| \alpha, \beta \in A \} = \emptyset.
\]


This follows from

\[
\begin{array}{c}
\{[\ast \ast b]\}, \{[\ast \ast c]\}, \{[\ast a]\}, \{[\ast \ast c]\}, \{[\ast \ast \ast]\}, \{[a \ast \ast]\}, \{[b \ast \ast]\}, \{[a \ast \ast]\}, \{[a \ast c]\}, \{[a \ast \ast]\}, \{[b \ast c]\}, \{[b \ast \ast]\}, \{[a \ast \ast]\}, \{[a \ast c]\}, \{[a \ast \ast]\}, \{[b \ast c]\}, \{[b \ast \ast]\}, \{[a \ast \ast]\}, \{[a \ast c]\}, \{[a \ast \ast]\}
\end{array}
\]

7. Pseudovariety of minimal bounded width clones

**Definition 9.** For each \(k \geq 1\), let \(G_k\) be the set of pairs \(f, g\) where \(g : \{1, \ldots, k\}^3 \to \{1, \ldots, k\}\) is an idempotent weak majority functions satisfying

\[
g(x, x, y) \approx g(x, y, x) \approx g(y, x, x) \approx f(x, y),
\]

\[
f(f(x, y), f(y, x)) \approx f(x, y),
\]

and

\[
g(g(x, y, z), g(y, z, x), g(z, x, y)) \approx g(x, y, z).
\]

Define a quasiordering \(\succeq\) on \(G_k\) by \(g' \succeq g\) if \(g' \in \text{Clo}(g)\). Also, define an action of \(S_2 \times S_k\) on \(G_k\) by having the nontrivial element of \(S_2\) take \(g \in G\) to \(\tilde{g}\) given by \(\tilde{g}(x, y, z) = g(x, z, y)\) and by having \(\sigma \in S_k\) take \(g\) to \(\sigma g\) given by \((\sigma g)(x, y, z) = \sigma(g(\sigma^{-1}(x), \sigma^{-1}(y), \sigma^{-1}(z)))\).

**Proposition 6.** The clones on \(\{1, \ldots, k\}\) of bounded width which are minimal with respect to containment are in bijection with the minimal equivalence classes of the quasiorder \(\succeq\) on \(G_k\). In particular, the number of such clones is at most \(|G_k|\), which is in turn trivially bounded by \(k^{3k}\). We have \(|G_2| = 2,728\) and \(|G_4| = 8,124,251,747,605\).

**Definition 10.** Say that \(g \in G_k\) is **minimal** if it is in a minimal equivalence class of the quasiorder \(\succeq\), say that an algebra \(A\) is a minimal bounded width algebra if it is isomorphic to an algebra which is term equivalent to \((\{1, \ldots, |A|\}, g)\) for some minimal \(g \in G_{|A|}\), and say that a variety \(V\) is a minimal bounded width variety if every finite algebra \(A \in V\) is minimal.

The proof of the next Theorem is based on an argument of Bulatov, specifically, the proof of Case 2.2 of Theorem 5 of [8].

**Theorem 7.** Let \(V\) be a locally finite idempotent bounded width variety such that every algebra in \(V\) is connected through two element semilattice and two element majority subalgebras, let \(A \in V\), and suppose there is a term \(m\) and a subset \(S \subseteq A\) which is closed under \(m\), such that \((S, m)\) is a bounded width algebra. Let \(V'\) be the reduct of \(V\) consisting of all terms \(t\) of \(V\) such that \(S\) is closed under \(t\). Then \(V'\) also has bounded width.

**Proof.** For any algebra \(F \in V\), let \(F'\) be the reduct of \(F\) having the same underlying set and having as operations all operations of \(V'\). For any algebra \(F\), define a graph \(G(F)\) on \(F\) by connecting two vertices \(a, b\) of \(F\) whenever there is a term \(t\) of \(F\) such that \(\{a, b\}\) is closed under \(t\) and such that \((\{a, b\}, t)\) has bounded width. It’s enough to show that for every finite \(F \in V\) and every subalgebra \(B' \subseteq F'\) we have \(G(B')\) connected, since every algebra in \(V'\) is a quotient of a subalgebra of the...
reduct $F'$ of some algebra $F$ of $V$, and since no affine algebra can have a two element bounded width algebra as a subalgebra. We will prove this by induction on the size of $B'$, which we may assume to be generated by two elements $a, b \in F$.

First we will show that every two element subalgebra of $F$ is in $G(F')$. We may as well suppose that $|F| = 2$, so that $F$ is either a semilattice or a majority algebra. If $F$ is semilattice then $m$ must act like a semilattice operation on $F$, and we are done. Now suppose that $F$ is a majority algebra. Let $p > |S|$ be prime. By one of the main results of $[3]$, there is a $p$-ary term $c \in \text{Clo}(m)$ such that $c|S$ is a cyclic term. Let $u$ be a $p$-ary term of $V$ such that its restriction to $F$ is a near-unanimity term. Setting

$$u'(x_1, ..., x_p) \approx u(c(x_1, ..., x_p), c(x_2, ..., x_p, x_1), ..., c(x_p, x_1, ..., x_{p-1})),$$

we see that $u'|S = c|S$, so $u'$ is a term of $V'$. Since $c|F$, being a term of a majority algebra, is either a projection or a near-unanimity operation, we see that $u'$ is a near-unanimity operation of $F'$, so $F'$ is term equivalent to a majority algebra.

Now let $a, b \in F$, and suppose that $B' = Sg_{F'}\{a, b\}$ has $G(B')$ disconnected, but that every proper subalgebra $C'$ of $B'$ has $G(C')$ connected. Let $\Sigma$ be the collection of ordered pairs $(p_i, q_i)$ of binary terms in the clone of $m$ such that $p_i|S = q_i|S$. Set

$$D_\Sigma = \{(p_i(c, d), q_i(c, d)) \mid c, d \in B', (p_i, q_i) \in \Sigma\},$$

and let $\theta$ be the congruence of $B' = Sg_{F'}\{a, b\}$ generated by $D_\Sigma$. Letting

$$T_\Sigma = \{(t(a, b, p_i(c, d)), t(a, b, q_i(c, d)) \mid c, d \in B', (p_i, q_i) \in \Sigma, t \text{ a term of } V'\}$$

and considering $T_\Sigma$ as a graph on $B'$, we see that $\theta$ is the transitive closure of $T_\Sigma$, since $T_\Sigma$ contains the image of $D_\Sigma$ under all unary polynomials of $B'$.

Let $(t(a, b, p_i(c, d)), t(a, b, q_i(c, d)) \in T_\Sigma$. Let $r$ be any binary term of $V$, and define a 4-ary term $r'$ by

$$r'(x, y, z, w) = r(t(x, y, p_i(z, w)), t(x, y, q_i(z, w))).$$

We clearly have $r'(x, y, z, w)|S = t(x, y, p_i(z, w))|S$, so $r' \in V'$, and thus

$$r(t(a, b, p_i(c, d)), t(a, b, q_i(c, d)) = r'(a, b, c, d) \in B'.$$

Since $r$ was an arbitrary binary term of $V$ we have

$$Sg_{F'}\{t(a, b, p_i(c, d)), t(a, b, q_i(c, d))\} \subseteq B'.$$

so, by Theorem $[4]$ and the fact that every two element subalgebra of $F$ is in $G(F')$, we see that $t(a, b, p_i(c, d)), t(a, b, q_i(c, d))$ are in the same connected component of $G(B')$. Since $G(B')$ was assumed to be disconnected, we see that $T_\Sigma$ is disconnected and therefore $B'/\theta$ has at least two elements. By the choice of $B'$, every congruence class of $\theta$ is a proper subalgebra of $B'$, and is therefore connected in $G(B')$.

By the choice of $\theta$, $p_i$ agrees with $q_i$ on $B'/\theta$ whenever $(p_i, q_i) \in \Sigma$, so the reduct $(B'/\theta, m)$ is in the algebra generated by $(S, m)$ (we only need to consider identities involving two variables since $B'$ is generated by two elements) and therefore has bounded width. Since $B'$ has bounded width and every proper subalgebra of $B'$ does have bounded width, there must be a congruence $\theta'$ such that $B'/\theta'$ is a nontrivial affine algebra, and by the choice of $B'$ every congruence class of $\theta'$ is connected in $G(B')$. Since it is both affine and bounded width, $B'/\theta' \cap \theta'$ must be a trivial algebra, so the transitive closure of $\theta \cup \theta'$ is $B'$, and we see that in fact $G(B')$ is connected.

An immediate consequence of Theorem $[7]$ is that every subalgebra and every quotient of a minimal bounded width algebra is also a minimal bounded width algebra. It’s easy to see that this also holds for powers, so in fact every minimal bounded width algebra generates a minimal bounded width variety. The same is not true for products, however: as we will see, up to the action of
$S_2 \times S_3$ there are two minimal elements $g \in G_3$ with $g \neq \tilde{g}$, and for either one of these the product $(\{1, 2, 3\}, g) \times (\{1, 2, 3\}, \tilde{g})$ has $(\{1, 2, 3\}, g)^2$ as a reduct and is thus not a minimal bounded width algebra.

**Theorem 8.** There exists an idempotent pseudovariety $V_{mbw}$ with basic operations $f, g$ satisfying the identities in Definition 7 such that every finite algebra in $V_{mbw}$ is a minimal bounded width algebra, and such that every finite core bounded width algebra has a reduct which is term equivalent to an algebra in $V_{mbw}$.

**Proof.** Fix an enumeration $A_1, A_2, \ldots$ of the collection of all finite algebras (up to isomorphism) in the variety with basic operations $f, g$ satisfying the identities in Definition 7. For each $i \geq 1$, let $g_i$ be a ternary term of $A_1 \times \cdots \times A_i$ such that when restricted to $A_1 \times \cdots \times A_{i-1}$ it is in the clone generated by $g_{i-1}$, and such that the reduct of $A_1 \times \cdots \times A_i$ with basic operation $g_i$ is a minimal bounded width algebra - that such a term $g_i$ exists follows from Theorem 7. Since there are only finitely many possible ternary terms $g_i|_{A_i}$ on $A_i$, one such must occur infinitely many times - call this term $g^1$. Similarly, we inductively define a sequence of ternary terms $g^i$ such that for each $i$, there are infinitely many $j > i$ with $g_{j|A_i} = g^k$ for all $k \leq i$. Finally, we let $V_{mbw}$ be the pseudovariety generated by the algebras $A_i = (A_i, f^i, g^i)$, where each $A_i$ is the underlying set of $A_i$, and $f^i$ is given by $f^i(x, y) = g^i(x, x, y)$.

From the construction of $V_{mbw}$, it is clear that every finite core bounded width algebra has a reduct which is term equivalent to an algebra in $V_{mbw}$. Now suppose that $A$ is a finite algebra in $V_{mbw}$. Then $A$ is in the variety generated by finitely many algebras $(A_i, g^i)$, so there is some $j$ such that $A$ is in the variety generated by the minimal bounded width algebra $(A_1 \times \cdots \times A_j, g_j)$, so by Theorem 7, $A$ must be a minimal bounded width algebra as well. □

### 8. Bounded width algebras of size three

We will draw a doodle for each bounded width algebra $A$ of size three as follows. We start by drawing a vertex for each element of $A$, and for $a, b \in A$ we draw a directed edge from $a$ to $b$ if $a \rightarrow b$, we draw a solid undirected edge connecting $a$ to $b$ if $\{a, b\}$ is a majority algebra. If $\{a, b\}$ is not a subalgebra of $A$, then we draw a dashed line connecting $a$ to $b$, and we record the values of $f(a, b), f(b, a)$ next to the dashed line (if $f(a, b) = f(b, a)$, then we only write their common value once). Finally, if $A$ has underlying set $\{a, b, c\}$, we draw a dashed circle around the set of elements $d \in A$ such that

$$(d, d) \in Sg_{A^2}\{(a, b), (b, c), (c, a)\}.$$ Throughout this section, we will also fix the following notation for maps $\alpha, \beta, \gamma : A^3 \rightarrow A^3$:

$$\alpha(x, y, z) = (f(x, y), f(y, z), f(z, x)), \quad \beta(x, y, z) = (f(x, z), f(y, z), f(z, y)), \quad \gamma(x, y, z) = (g(x, y, z), g(y, z, x), g(z, x, y)).$$

Note that if $A \in V_{mbw}$ has underlying set $\{a, b, c\}$, then $g$ is completely determined by $f$ and $\gamma(a, b, c), \gamma(a, c, b)$, and that $f$ is completely determined by $\alpha(a, b, c)$ and $\beta(a, b, c)$. Figure 1 is a summary of the main classification result proved in this section.

**Lemma 4.** If $V$ is a locally finite idempotent variety of bounded width, then $V$ has terms $f, g$ as in Definition 7 such that for every three element algebra $A$ in $V$ we have $f^A(g^A, \tilde{g}^A) = g^A$ and either have $g^A$ cyclic or $A \subseteq \text{Aut}(g^A)$ and $g^A(a, b, c) = a$ whenever $a, b, c$ are distinct elements of $A$.

**Proof.** Since $V$ is locally finite, it has finitely many terms of arity 3. Choose a term $g$ as above such that a maximal number of triples $(a, b, c)$ in the finitely many isomorphism classes of three element algebras have $g(a, b, c) = g(b, c, a)$, $g(a, c, b) = g(b, a, c)$ and $f(g(a, b, c), g(a, c, b)) = g(a, b, c)$,
<table>
<thead>
<tr>
<th>Doodle</th>
<th>f</th>
<th>$\gamma$</th>
<th>Aut</th>
<th>Quotients</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (a, b, c)$</td>
<td>$S_3$</td>
<td>Simple</td>
</tr>
<tr>
<td><img src="image2.png" alt="Diagram 2" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (c, c, c)$</td>
<td>${1, (a b)}$</td>
<td>${a} \rightarrow {b, c}$, ${b} \rightarrow {a, c}$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (a, a, a)$</td>
<td>${1, (a b)}$</td>
<td>${c} \rightarrow {a, b}$</td>
</tr>
<tr>
<td><img src="image4.png" alt="Diagram 4" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (c, c, c)$</td>
<td>$1$</td>
<td>${a} \rightarrow {b, c}$, ${a, b} \rightarrow {c}$</td>
</tr>
<tr>
<td><img src="image5.png" alt="Diagram 5" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (c, c, c)$</td>
<td>${1, (a b)}$</td>
<td>${a} \rightarrow {b, c}$, ${b} \rightarrow {a, c}$, ${a, b} \rightarrow {c}$</td>
</tr>
<tr>
<td><img src="image6.png" alt="Diagram 6" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (a, b, c)$</td>
<td>$A_3$</td>
<td>Simple</td>
</tr>
<tr>
<td><img src="image7.png" alt="Diagram 7" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (c, c, c)$</td>
<td>$1$</td>
<td>${a} \rightarrow {b, c}$, ${b} \rightarrow {a, c}$</td>
</tr>
<tr>
<td><img src="image8.png" alt="Diagram 8" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (b, b, b)$</td>
<td>$1$</td>
<td>Simple</td>
</tr>
<tr>
<td><img src="image9.png" alt="Diagram 9" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (b, b, b)$</td>
<td>$1$</td>
<td>${c} \rightarrow {a, b}$</td>
</tr>
<tr>
<td><img src="image10.png" alt="Diagram 10" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (c, c, c)$</td>
<td>${1, (a b)}$</td>
<td>${a, b} \rightarrow {c}$</td>
</tr>
<tr>
<td><img src="image11.png" alt="Diagram 11" /></td>
<td>$f(a, b, c)$</td>
<td>$\gamma(a, b, c) = (a, a, a)$</td>
<td>${1, (a b)}$</td>
<td>${c} \rightarrow {a, b}$</td>
</tr>
</tbody>
</table>

**Figure 1.** The eleven minimal bounded width algebras of size three, up to isomorphism and term equivalence.
\[ f(g(a, c, b), g(a, b, c)) = g(a, c, b). \] Among such terms, choose a \( g \) such that the image of the map \( \gamma : (a, b, c) \mapsto (g(a, b, c), g(b, c, a), g(c, a, b)) \) is minimal. Note that if we have \( g(a, b, c) = g(b, c, a) \) and \( g(a, c, b) = g(b, a, c) \) then we must also have \( f(g(a, b, c), g(a, c, b)) = g(a, b, c) \), \( f(g(a, c, b), g(a, b, c)) = g(a, c, b) \), since otherwise we may replace \( g \) by \( g'(x, y, z) = f(g(x, y, z), g(x, z, y)) \).

Now suppose \( t \in \text{Clo}(g) \) is any term, and \( a, b, c \) are distinct elements of a three element algebra, such that \( |\{t(a, b, c), t(b, c, a), t(c, a, b)\}| \leq 2 \). Setting \( g'(x, y, z) = t(g(x, y, z), g(y, z, x), g(z, x, y)) \) and replacing \( g \) with \( g''(x, y, z) = g'(g'(x, y, z), g'(y, z, x), g'(z, x, y)) \) shows that in order for the image of \( \gamma \) to be minimal, we must have \( (a, b, c) \) not in the image of \( \gamma \). In particular, if \( a, b, c \) are distinct elements of a three element algebra then we may not have \( \gamma(a, b, c) = (a, c, b) \), as otherwise one of \( t = f(g, \tilde{g}) \) or \( t = g(g(x, y, z), x, y) \) contradicts the above. Defining maps \( \alpha, \beta \) by \( \alpha(a, b, c) = (f(a, b), f(b, c), f(c, a)) \) and \( \beta(a, b, c) = (f(a, c), f(b, a), f(c, b)) \), we see that if \( a, b, c \) are distinct elements of a three element algebra and \( \gamma(a, b, c) = (a, c, b) \), then we must also have \( \gamma(a, c, b) = (a, c, b) \) and \( \alpha(a, b, c), \beta(a, b, c) \in \{(a, b, c), (b, c, a), (c, a, b)\} \), from which we see that \( g \) commutes with cyclic permutations of \( \{a, b, c\} \).

**Lemma 5.** If \( A \in \mathcal{V}_{\text{mbw}} \) and \( a, b \in A \) have \( f(a, b) = b \), then \( a \to b \).

**Proof.** Let \( t(x, y) = f(x, f(x, y)) \). Then

\[
t(a, b) = f(a, f(a, b)) = f(a, b) = b
\]

and

\[
t(b, a) = f(b, f(b, a)) = f(f(a, b), f(b, a)) = f(a, b) = b,
\]

so by Lemma 2 we have \( a \to b \).

**Lemma 6.** If \( A \in \mathcal{V}_{\text{mbw}} \) and \( a, b, c \in A \) have \( f(a, b) = a, f(b, a) = c \), and at least one of \( f(b, c), f(f(b, c), c), f(f(f(b, c), c), c), \ldots \) is equal to \( b \), then \( \{a, b\} \) is a majority algebra.

**Proof.** By Theorem 7 we just have to find a ternary function \( g' \in \text{Clo}(g) \) which acts as the majority operation on \( \{a, b\} \). Define \( f^i(x, y) \) by \( f^0(x, y) = y, f^1(x, y) = f(x, y), f^{i+1}(x, y) = f(f^i(x, y), y) \), and choose \( k \geq 1 \) such that \( f^{i+1}(b, c) = b \). Define \( h^i(x, y, z) \) by \( h^0(x, y, z) = g(x, y, z) \) and

\[
h^{i+1}(x, y, z) = g(f(x, y), f(y, x), h^i(x, y, z)).
\]

Then by induction on \( i \), we see that

\[
h^i(x, x, y) = f^i(x, f(x, y))
\]

and

\[
h^i(x, y, x) = h^i(y, x, x) = f(x, y).
\]

Upon setting

\[
i(x, y, z) = g(h^k(x, y, z), h^k(z, x, y), f^k(x, g(x, y, z))),
\]

21
we see that
\[
i(x, x, y) = g(h^k(x, x, y), h^k(y, x, x), f^k(x, f(x, y)))
\]
\[
= g(f^k(x, f(x, y)), f(x, y), f^k(x, f(x, y)))
\]
\[
= f^{k+1}(x, f(x, y)),
\]
\[
i(x, y, x) = g(h^k(x, y, x), h^k(x, x, y), f^k(x, f(x, y)))
\]
\[
= g(f(x, y), f^k(x, f(x, y)), f^k(x, f(x, y)))
\]
\[
= f^{k+1}(x, f(x, y)),
\]
\[
i(y, x, x) = g(h^k(y, x, x), h^k(x, y, x), f^k(y, f(x, y)))
\]
\[
= g(f(x, y), f(x, y), f^k(y, f(x, y)))
\]
\[
= f(f(x, y), f^k(y, f(x, y))).
\]

Now we set
\[
g'(x, y, z) = g(i(x, y, z), i(y, z, x), i(z, x, y))
\]
and
\[
f'(x, y) = f(f^{k+1}(x, f(x, y)), f(f(x, y), f^k(y, f(x, y))),
\]
so that
\[
g'(x, y, x) \approx g'(y, x, x) \approx g'(y, x, x) \approx f'(x, y).
\]

We just need to check that \(f'(a, b) = a\) and \(f'(b, a) = b\). We have
\[
f'(a, b) = f(f^{k+1}(a, f(a, b)), f(f(a, b), f^k(b, f(a, b)))
\]
\[
= f(f^{k+1}(a, c), f(a, f^k(b, a)))
\]
\[
= f(a, a, f^{k-1}(f(b, a), a))
\]
\[
= f(a, a, f^{k-1}(c, a)) = a
\]
since \(\{a, c\} = \{f(a, b), f(b, a)\}\) is a majority algebra, and
\[
f'(b, a) = f(f^{k+1}(b, f(b, a)), f(f(b, a), f^k(a, f(b, a))))
\]
\[
= f(f^{k+1}(b, c), f(c, f^k(a, c)))
\]
\[
= f(f^{k+1}(b, c), c)
\]
\[
= f^{k+2}(b, c) = b. \qedhere
\]

Lemma 7. If \(\mathbb{A} \in \mathbb{V}_{mbw}\) and \(a, b, c\) are distinct elements of \(\mathbb{A}\) such that \(\{a, c\}\) is a majority subalgebra, \(b \rightarrow c\), and \((a, c) \in Sg_{\mathbb{A}^2}\{(a, b), (b, a)\}\), then \(\{a, b, c\}\) is a subalgebra of \(\mathbb{A}\) which is isomorphic to the subdirect product of a two element majority algebra and a two element semilattice.

Proof. Since \((a, c) \in Sg_{\mathbb{A}^2}\{(a, b), (b, a)\}\), there is a term \(t\) such that \(t(a, b) = a\) and \(t(b, a) = c\). Let \(f'(x, y) = t(t(x, y), x)\). Since \(t\) either acts as first projection or second projection on \(\{a, c\}\), \(f'\) acts as the first projection on \(\{a, c\}\), that is, \(f'(a, c) = a\) and \(f'(c, a) = c\). Furthermore, we have
\[
f'(a, b) = t(t(a, b), a) = t(a, a) = a
\]
and
\[
f'(b, a) = t(t(b, a), b) = t(c, b) = c
\]
since \( b \to c \). Thus \( \{a, b, c\} \) is closed under \( f' \), and the restriction of \( f' \) to \( \{a, b, c\} \) agrees with the \( f \) for the unique subdirect product of a two element majority algebra and a two element semilattice with size three. We define \( \alpha' : \mathbb{A}^3 \to \mathbb{A}^3 \) by
\[
\alpha'(x, y, z) = (f'(x, y), f'(y, z), f'(z, x)),
\]
and define \( g' \) by
\[
g'(x, y, z) = g(\alpha'(\alpha'(x, y, z))).
\]
We have
\[
\alpha'(a, b, c) = (a, c, c),
\alpha'(a, c, b) = (a, c, c),
\alpha'(a, b, b) = (a, b, c),
\alpha'(a, a, b) = (a, a, c),
\]
so for any \( x, y, z \in \{a, b, c\} \), \( \alpha'(\alpha'(x, y, z)) \) is either diagonal or a cyclic permutation of \( (a, a, c) \) or \( (a, c, c) \). The value of \( g \) is known on any such triple (since \( g \) is idempotent and \( \{a, c\} \) is a majority algebra), and one easily checks that the restriction of \( g' \) to \( \{a, b, c\} \) agrees with the \( g \) for the subdirect product of a two element majority algebra and a two element semilattice with size three. Now we apply Theorem 7.

\[\square\]

**Lemma 8.** If \( \mathbb{A} \in \mathcal{V}_{\text{mb}} \) and \( a, b, c \) are distinct elements of \( \mathbb{A} \) such that \( \{a, c\} \) and \( \{b, c\} \) are subalgebras of \( \mathbb{A} \) and \( (c, c) \in \text{Sg}_{\mathbb{A}^2}(\{(a, b), (b, a)\}) \), then \( \{a, b, c\} \) is a subalgebra of \( \mathbb{A} \) which is isomorphic to the free semilattice on the two generators \( a, b \).

**Proof.** Suppose first, for a contradiction, that \( \{a, c\} \) is a majority algebra. If \( b \to c \), then
\[
(a, c) = (f(a, c), f(b, c)) = f((a, b), (c, c)) \in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\},
\]
so by Lemma 7 we see that \( \{a, b, c\} \) is isomorphic to the subdirect product of a two element majority algebra and a two element semilattice, but then \( (c, c) \not\in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} = \{(a, b), (b, a), (a, c), (c, a)\} \), a contradiction.

If \( c \to b \), then
\[
(b, c) = (f(c, b), f(c, a)) = f((c, c), (b, a)) \in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\},
\]
and similarly \( (c, b) \in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \), so
\[
(b, b) = (f(c, b), f(c, b)) = f((b, c), (c, b)) \in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\},
\]
so by Lemma 2 \( a \to b \), but then \( (c, c) \not\in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} = \{(a, b), (b, a), (b, b)\} \), a contradiction.

Thus if \( \{a, c\} \) is a majority algebra, then we must also have \( \{b, c\} \) a majority algebra. From \( (c, c) \in \text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \), there must be some binary term \( t \) such that \( t(a, b) = t(b, a) = c \), and we may assume without loss of generality that \( t \) acts as first projection on any majority subalgebra of \( \mathbb{A} \). Define terms \( h, g' \) by
\[
h(x, y, z) = g(x, t(x, y), t(x, z))
\]
and
\[
g'(x, y, z) = g(h(x, y, z), h(y, z, x), h(z, x, y)).
\]
Then
\[
h(a, a, b) = g(a, a, c) = a,
\]
\[
h(a, b, a) = g(a, c, a) = a,
\]
\[
h(b, a, a) = g(b, c, c) = c,
\]
so
\[
g'(a, a, b) = g(a, a, c) = a.
\]
and similarly \( g'(a, b, a) = g'(b, a, a) = a \). By interchanging the roles of \( a \) and \( b \) in this argument, we see that \( g' \) acts as the majority operation on \( \{a, b\} \), so by Theorem 7 \( \{a, b\} \) is a majority algebra, so \( (c, c) \not\in S_{\alpha, \beta} \{a, b\} \) \( \{a, b\}, \{a, b\} \}, \) a contradiction.

We have shown that neither of \( \{a, c\}, \{b, c\} \) can be a majority algebra, so they must both be semilattice subalgebras. If \( a \to c \to b \), then \( (c, b) = f((a, b, (c, c)) \in S_{\alpha, \beta} \{a, b\}, (c, a) \}, \) \( (b, b) = f((b, c), (b, c)) \in S_{\alpha, \beta} \{a, b\}, (a, a) \}, \) so \( a \to b \), a contradiction. Similarly we can’t have \( b \to c \to a \).

Let \( t \) be a binary term such that \( t(a, b) = t(b, a) = c \). Suppose that \( c \to a, c \to b \), and define \( g' \) by

\[
g'(x, y, z) = g(t(x, y), t(y, z), t(z, x)).
\]

Then

\[
g'(a, a, b) = g(a, c, c) = a,
\]

and similarly \( g'(a, b, a) = g'(b, a, a) = a \) as well as \( g'(a, b, b) = g'(b, a, b) = g'(b, b, a) = b \), so by Theorem 7 \( \{a, b\} \) is a majority algebra, a contradiction.

Thus we must have \( a \to c \) and \( b \to c \), so \( \{a, b, c\}, t \) is the free semilattice on the generators \( a, b, c \), and we are done by Theorem 7.

\[
\text{Theorem 9. If } \mathcal{A} \in V_{mbw}, \{a, b, c\} \subseteq \mathcal{A} \text{ is closed under } f, \text{ and not all three of } \{a, b\}, \{b, c\}, \{a, c\} \text{ are majority subalgebras of } \mathcal{A}, \text{ then } \{a, b, c\} \text{ is a subalgebra of } \mathcal{A} \text{ and the restriction of } g \text{ to } \{a, b, c\} \text{ is determined by the restriction of } f \text{ to } \{a, b, c\}. \\
\]

\[
\text{Proof. First we show that the fact that } g \text{ is determined by } f \text{ follows from the fact that } \{a, b, c\} \text{ is necessarily a subalgebra. If there were two different algebras } \{\alpha, \beta\}, f, g \text{ and } \{\alpha, \beta\}, f, g' \text{ in } V_{mbw}, \text{ then the subset } \{\alpha, \beta\}, \{\alpha, \beta\}, \{\alpha, \beta\} \text{ of their product would be closed under } f \text{ and not all three of } (a, b), (b, b), (c, c) \} \text{ would be majority subalgebras, so it would be a subalgebra of the product. Thus it would be the graph of an isomorphism, and we would have to have } g = g'. \\
\]

By Theorem 4 at least two of \( \{a, b\}, \{b, c\}, \{a, c\} \) are subalgebras of \( \mathcal{A} \). If all three are semilattice subalgebras, then \( \{a, b, c\} \) is a 2-semilattice and we are done by Theorem 7.

If all three are subalgebras but not all three have to same type, then we may assume without loss of generality that \( a \to b \) and that \( f(b, c) = b \). Then since \( \alpha(a, b, c) = (b, b, f(c, a)) \) has two of its coordinates equal and \( \alpha(a, c, b) = (f(a, c), f(c, b), b) \) either has two of its coordinates equal or is equal to \( (a, c, b) \), we see that \( \beta(\alpha(x, y, z)) \) always has two of its coordinates equal for any \( x, y, z \in \{a, b, c\} \). Thus, if we set \( g'(x, y, z) = g(\beta(\alpha(x, y, z))) \), then for \( x, y \in \{a, b, c\} \) we have

\[
g'(x, x, y) = g'(x, y, x) = g'(y, y, x) = f(x, y)
\]

and \( \{a, b, c\} \) is closed under \( g' \), so we are done by Theorem 7.

Now suppose that \( \{a, b\} \) is not a subalgebra of \( \mathcal{A} \), in which case \( \{a, c\}, \{b, c\} \) must be subalgebras by Theorem 4. If \( f(a, b) = f(b, a) = c \), then by Lemma 8 \( \{a, c\} \) is a subalgebra of \( \mathcal{A} \) which is isomorphic to a free semilattice on the generators \( a, b, c \).

If \( f(a, b) \neq f(b, a) \), then one of them must be \( c \) and we may assume without loss of generality that the other one is \( a \), so \( \{a, c\} \) is a majority subalgebra of \( \mathcal{A} \). If \( f(b, a) = a \), then by Lemma 5 we have \( b \to a \), contradicting the assumption that \( \{a, b\} \) was not a subalgebra. Thus we must have \( f(a, b) = a, f(b, a) = c \). If \( f(b, c) = b \), then by Lemma 6 we have \( \{a, c\} \) a majority algebra, which is again a contradiction. Since \( \{b, c\} \) is a subalgebra and \( f(b, c) \neq b \) we must have \( b \to c \), and by Lemma 7 \( \{a, b, c\} \) is a subalgebra of \( \mathcal{A} \) which is isomorphic to a subdirect product of a two element majority algebra and a two element semilattice.

\[
\text{Theorem 10. We may choose the operation } g \text{ of } V_{mbw} \text{ such that every three element algebra of } V_{mbw} \text{ is isomorphic to one of the eleven algebras shown in Figure 7.}
\]
Proof. Let $\mathcal{A}_1, ..., \mathcal{A}_{11}$ be the algebras given in the corresponding rows of Figure 1. We start by choosing $f, g, f_3$ as in Corollary 1 and then we modify $g$ to make it satisfy the conclusion of Lemma 4 (note that in doing so, we do not change the common value of $g(x, x, y) \approx g(x, y, x) \approx g(y, x, x)$).

First we will show that on every three element algebra $\mathcal{A} = (A, f, g)$ either $\mathcal{A}$ or $\tilde{\mathcal{A}} = (A, f, \tilde{g})$ is isomorphic with one of $\mathcal{A}_4, \mathcal{A}_6$. If $\mathcal{A}$ is not conservative, then by the proof of Theorem 9 $\mathcal{A}$ is either isomorphic to $\mathcal{A}_5$, the free semilattice on two generators, or $\mathcal{A}_7$, the subdirect product of a two element majority algebra and a two element semilattice.

If $\tilde{\mathcal{A}}$ is conservative and is not a majority algebra or the strongly connected 2-semilattice, then by the proof of Theorem 9 the restriction of $g$ to $\tilde{\mathcal{A}}$ is in the clone of the restriction of $g(\beta(\alpha(x, y, z)))$ to $\tilde{\mathcal{A}}$, and the reduct corresponding to $g(\beta(\alpha(x, y, z)))$ is isomorphic to one of $\mathcal{A}_4, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}$. We need to show that in this case, either $g$ or $\tilde{g}$ agrees with $g(\beta(\alpha(x, y, z)))$. To see this, note that by Lemma 4 the restriction of $g$ to $\tilde{\mathcal{A}}$ is cyclic, so we must have $\gamma(a, b, c), \gamma(c, a, b) = (a, c, b)$ in the intersection of the diagonal of $\tilde{\mathcal{A}}^2$ with $S_{\tilde{\mathcal{A}}^2}\{(a, b), (b, c), (c, a)\}$, that is, in the circled set of vertices of the corresponding doodle. By examining the possibilities, we see that in each case the circled set of vertices either has size 1, in which case $g, \tilde{g}$, and $g(\beta(\alpha(x, y, z)))$ all agree on $\mathcal{A}$, or else there is an automorphism $\iota$ of $\mathcal{A}$ with order two which interchanges the two circled vertices. Then since $\gamma$ is cyclic we must have

$$\gamma(a, c, b) = \iota(\gamma(a, c, b)) = \iota(\gamma(a, b, c)),$$

so $g$ and $\tilde{g}$ do not agree with each other on $\mathcal{A}$, and exactly one of them gives a reduct isomorphic with $\mathcal{A}_{11}$.

Finally we have the case where $\tilde{\mathcal{A}}$ is a majority algebra with $g$ cyclic. There are two cases: either $\gamma(a, b, c) = \gamma(a, c, b)$, or $\gamma(a, b, c) \neq \gamma(a, c, b)$. In the first case, $\tilde{\mathcal{A}}$ is the median algebra $\mathcal{A}_2$. In the second case, letting $\{\gamma(a, b, c), \gamma(c, a, b)\} = \{a, b\}$, we see that either $\mathcal{A}$ or $\tilde{\mathcal{A}}$ is isomorphic to $\mathcal{A}_3$.

Next we show that when $\mathcal{A}$ is a minimal bounded width algebra of size three with $A$ not isomorphic to $\tilde{\mathcal{A}}$, then $\tilde{\mathcal{A}}^2$ is term equivalent to a proper reduct of $\mathcal{A} \times \tilde{\mathcal{A}}$, so that by Theorem 7 at most one of $\mathcal{A}, \tilde{\mathcal{A}}$ can be an algebra of the pseudovariety $\mathcal{V}_{mbw}$. If $\mathcal{A}$ or $\tilde{\mathcal{A}}$ is isomorphic to $\mathcal{A}_{11}$ then this is easy: the reduct of $\mathcal{A}_{11} \times \mathcal{A}_{11}$ given by $g'(x, y, z) = g(\alpha(x, y, z))$ is term equivalent with $\mathcal{A}_{11}^2$. By replacing $g$ with $\tilde{g}$ if necessary, we may assume without loss of generality that $\mathcal{A}_{11} \in \mathcal{V}_{mbw}$ and $\mathcal{A}_{11} \not\in \mathcal{V}_{mbw}$.

One may easily check that in any algebra $\mathcal{A}$, if $a$ is such that $f(a, b) = a$ for all $b \in A$ then we must have $f_3(a, b, c) = a$ for any $b, c \in A$. In particular, $f_3$ agrees with first projection in any majority algebra. In $\mathcal{A}_{11}$, this gives us

$$f_3^{\mathcal{A}_{11}}(a, b, c) = a,$$

$$f_3^{\mathcal{A}_{11}}(b, c, a) = b,$$

and we have

$$f_3^{\mathcal{A}_{11}}(a, b, c) \in \{a, b\},$$

since $\mathcal{A}_{11}$ has a quotient isomorphic to a semilattice directed from $c$ to $\{a, b\}$. By possibly replacing $f_3(x, y, z)$ with $f_3(x, z, y)$, we may assume without loss of generality that $f_3^{\mathcal{A}_{11}}(c, a, b) = a$.

In order to prove that $\mathcal{A}_3^2$ is term equivalent to a reduct of $\mathcal{A}_3 \times \mathcal{A}_3$, we define $h$ by

$$h(x, y, z) = g(f_3(x, y, z), g(x, y, z), g(x, z, y))$$

and then define $g'$ by

$$g'(x, y, z) = g(h(x, y, z), h(y, z, x), h(z, x, y)).$$
Then
\[ g^{\mathcal{K}_3}(a, b, c) = g^{\mathcal{K}_3}(g^{\mathcal{K}_3}(a, a, b), g^{\mathcal{K}_3}(b, a, b), g^{\mathcal{K}_3}(c, a, b)) = g^{\mathcal{K}_3}(a, b, a) = a \]
and
\[ g^{\mathcal{K}_3}(a, b, c) = g^{\mathcal{K}_3}(g^{\mathcal{K}_3}(b, a, b), g^{\mathcal{K}_3}(b, b, a), g^{\mathcal{K}_3}(c, b, a)) = g^{\mathcal{K}_3}(a, b, a) = a, \]
and from the definition of \( g' \) it is clear that \( g^{\mathcal{K}} \) is cyclic whenever \( g^{\mathcal{K}} \) is cyclic, so this gives us the desired reduct. Additionally, we have
\[ g^{\mathcal{K}_{11}}(a, b, c) = g^{\mathcal{K}_{11}}(g^{\mathcal{K}_{11}}(a, a, b), g^{\mathcal{K}_{11}}(b, a, b), g^{\mathcal{K}_{11}}(a, a, b)) = g^{\mathcal{K}_{11}}(a, b, a) = a, \]
so \( g' \) agrees with \( g \) on \( \mathcal{K}_{11} \).
Moreover, since
\[ h(x, y, z) = h(x, y, x) \approx g(f(x, y), f(x, y), f(x, y)) \approx f(x, y) \]
and
\[ h(y, x, x) \approx g(f(y, x), f(x, y), f(x, y)) \approx f(x, y), \]
the function \( g' \) given above has the additional property that
\[ g'(x, y, z) \approx g'(x, y, x) \approx g'(y, x, x) \approx f(x, y), \]
so we may replace \( g \) with \( g' \) in \( \mathcal{V}_{mcbw} \).

\[ \square \]

**Corollary 3.** The number of minimal conservative elements of \( \mathcal{G}_k \), up to term equivalence, is
\[ \sum_{E \subseteq \{1, ..., k\}} 7^{\Delta(E)} 2^{k-|E|} = 7^{\binom{k}{3}} (1 + o(1)), \]
where \( \Delta(E) \) is the number of triangles in the graph \( \langle \{1, ..., k\}, E \rangle \).

**Definition 11.** We define the variety \( \mathcal{V}_{mcbw} \) to be the variety with one basic operation \( g \), which is generated by idempotent conservative bounded width algebras \( A = (A, g^{A}) \) such that on every three element subset \( S \subseteq A \), \( (S, g^{A} |_{S}) \) is isomorphic to one of the nine conservative examples in Figure II.

The variety \( \mathcal{V}_{mcbw} \) is easily seen to be locally finite. One may compute by hand that \( |\mathcal{F}_{\mathcal{V}_{mcbw}}(x, y)| = 4 \), and by computer that \( |\mathcal{F}_{\mathcal{V}_{mcbw}}(x, y, z)| = 2547 \). A few of the two and three variable identities satisfied in \( \mathcal{V}_{mcbw} \) are as follows:
\[ f(x, y) \approx f(x, f(x, y)) \]
\[ \approx f(x, f(y, x)) \]
\[ \approx f(f(x, y), x) \]
\[ \approx f(f(x, y), y) \]
\[ \approx g(x, y, f(x, y)) \]
\[ \approx g(x, f(x, y), f(y, x)) \]
\[ g(x, y, z) \approx g(x, f(y, z), z) \]
\[ \approx g(x, f(z, x), f(z, x)) \]
\[ f(f(x, y), z) \approx g(z, x, f(x, y)) \]
\[ f(f(x, y), f(x, z)) \approx g(x, f(x, y), f(x, z)). \]

In later sections, we will give two examples of minimal elements of \( \mathcal{G}_4 \) for which the identity \( f(x, f(y, x)) \approx f(x, y) \) can not be satisfied, as well as an example of a minimal element of \( \mathcal{G}_6 \) for which none of the identities listed above which only involve \( f \) can be satisfied.
We need a result proved implicitly by Bulatov in [7]. Note that by Theorem 3, a pair \(a, b\) in a minimal bounded width algebra forms a semilattice subalgebra directed from \(a\) to \(b\) if and only if we have \(f(a, b) = b\), since in this case the function \(t(x, y) \approx f(x, f(x, y))\) has \(t(a, b) = f(a, f(a, b)) = f(a, b) = b\) and \(t(b, a) = f(b, f(b, a)) = f(f(a, b), f(b, a)) = f(a, b) = b\). Thus, by Theorem 7, a pair \(a, b\) with \(f(a, b) = b\) is the same as what Bulatov calls a “thin semilattice edge” or a “thin red edge” in [8] and [7].

From the above discussion, we see that reachability of \(y\) from \(x\) is the same as the existence of a (directed) thin red path from \(x\) to \(y\) in Bulatov’s colored graph attached to \(A\), and that being maximal in \(A\) is the same as being contained in what Bulatov calls a “maximal strongly connected component” of \(A\). The key result we need from Bulatov’s work is his “yellow connectivity property”.

**Definition 12.** We say that a subset \(S\) of an algebra \(A\) which has a partial semilattice operation \(s\) is upwards closed if whenever \(a \in S\) and \(a' \in A\) have \(a \rightarrow_s a'\), we also have \(a' \in S\).

**Proposition 7** (Yellow Connectivity Property). If \(A\) is a minimal bounded width algebra and \(A, B\) are upwards closed subsets of \(A\), then there are \(a \in A\) and \(b \in B\) such that \((a, b)\) is a majority subalgebra of \(A\).

**Proof.** This follows from Proposition 6 of [7], since every minimal bounded width algebra is necessarily what Bulatov calls a conglomerate algebra and since every upwards closed set contains a maximal strongly connected component. For the sake of completeness we give a proof of a slight refinement of the yellow connectivity property, based on Bulatov’s argument, in the appendix. \(\square\)

We will need the following easy result.

**Proposition 8.** Suppose that \(R \subseteq A \times B\) is subdirect and closed under a partial semilattice operation \(s\), and that \(s\) acts as first projection on \(A\). Then for any \(a \in A\), the set \(\pi_2(R \cap \{(a) \times B\})\) is upwards closed in \(B\).

**Proof.** Suppose that \((a, b) \in R\) and that \(b' \in B\) with \(b \rightarrow b'\). Since \(R\) is subdirect, there is \(a' \in A\) with \((a', b') \in R\). Then

\[(a, b') = (s(a, a'), s(b, b')) = s((a, b), (a', b')) \in R.\]

**Lemma 9.** Suppose that \(A\) is generated by \(a, b \in A\), and let \(R\) be a subalgebra of \(A^3\) which contains \(Sg_A \{a, b\}, \{a, b, a\}, \{b, a, a\}\). Let \(U, V, W\) be any three maximal strongly connected components of \(A\). If \(R = U \times V \times W \neq \emptyset\), then \(U \times V \times W \subseteq R\).

**Proof.** By the definition of \(R\) and the assumption \(A = Sg\{a, b\}\), we see that \(\{a\} \times A \subseteq \pi_{1,2}R\). Choose \(u \in U\) such that the set \(T\) of \(v \in V\) with \((u, v) \in \pi_{1,2}R\) is maximal. Suppose for a contradiction that \(T \neq V\), so that there are \(v, v' \in V\) with \(v \in T\), \(v' \notin T\), and \(v \rightarrow v'\). Then \((u, v), (a, v') \in \pi_{1,2}R\), so

\[(s(u, a), v') = (s(u, a), s(v, v')) = s((u, v), (a, v')) \in \pi_{1,2}R,\]

and from \(u \rightarrow s(u, a)\) we have \(s(u, a) \in U\). Furthermore, for any \(t \in T\) we have \((a, t) \in \pi_{1,2}R\), so \((s(u, a), t) = s((u, t), (a, t)) \in \pi_{1,2}R\). Thus \(\{s(u, a)\} \times (T \cup \{v'\}) \subseteq \pi_{1,2}R\), contradicting the choice of \(u\). Thus \(\{u\} \times V \subseteq \pi_{1,2}R\).

Let \(S\) be the set of all \(u \in U\) such that \(\{u\} \times V \subseteq \pi_{1,2}R\). If \(S \neq U\), then since \(S \neq \emptyset\) and \(U\) is strongly connected there are \(u, u' \in U\) with \(u \in S, u' \notin S\), and \(u \rightarrow u'\). Let \(T\) be the set of \(v \in A\) such that \((u', v) \in \pi_{1,2}R\). From the assumption \(R \cap (U \times V \times W) \neq \emptyset\) we easily see that \(T \neq \emptyset\). If \(T \neq V\), then there are \(v, v' \in A\) with \(v \in T, v' \notin T\), and \(v \rightarrow v'\). Then since \((u, v'), (u', v) \in \pi_{1,2}R\), we have \((u', v') = (s(u, u'), s(v, v')) = s((u, v'), (u', v)) \in \pi_{1,2}R\). Thus in fact we must have \(v' \in T\), contradicting our choice of \(v'\), so we must \(T = V\) and therefore \(u' \in S\), contradicting the choice
of $u'$, which shows that $S = U$. We have shown that $U \times V \subseteq \pi_{1,2}R$, and similarly we also have $U \times W \subseteq \pi_{1,3}R$, and $V \times W \subseteq \pi_{2,3}R$.

Let $P$ be the set of pairs $(u, v) \in U \times V$ such that there exists $w \in W$ with $(u, v, w) \in R$. Suppose for a contradiction that $P \neq U \times V$. By assumption we have $P \neq \emptyset$, so there exist $(u, v) \rightarrow (u', v')$ with $(u, v) \in P$ and $(u', v') \notin P$. By the definition of $P$, there is some $w \in W$ with $(u, v, w) \in R$, and from $U \times V \subseteq \pi_{1,2}R$ there is some $z \in A$ with $(u', v', z) \in R$. Then

$$(u', v', s(w, z)) = s((u, v, w), (u', v', z)) \in R$$

and $w \rightarrow s(w, z) \in W$ and we see that $(u', v') \notin P$, a contradiction. Thus $P = U \times V$.

Now letting $R = \mathbb{R} \cap (U \times V \times W)$, we see that $R$ is closed under $s$ and $\pi_{1,2}R = U \times V, \pi_{1,3}R = U \times W, \pi_{2,3}R = V \times W$, so by Theorem 7 we have $R = U \times V \times W$. \hfill \Box

**Lemma 10.** If $A$ is a minimal bounded width algebra and $a, b \in A$ are such that $a$ is a maximal element of $Sg_A\{a, b\}$, then $(a, a, a) \in Sg_{A^3}\{(a, a, b), (a, b, a), (b, a, a)\}$.

**Proof.** We may assume without loss of generality that $A = Sg\{a, b\}$. Let $A$ be a maximal strongly connected component of $A$ which contains $a$, and set $\mathbb{R} = Sg_{A^3}\{(a, a, b), (a, b, a), (b, a, a)\}$. The main step of the proof is to show that $\mathbb{R} \cap (A \times A \times A) \neq \emptyset$. Although this follows from Bulatov’s results on quasi-2-decomposability (see Proposition 7 of [7]), we will give a direct argument using the yellow connectivity property, which follows the last few steps of the proof of Lemma 16 of [7]. Throughout the proof, we will fix a partial semilattice operation $s$ which is adapted to $A$.

Letting $B$ be a maximal strongly connected component which is reachable from $b$, we see from Lemma 9 that $A \times A \times B \subseteq \mathbb{R}$. In particular, we have $A \times A \subseteq \pi_{1,2}R$, and similarly we have $A \times A \subseteq \pi_{1,3}R, \pi_{2,3}R$.

Now we will prove that $\mathbb{R} \cap (A \times A \times A) \neq \emptyset$. First we let $C$ be the set of elements $c \in A$ such that $\mathbb{R} \cap \{(c) \times A \times \{a\}\} \neq \emptyset$. Since $b \in C$, we know that $C \neq \emptyset$. Now suppose that $c \in C$ and $c \rightarrow c'$, and we will show that $c' \in C$. Say that $(c, a', a) \in \mathbb{R} \cap \{(c) \times A \times \{a\}\}$ and pick some $(c', u, a) \in \mathbb{R}$ (we can do this since $(c', a) \in \pi_{1,3}R$). Then

$$(c', s(a', u), a) = s((c, a', a), (c', u, a)) \in \mathbb{R}$$

and $a' \rightarrow s(a', u)$, so $(c', s(a', u), a) \in \mathbb{R} \cap \{(c') \times A \times \{a\}\}$, that is, $c' \in C$. Thus, by the yellow connectivity property, there exists some $c \in C$ and some $c' \in A$ such that $(c, c')$ is a majority algebra. Fix such a pair $c, c'$.

Now we let $D$ be the set of elements $d \in A$ such that $(c', d, a) \in \mathbb{R}$, and let $D'$ be the set of $d' \in A$ such that $(c, d', a) \in \mathbb{R}$, and consider them both as subsets of the algebra $D = \pi_2(\mathbb{R} \cap \{(c, c') \times A \times \{a\}\})$. Note that $D$ is nonempty since $(c', a) \in \pi_{1,2}R$ and $D'$ contains an element of $A$ since $c \in C$. By Proposition 8, $D, D'$ are upwards closed in $\mathbb{R}$. By the yellow connectivity property applied to $D$, we can choose $d \in D, d' \in D' \cap A$ with $(d, d')$ a majority algebra. Fix such a pair $d, d'$.

Now we let $E$ be the set of $e \in A$ such that $(c', d', e) \in \mathbb{R}$, let $E'$ be the set of $e' \in A$ such that $(c, d', e') \in \mathbb{R}$, and consider them both as subsets of the algebra $E = \pi_3(\mathbb{R} \cap \{(c, c') \times \{d', d'\} \times A\})$. By Proposition 8, both $E$ and $E'$ are upwards closed in $\mathbb{R}$. $E$ is nonempty since $c', d' \in A$ and $A \times A \subseteq \pi_{1,2}R$, and $E' \cap A$ is nonempty since it contains $a$ by the definitions of $D, D'$. We see from the yellow connectivity property that there are $e \in E$ and $e' \in E' \cap A$ with $(e, e')$ a majority algebra.

At this point we have constructed $c, d, e \in A$ and $c', d', e' \in A$ such that $(c', d', e), (c', d', e'), (c, d', e') \in \mathbb{R}$, and each of $(c, c'), (d, d'), (e, e')$ is a majority algebra, so

$$(c', d', e') = (g(c', c'), g(d', d'), g(e, e', e')) = g((c', d', e), (c', d', e'), (c, d', e')) \in \mathbb{R} \cap (A \times A \times A).$$

Now we can apply Lemma 9 to see that we have $A \times A \times A \subseteq \mathbb{R}$. Thus $(a, a, a) \in \mathbb{R}$. \hfill \Box

**Theorem 11.** If $a, b$ are distinct elements of a minimal bounded width algebra $A$ such that $(a, b)$ is a maximal element of $Sg_A\{(a, b), (b, a)\}$, then $(a, b)$ is a majority subalgebra of $A$. 28
Theorem 4, there exist binary terms $p$, $g$ of a two element semilattice, and so $x, p \in A$ two element majority subalgebras, this must in fact be a semilattice subalgebra, so it is also a

Now we set $f$, that one sees, by induction on $n$, define terms $x, y, p \in A$.

If a minimal bounded width algebra has no semilattice subalgebra, then it is a majority algebra.

Let $p_1(x, y) = q_i(x, y) = q_i(q_{i-1}(x, y), p_{i-1}(y, x))$, and set $q_i(x, y) = s(x, q_i(x, y))$, so that $p_1(x, y) \approx q_i(q_{i-1}(x, y), p_{i-1}(y, x))$ and $q_i(x, q_i(x, y)) \approx q_i(q_i(x, y), x) \approx q_i(x, y)$. Now we inductively define terms $r_0(x, y, z)$ by $r_0(x, y, z) = x$, and

$$r_{i+1}(x, y, z) = q_i(q_i(r_i(x, y, z), x, y, z), r_i(y, z, x)).$$

Then one sees, by induction on $i$, that

$$r_i(x, y, z) \approx r_i(x, y, x) \approx p_i(x, y), \quad r_i(x, y, x) \approx p_i(x, y).$$

Now we set $f'(x, y) = p_k(x, y)$ and $g'(x, y, z) = g(r_k(x, y, z), r_k(y, z, x), r_k(z, x, y))$. These satisfy the identities

$$f'(x, y) \approx g'(x, y, z) \approx g'(x, y, x) \approx f'(f'(x, y), f'(y, x)).$$

and $f'(x, y)$ is reachable from $x$ by construction.

Corollary 5. If a minimal bounded width algebra has no semilattice subalgebra, then it is a majority algebra.

Corollary 6. Every minimal bounded width algebra has a partial semilattice term $s(x, y)$ such that there is a sequence of terms $p_0(x, y), ..., p_k(x, y)$ with $p_0(x, y) \approx y$, $p_k(x, y) \approx s(x, y)$, and for each $i$ either $p_i(x, y) \rightarrow p_{i+1}(x, y)$ or $\{p_i(x, y), p_{i+1}(x, y)\}$ is a majority algebra.

10. Spirals

Suppose that $\mathbb{A} = (A, f, g)$ is a minimal bounded width algebra with no two element majority subalgebra. Since $\{f(x, y), g(x, y)\}$ is always a majority subalgebra, this immediately implies that we have $f(x, y) \approx f(y, x)$ in $\mathbb{A}$. Let $\mathbb{A}' = (A, f)$ be the reduct obtained by dropping $g$. By Theorem 4, there exist binary terms $p_0, ..., p_n$ of $\mathbb{A}'$ such that for any $x, y \in \mathbb{A}'$ we have $p_0(x, y) = x, p_n(x, y) = y$, and for each $i$, the set $\{p_i(x, y), p_{i+1}(x, y)\}$ is a subalgebra of $\mathbb{A}$. Since $\mathbb{A}$ has no two element majority subalgebras, this must in fact be a semilattice subalgebra, so it is also a semilattice subalgebra of $\mathbb{A}'$. Thus any quotient of $\mathbb{A}'$ which does not identify $x$ and $y$ must contain a two element semilattice, and so $\mathbb{A}'$ has bounded width. Since $\mathbb{A}$ was assumed minimal, we have proved the following claim.

Proposition 9. If $\mathbb{A} = (A, f, g)$ is a minimal bounded width algebra with no majority subalgebra, then $g \in \text{Clo}(f)$ and $f(x, y) \approx f(y, x)$. 

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Suppose section is the algebra depicted in Figure 2. Let $f$ be chosen as in Corollary 4. Let $S$ be the set of all maximal elements of $A$. Then for any $x \in A$ and any $y \in S$, $y$ is reachable from $x$. Moreover, $S$ is a subalgebra of $A$, and for any $x \in A \setminus S$ the set $S \cup \{x\}$ is a subalgebra of $A$ which has a congruence $\theta$ given by the partition $\{S, \{x\}\}$ such that the quotient $(S \cup \{x\})/\theta$ is a semilattice directed from $\{x\}$ to $S$.

**Proof.** Since $f$ is commutative, for any $x, y \in A$ we see that $f(x, y) = f(y, x)$ is reachable from both $x$ and $y$. Thus, if $y \in S$ then we see from the definition of maximality that $y$ is reachable from $f(x, y)$, which is reachable from $x$, so $y$ is reachable from $x$. Since anything which is reachable from an element of $S$ is necessarily in $S$, we see that for any $x \in A$ and any $y \in S$ we have $f(x, y) = f(y, x) \in S$.

**Lemma 11.** Suppose $A$ is a minimal bounded width algebra with no majority subalgebra, and that $f$ is chosen as in Corollary 4. Let $S$ be the set of all maximal elements of $A$. Then for any $x \in A$ and any $y \in S$, $y$ is reachable from $x$. Moreover, $S$ is a subalgebra of $A$, and for any $x \in A \setminus S$ the set $S \cup \{x\}$ is a subalgebra of $A$ which has a congruence $\theta$ given by the partition $\{S, \{x\}\}$ such that the quotient $(S \cup \{x\})/\theta$ is a semilattice directed from $\{x\}$ to $S$.

**Lemma 12.** Suppose $A$ is a minimal bounded width algebra with no majority subalgebra which is generated by two elements $a, b \in A$. Let $S$ be the set of all maximal elements of $A$. Then $A = S \cup \{a, b\}$.

**Proof.** If $S$ contains $a$ or $b$, then we are done by the previous Lemma. Otherwise, since $S \cup \{a\}$ is a minimal bounded width algebra as well (by Theorem 4), we may apply Theorem 4 to see that there is an element $p(a, b) \in S$ such that $\{a, p(a, b)\}$ is a semilattice. Letting $s$ be as in the proof of Corollary 4 and setting $p'(x, y) = s(x, p(x, y))$, we get $p'(a, b) = p(a, b)$ and $\{x, p'(x, y)\}$ is a semilattice for all $x, y \in A$. Then we may use the proof of Corollary 4 taking $p_1(x, y) = p'(x, y)$ and $p_2(x, y) = s(p'(x, y), p'(y, x))$, to see that we may choose $f$ such that $f(x, y) = S_{\mathcal{F}_A}(x, y)\{p_2(x, y), p_2(y, x)\}$, so that in particular we have $f(a, b) \in S_{\mathcal{G}_A}\{s(p'(a, b), p'(b, a)), s(p'(b, a), p'(a, b))\}$. Since $p'(a, b) \in S$ and $S \cup \{p'(b, a)\}$ has the properties described in the previous lemma, we see that $s(p'(a, b), p'(b, a))$ and $s(p'(b, a), p'(a, b))$ are in $S$, and so $f(a, b) \in S$. Thus $S \cup \{a, b\}$ is a subalgebra of $A$, and so it must be equal to all of $A$.

In the proof of the next theorem, we will need to make use of the Absorption Theorem of 8.

**Proposition 10** (Absorption Theorem). Suppose that $A, B$ are both finite algebras in some variety which omissions type 1 and that $R$ is a subdirect product of $A$ and $B$ which is linked (that is, if we let $\pi_1, \pi_2$ be the projection maps to $A, B$ then $ker \pi_1 \lor ker \pi_2$ is the total congruence). Then either $R = A \times B$ or one of $A, B$ has a proper absorbing subalgebra (a subalgebra $S$ of $A$ is absorbing if there is a term $t$ of some arity $k$, such that for any $a_1, ..., a_k \in A$ with $a_i \in S$ for all but at most one $i$, we have $t(a_1, ..., a_k) \in S$).

**Theorem 12.** Suppose $A$ is a minimal bounded width algebra with no majority subalgebra which is generated by two elements $a, b \in A$. Then either $A$ is a two element semilattice, or on letting $S$ be the set of maximal elements of $A$ we have $A = S \cup \{a, b\}$ with $S \cap \{a, b\} = \emptyset$, and $A$ has a
congruence $\theta$ corresponding to the partition $\{S, \{a\}, \{b\}\}$ such that $\mathbb{A}/\theta$ is isomorphic to the free semilattice on the generators $\{a\}, \{b\}$.

Proof. By the previous lemmas, we just need to show that if $\mathbb{A}$ is not a two element semilattice then $S \cap \{a, b\} = \emptyset$. Since for any pair of elements $x, y \in S$, $y$ is reachable from $x$, we easily see that $S$ has no proper absorbing subalgebras. Suppose first that $\{a, b\} \subset S$. In this case, no quotient of $\mathbb{A}$ can be a two element semilattice, so we may assume without loss of generality that $\mathbb{A}$ is simple. Letting $\mathbb{R} = Sg_{\mathbb{A}}\{\{a, b\}, \{b, a\}\}$, we see from the simplicity of $\mathbb{A}$ that either $\mathbb{R}$ is linked or else is the graph of an automorphism of $\mathbb{A}$. If $\mathbb{R}$ is linked, then by the Absorption Theorem and the fact that $\mathbb{A} = S$ has no proper absorbing subalgebra we have $\mathbb{R} = \mathbb{A} \times \mathbb{A}$, so $(a, a) \in \mathbb{R}$, and this implies that there is a term $t$ such that $t(a, b) = t(b, a) = a$, so $\{a, b\}$ is a two element semilattice, which is a contradiction. On the other hand, if $\mathbb{R}$ is the graph of an automorphism of $\mathbb{A}$, then since $a$ is maximal in $\mathbb{A}$ we have $(a, b)$ maximal in $\mathbb{R} = Sg_{\mathbb{A}}\{\{a, b\}, \{b, a\}\}$, so by Theorem 11 $\{a, b\}$ must be a majority subalgebra of $\mathbb{A}$, a contradiction.

Now suppose that $S \cap \{a, b\} = \{b\}$, and assume without loss of generality that no nontrivial congruence of $S$ extends to a nontrivial congruence of $\mathbb{A}$. Let $\mathbb{R} = Sg_{\mathbb{A}}\{\{a, b\}, \{b, a\}\} \cap S^2$. Since $(f(a, b), f(b, a)) \in \mathbb{R}$ and every element of $Sg_{\mathbb{A}}\{\{a, b\}, \{b, a\}\}$ which is reachable from $(f(a, b), f(b, a))$ is also in $\mathbb{R}$, we see that $\mathbb{R}$ is a subdirect in $S^2$. Letting $\pi_1, \pi_2$ be the projections from $\mathbb{R}$ to $S$, we see that $\pi_1 \cup \pi_2$ restricts to a congruence $\theta$ of $S$, which is generated by pairs $(x, y)$ such that there exists $z \in S$ with $(x, z), (y, z) \in S$. Since $(x, z), (y, z) \in \mathbb{R}$ implies that $(f(a, x), f(b, z))$ and $(f(a, y), f(b, z))$ are also in $\mathbb{R}$, we see that if $x/\theta = y/\theta$ then $f(a, x)/\theta = f(a, y)/\theta$. Thus if $\theta$ is a nontrivial congruence of $S$ then it extends to a nontrivial congruence of $\mathbb{A}$, so in fact $\mathbb{R}$ must either be linked or the graph of an automorphism of $S$. If $\mathbb{R}$ is linked, then by the Absorption Theorem we have $\mathbb{R} = S^2$, so $(b, b) \in \mathbb{R}$ and $\{a, b\}$ must be a two element semilattice. Otherwise, suppose $\mathbb{R}$ is the graph of an automorphism $\iota : S \rightarrow S$. For any $x \in S$, we have $(f(a, x), f(b, \iota(x))) \in \mathbb{R}$ and $(f(b, x), f(b, \iota(x))) \in \mathbb{R}$, so we must have $f(a, x) = f(b, x)$ for all $x \in S$. But then since $b$ and $a$ generate $\mathbb{A}$ we see that $b$ and $\iota(b)$ generate $S$, so by the previous part of the argument we must have $b = \iota(b)$ and $S = \{b\}$, and $\mathbb{A}$ is a two element semilattice again.

Definition 13. An algebra $\mathbb{A} = (A, f)$ is a spiral if $f$ is a commutative idempotent binary operation and every subalgebra of $\mathbb{A}$ which is generated by two elements either has size two or has a surjective homomorphism to the free semilattice on two generators. $\mathbb{A}$ is a weak spiral if $f$ is a commutative idempotent binary operation such that for every $x, y \in A$, the sequence $f(x, y), f(x, f(x, y)), f(x, f(x, f(x, y))), ...$ is eventually constant.

Corollary 7. If $\mathbb{A}$ is a minimal bounded width algebra with no majority subalgebra, then $\mathbb{A}$ is a spiral. Every spiral is a weak spiral.

Proof. The first statement follows directly from Theorem 12, so we just need to show that every spiral is a weak spiral. Let $y_0 = y, y_1 = f(x, y), y_{i+1} = f(x, y_i)$. If $f(x, y_i) \neq y_i$ for all $i$, then by Theorem 12 we see that $y_i \notin Sg_{\mathbb{A}}\{x, f(x, y_i)\} = Sg_{\mathbb{A}}\{x, y_{i+1}\}$, so the size of $Sg_{\mathbb{A}}\{x, y_i\}$ is a strictly decreasing function of $i$ which always takes positive integer values, a contradiction.

Proposition 11. Every weak spiral has bounded width.

Proof. Define a sequence of terms $f_i(x, y)$ by $f_0(x, y) = y, f_1(x, y) = f(x, y)$, and $f_{i+1}(x, y) = f(x, f_i(x, y))$. We need to show that the identities $f(x, x) \approx x, f(x, y) \approx f(y, x)$ and $f_k(x, y) \approx f_{k+1}(x, y)$ can’t simultaneously be satisfied in a nontrivial affine algebra. So suppose for a contradiction that they do hold in some affine algebra, with $f(x, y) \approx \alpha x + \beta y$. From $f(x, x) \approx x$, we see that $(\alpha + \beta)x \approx x$, and from $f(x, y) \approx f(y, x)$ we see that $(\alpha - \beta)x \approx (\alpha - \beta)y$. Thus $2\alpha x \approx 2\beta x \approx x$, and by induction on $n$ we see that $2^n f_n(x, y) \approx (2^n - 1)x + y$. Then by multiplying both sides of $f_k(x, y) \approx f_{k+1}(x, y)$ by $2^{k+1}$, we see that $2(2^k - 1)x + 2y \approx (2^{k+1} - 1)x + y$, so $y \approx x$ and our affine algebra is in fact trivial.
Corollary 8. Every nontrivial reduct of a spiral has bounded width. In particular, if \( \mathbb{A} \) is a minimal spiral and \( t \) is any term of \( \mathbb{A} \) which is not a projection, then \( f \in \text{Clo}(t) \).

Proof. Since \( t \) is a nontrivial element of \( \text{Clo}(f) \), upon restricting \( t \) and \( f \) to any two element semilattice contained in \( \mathbb{A} \) we see that there must be some two variable restriction \( p(x,y) \) of \( t \) which acts as the semilattice operation on any two element semilattice of \( \mathbb{A} \).

We just need to show that for every subset \( P \) of \( \mathbb{A} \) which is closed under \( p \), the graph on \( P \) with edges corresponding to two element semilattices of \( \mathbb{A} \) is connected. So suppose that \( x,y \in P \) are not connected via a chain of two element semilattices, and that \( P \) is minimal such that such a pair \( x,y \) exist. Since \( \{x,y\} \) is not itself a two element semilattice, we see from the definition of a spiral that we can write \( Sg_{\mathbb{A}} \{x,y\} = S \cup \{x,y\} \) for some \( S \) which is closed under \( f \), \( S \cap \{x,y\} = \emptyset \), and that \( p(x,y) \in S \) since \( p \) acts as the semilattice operation on the three element quotient of \( S \cup \{x,y\} \). Thus \( P \cap (S \cup \{x\}) \) is also closed under \( p \), contains \( p(x,y) \), and does not contain \( y \), so by the minimality of \( P \) it contains a chain of two element semilattices connecting \( x \) to \( p(x,y) \). Similarly \( P \cap (S \cup \{y\}) \) contains a chain of two element semilattices connecting \( p(x,y) \) to \( y \), so \( x,y \) are connected by a chain of two element semilattices contained in \( P \).

\( \square \)

Corollary 9. A bounded width algebra \( \mathbb{A} \) has no nontrivial proper reducts if and only if \( \mathbb{A} \) is either a minimal majority algebra or a minimal spiral.

Proof. If \( \mathbb{A} \) has no majority subalgebra then this follows from the results of this section. Otherwise, if \( \mathbb{A} \) has a majority subalgebra \( \{a,b\} \) then the restriction of \( f \) to \( \{a,b\} \) is first projection, so \( \text{Clo}(f) \) does not have bounded width and is therefore a proper reduct. Thus \( f \) must be first projection, and \( \mathbb{A} \) is a minimal majority algebra.

It remains to show that any nontrivial reduct of a majority algebra is also a majority algebra. Suppose that \( t \) is a nontrivial term of the majority algebra \( \mathbb{A} \) with basic operation \( g \). The restriction of \( t \) to any two element subset of \( \mathbb{A} \) is then necessarily either a projection or a near-unanimity operation, and in the second case there is a majority term in the clone of \( t \). Thus we just need to show that \( t \) can’t be a semiprojection. Suppose for a contradiction that \( t(x,y,y,...,y) \approx x \) but \( t \) is not first projection. Since \( t \) is not a basic operation of \( \mathbb{A} \) or a projection, we can write \( t = g(t_1,t_2,t_3) \), with \( t_1,t_2,t_3 \) defined by shorter expressions than \( t \). Since \( \mathbb{A} \) is conservative, we have \( t_i(x,y,y,...,y) \in \{x,y\} \) for \( i = 1,2,3 \), so since \( g \) is a majority operation at least two of the \( t_i \)s have \( t_i(x,y,y,...,y) \approx x \), and we may suppose without loss of generality that these are \( t_1,t_2 \). By induction, we see that \( t_1,t_2 \) are first projection, so \( t \approx g(\pi_1,\pi_1,t_3) \approx \pi_1 \), and we see that \( t \) is first projection as well.

\( \square \)

Theorem 13. If \( \mathbb{A} \) is a minimal spiral, then for any nontrivial binary terms \( p(x,y), q(x,y), r(x,y) \), \( \mathbb{A} \) has a term \( w \) such that

\[
 w(x,x,y) \approx p(x,y), \quad w(x,y,x) \approx q(x,y), \quad w(y,x,x) \approx r(x,y).
\]

Proof. Let \( \mathbb{F} = \mathcal{F}_{\mathbb{A}}(x,y) \) be the free algebra on two generators in the variety generated by \( \mathbb{A} \). By Theorem [7], we see that \( \mathbb{F} \) is also minimal. By Theorem [12], we can write \( \mathbb{F} = S \cup \{x,y\} \), where \( S \) is the maximal strongly connected component of \( \mathbb{F} \), and the partition \( \{S, \{x\}, \{y\}\} \) of \( \mathbb{F} \) defines a congruence such that the quotient is the free semilattice on two generators. In particular, we must have \( f(x,y), p(x,y), q(x,y), r(x,y) \in S \). Let \( \mathbb{R} = Sg_{\mathbb{F}}\{(x,x,y),(x,y,x),(y,y,x)\} \), and let \( \mathbb{R} = \mathbb{R} \cap (S \times S \times S) \). We just need to show that \( \mathbb{R} = S \times S \times S \).

We have \( \{x\} \times \mathbb{F} \subseteq \pi_1,2 \mathbb{R} \), and \( (f(x,y), f(x,y), f(x,y)) \in \mathbb{R} \). Letting \( s \) be a partial semilattice operation adapted to \( \mathbb{A} \), we see that \( \mathbb{R} \) is closed under \( s \). A straightforward argument shows that \( \{s(f(x,y),x)\} \times S \subseteq \pi_1,2 \mathbb{R} \), and from this one can easily show that \( \pi_1,2 \mathbb{R} = S \times S \), and similarly \( \pi_1,3 \mathbb{R} = \pi_2,3 \mathbb{R} = S \times S \). Thus by Theorem [12] we have \( \mathbb{R} = S \times S \times S \).

Note that the definition of a spiral makes sense for infinite algebras as well.
Theorem 14. An arbitrary product of spirals is a (possibly infinite) spiral, and any subalgebra of a spiral is a spiral. If \( \mathcal{A} \) is a (possibly infinite) spiral and \( \theta \) is a congruence of \( \mathcal{A} \) such that the intersection of every class of \( \theta \) with every finitely generated subalgebra of \( \mathcal{A} \) is finite, then \( \mathcal{A}/\theta \) is also a spiral.

Proof. The first two claims follow directly from the definition of a spiral, so we only have to prove the claim about quotients. Let \( x/\theta, y/\theta \in \mathcal{A}/\theta \), and suppose without loss of generality that \( y \in y/\theta \) is chosen such that \( S_{g_{\mathcal{A}}}(x, y) \cap (y/\theta) \) is minimal. Suppose that \( S_{g_{\mathcal{A}}}/\theta(x/\theta, y/\theta) \) has more than two elements, so that in particular \( S_{g_{\mathcal{A}}}(x, y) \) also has more than two elements and thus maps to the free semilattice on two generators. We need to show that if \( a/\theta, b/\theta \in S_{g_{\mathcal{A}}}/\theta(x/\theta, y/\theta) \) with \( a/\theta \neq y/\theta \), then \( f(a, b)/\theta \neq y/\theta \). We may assume without loss of generality that \( a, b \in S_{g_{\mathcal{A}}}(x, y) \).

Suppose for a contradiction that \( f(a, b) \in y/\theta \), then by the choice of \( y \) we have \( y \in S_{g_{\mathcal{A}}}(x, f(a, b)) \). But this is a contradiction, since \( f(a, b) \neq y \) and \( (S_{g_{\mathcal{A}}}(x, y)) \setminus \{y\} \) is a subalgebra of \( \mathcal{A} \).

This shows that the collection of finite spirals forms a pseudovariety. The next result shows that it does not form a variety.

Proposition 12. For any odd \( p \), the variety generated by the collection of finite spirals contains the affine algebra \((\mathbb{Z}/p, f)\) given by \( f(a, b) = \frac{a+b}{2} \).

Proof. For each \( n \geq 1 \), we define a finite spiral \( \mathcal{A}_n = (\{1, \ldots, n\}, f) \) by

\[
 f(x, y) = \begin{cases} 
 x & x = y \\
 \min\left(\frac{p+1}{2}(x+y), n\right) & x \neq y 
\end{cases}
\]

To see that this is a spiral, note that for \( x \neq y \) with \( x, y < n \), we have \( f(x, y) > x, y \), and for any \( x \in \{1, \ldots, n\} \) we have \( x \rightarrow n \). Inside the product of the \( \mathcal{A}_n \)'s, we have an isomorphic copy of the infinite spiral \( \mathcal{A}_\mathbb{N} = (\mathbb{N}, f) \) given by

\[
 f(x, y) = \begin{cases} 
 x & x = y \\
 \frac{p+1}{2}(x+y) & x \neq y 
\end{cases}
\]

Now we look \( \text{(mod } p) \). □

Since the collection of finite spirals forms a pseudovariety, it should be defined by a sequence of identities such that all but finitely many of the identities hold in any finite spiral. The next result gives such a collection of identities.

Theorem 15. For any nontrivial term \( p(x, y) \) built out of \( f(x, y) \), let \( k_p \) be the number of times \( f \) occurs in the definition of \( p \). Define from \( p \) a sequence of terms \( p_0(x, y) = y, p_1(x, y) = p(x, y), \) and \( p_{i+1}(x, y) = p(x, p_i(x, y)) \). Then in any finite spiral \( \mathcal{A} \), the identities

\[
 f(x, p_k(x, y)) \approx p_k(x, y)
\]

with \( p \) a binary term built out of \( f \) and \( k \geq |\mathcal{A}| - 1 \) hold in \( \mathcal{A} \). Conversely, if \( \mathcal{A} = (A, f) \) with \( f \) a commutative idempotent binary operation such that all but finitely many of the identities \( f(x, p_k(x, y)) \approx p_k(x, y) \) with \( p \) a nontrivial binary term built out of \( f \) and \( k \geq k_p \) hold, then \( \mathcal{A} \) is a spiral.

Proof. Let \( \mathcal{A} \) be a finite spiral, and let \( a, b \in \mathcal{A} \). By the Pigeonhole Principle, there are \( 0 \leq i < j \leq |\mathcal{A}| \) with \( p_i(a, b) = p_j(a, b) \). Since \( p_i(a, b) = p_j(a, b) = p_{j-i}(a, p_i(a, b)) \) and \( p_{j-i} \) is nontrivial, by the definition of a spiral we must have \( a \rightarrow p_i(a, b) \), so for all \( k \geq i \) we have \( p_k(a, b) = p_i(a, b) \) and \( f(a, p_i(a, b)) = p_i(a, b) \).

Now suppose that \( \mathcal{A} = (A, f) \) with \( f \) a commutative idempotent binary operation such that all but finitely many of the identities \( f(x, p_k(x, y)) \approx p_k(x, y) \) with \( p \) a nontrivial binary term built out...
of $f$ and $k \geq k_p$ hold. In order to show that $\mathbb{A}$ is a spiral, we need to show that for any $a, b \in \mathbb{A}$, if there exist $c, d \in Sg_{\mathbb{A}}(a, b)$ with $c \neq b$ and $f(c, d) = b$, then $a \rightarrow b$. Note that since $c \neq b$ there is a nontrivial binary term $p(x, y)$ with $f(c, d) = p(a, b)$. From $p(a, b) = b$, we see that for all $k \geq 0$ we have $p_k(a, b) = b$, so taking $k$ sufficiently large we see that $f(a, b) = f(a, p_k(a, b)) = p_k(a, b) = b$, so $a \rightarrow b$. \hfill \Box

11. Examples generated by two elements

We start by giving a classification of minimal bounded width algebras of size four which are generated by two elements. The classification is summarized in Figure 3.

Lemma 13. Let $\mathbb{A} = \{(a, b, c, d), f, g\}$ be a minimal bounded width algebra with $f$ chosen as in Corollary 7 and suppose that $\mathbb{A}$ is generated by $a$ and $b$. Then $f(a, b) \neq f(b, a)$.

Proof. Suppose for contradiction that $f(a, b) \in \{a, b\}$. If $f(a, b) = b$ then $a \rightarrow b$, contradicting the assumption that $\mathbb{A}$ is generated by $a, b$. Similarly, we have $f(b, a) \neq a$, so we may assume without loss of generality that $f(a, b) = a$ and $f(b, a) = c$. In this case, $\{a, c\}$ is a majority algebra, so $f(a, c) = a, f(c, a) = c$. By Theorem 9 if $f(b, c)$ and $f(c, b)$ are both in $\{a, b, c\}$ then $\{a, b, c\}$ is a proper subalgebra of $\mathbb{A}$ containing $a, b$, contradicting the assumption that $a, b$ generate $\mathbb{A}$. Thus at least one of $f(b, c), f(c, b)$ must be $d$. In particular, $\{b, c\}$ is not a subalgebra of $\mathbb{A}$.

By our choice of $f, c = f(b, a)$ is reachable from $b$. The only way that this is possible is if we have $b \rightarrow d \rightarrow c$. If $f(b, c) = f(c, b) = d$, then $\{b, c, d\}$ is closed under $f$, and in fact by Lemma 8 this case is impossible. Thus $\{f(b, c), f(c, b)\}$ is a two element majority algebra which contains $d$, so it must be $\{a, d\}$. But then since $a$ is either $f(b, c)$ or $f(c, b)$, $a$ must be reachable from either $b$ or $c$. From $f(b, a) = f(c, a) = c$ and $f(d, a) = d$ we see that $a$ isn’t reachable from any of $b, c, d$, a contradiction. \hfill \Box

Lemma 14. Let $\mathbb{A} = \{(a, b, c, d), f, g\}$ be a minimal bounded width algebra with $f$ chosen as in Corollary 7 and suppose that $\mathbb{A}$ is generated by $a$ and $b$. Then $f(a, b) \neq f(b, a)$.

Proof. Suppose for contradiction that $f(a, b) = f(b, a) = c$. By our choice of $f, c$ must be reachable from both $a$ and $b$. If we have $a \rightarrow c$ and $b \rightarrow c$, then by Lemma 9 $\{a, b, c\}$ is a subalgebra of $\mathbb{A}$, a contradiction.

Suppose now that neither of $a \rightarrow c$ nor $b \rightarrow c$ holds. Then in order for $c$ to be reachable from $a$ and $b$, we must have $a \rightarrow d \rightarrow b \rightarrow d$, and $d \rightarrow c$. By Lemma 9, at least one of $f(a, c), f(c, a), f(b, c), f(c, b)$ must be $d$, and we may suppose without loss of generality that either $f(a, c)$ or $f(c, a)$ is $d$. Then $\{f(a, c), f(c, a)\}$ is a majority subalgebra containing $d$, and since every two element algebra containing $d$ is a semilattice, we must have $f(a, c) = f(c, a) = d$. However, this together with $a \rightarrow d \rightarrow c$ contradicts Lemma 8.

Thus exactly one of $a \rightarrow c, b \rightarrow c$ holds, and we may assume without loss of generality that $a \rightarrow c$. In order for $c$ to be reachable from $b$, we must then have $b \rightarrow d$ and at least one of $d \rightarrow a, d \rightarrow c$. Also, by Lemma 9, at least one of $f(b, c), f(c, b)$ must be $d$. If $f(b, c) = f(c, b) = d$, then $d$ is reachable from $c$, so $c \rightarrow d \rightarrow b \rightarrow d \rightarrow a \rightarrow c$, and we see in this case that if $f$ is commutative, so $\mathbb{A}$ is a minimal spiral and so by Theorem 12 $\{c, d\}$ has to strongly connected, which is impossible. Thus $\{f(b, c), f(c, b)\}$ must be a majority algebra containing $d$. If $\{f(b, c), f(c, b)\} = \{a, d\}$, then $a$ is reachable from one of $b, c$, which is impossible.

Thus $\{f(b, c), f(c, b)\} = \{c, d\}$, and by Lemma 7 $\{b, c, d\}$ is a subalgebra of $\mathbb{A}$ which is isomorphic to the subdirect product of a two element majority algebra and a two element semilattice. Also, since $c$ is reachable from $b$, we have $b \rightarrow d \rightarrow a \rightarrow c$. At this point we have completely determined $f$. In order to get a contradiction, we will show that $(a, a) \in Sg_{\mathbb{A}}\{(a, b), (b, a)\}$. We have $(c, c) = f((a, b), (b, a)), (c, d) = f((a, b), (c, c)), (d, a) = f((b, a), (c, d))$, and similarly
(a, d) ∈ Sg_{A^{2z}}{(a, b), (b, a)}. But then (a, a) = f((a, d), (d, a)) ∈ Sg_{A^{2z}}{(a, b), (b, a)}, so b → a, a contradiction.

Lemma 15. Let $A = \{(a, b, c, d), f, g\}$ be a minimal bounded width algebra with $f$ chosen as in Corollary 4 and suppose that $f(a, b) = c, f(b, a) = d$. Then $a \to c$ and $b \to d$.

Proof. Suppose for a contradiction that we do not have $a \to c$. Then in order for $c = f(a, b)$ to be reachable from $f$, we must have $a \to d \to b \to c$. In this case we do not have $b \to d$, so in order for $d = f(b, a)$ to be reachable from $b$ we must have $b \to c \to a \to d$. This completely pins down $f$. Note that by Lemma 9, the restriction of $g$ to $\{a, c, d\}$ and the restriction of $g$ to $\{b, c, d\}$ are determined by $f$.

Defining $\alpha(x, y, z) = (f(x, y), f(y, z), f(z, x))$, we have

\[
\begin{align*}
\alpha(a, b, c) &= (c, c, a), \\
\alpha(a, c, b) &= (a, c, d), \\
\alpha(a, a, b) &= (a, c, d),
\end{align*}
\]

so $g'(x, y, z) = g(\alpha(x, y, z))$ is completely determined by $f$. By minimality of $A$, we see that $g \in \text{Clo}(g')$, so in particular $A$ has $(a, b)(c, d)$ as an automorphism. Thus, we have

\[Sg_{A^{2z}}{(a, b), (b, a)} = \{(a, b), (b, a), (c, d), (d, c)\},\]

which is isomorphic to $A$. Since $A$ is strongly connected, this shows that $(a, b)$ is a maximal element of $Sg_{A^{2z}}{(a, b), (b, a)}$, so by Theorem 11 $(a, b)$ is a majority subalgebra of $A$, contradicting the assumption that $f(a, b) = c, f(b, a) = d$. \qed

Theorem 16. Every minimal bounded width algebra $A$ of size four which is generated by two elements is term equivalent to an algebra isomorphic to one of the three examples in Figure 3 and conversely each algebra in Figure 3 is a minimal bounded width algebra.

Proof. Write $A = \{(a, b, c, d), f, g\}$, with $f$ chosen as in Corollary 4, and suppose $A$ is generated by $a$ and $b$. By the previous results, we may assume without loss of generality that $f(a, b) = c, f(b, a) = d$, and that $a \to c, b \to d$.
Suppose first that \( a \in \{ f(b,c), f(c,b) \} \). Then \( \mathbb{A} \) is generated by \( b \) and \( c \), so by the previous results we have \( \{ f(b, c), f(c, b) \} = \{ a, d \} \) and either \( b \to a, c \to d \) or \( b \to d, c \to a \), and both possibilities are clearly impossible. Thus, \( \{ b, c, d \} \) is closed under \( f \), so by Lemma 9 \( \{ b, c, d \} \) is a three element subalgebra of \( \mathbb{A} \). Similarly, \( \{ a, c, d \} \) is also a three element subalgebra of \( \mathbb{A} \).

Since \( \{ b, c, d \} \) is a three element subalgebra of \( \mathbb{A} \) with \( \{ c, d \} \) a majority algebra and \( b \to d \), we either have \( b \to c, c \to b, \{ b, c \} \) a majority algebra, or \( f(b, c) = d, f(c, b) = c \). Similarly, there are four possible ways to assign values to \( f(a, d), f(d, a) \), so up to symmetry there are just \( 4^2 + 4 = 10 \) possible functions \( f \) to consider.

We briefly summarize why seven of these cases are not minimal bounded width algebras. If both \( c \to b \) and \( d \to a \), then \( \mathbb{A} \) is strongly connected and has an automorphism interchanging \( a \) and \( b \), so by Theorem 11 \( \{ a, b \} \) is a majority algebra. If \( f(b, c) = d, f(c, b) = c \), and \( d \to a \), then \( (a, a) \in S_{g_{b,d}} \{ (a, b), (b, a) \} \), so \( b \to a \). If \( f(b, c) = d, f(c, b) = c \), and \( \{ a, d \} \) is a majority algebra, then \( (a, d) \in S_{g_{b,c}} \{ (a, b), (b, a) \} \), so by Lemma 2 \( \{ a, b, d \} \) is a subdirect product of a two element majority algebra and a two element semilattice. In the four remaining cases which don’t work, the reader may check that we have either \( (c, c) \in S_{g_{b,c}} \{ (a, b), (b, a) \} \) with \( \{ b, c \} \) a subalgebra, or the analogous statement with \( d \) instead of \( c \), and we can apply Lemma 8 to see that either \( \{ a, b, c \} \) or \( \{ a, b, d \} \) is a free semilattice on the generators \( a, b \).

Now we check that the three algebras displayed in Figure 3 are minimal bounded width algebras. Call them \( \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \). \( \mathbb{A}_1 \) is isomorphic to \( \mathcal{F}_{\text{var}}(x, y) \), the free algebra on two generators in the variety generated by conservative bounded width algebras, so it is minimal. Due to the semilattice quotients of \( \mathbb{A}_2 \), we see that for any nontrivial binary term \( p(x, y) \), we must have \( p(a, b), p(b, a) \in \{ c, d \} \), and because \( \mathbb{A}_2 \) has the automorphism \( (a, b)(c, d) \), we have \( p(a, b) \neq p(b, a) \), so either \( p(x, y) \approx f(x, y) \) or \( p(x, y) \approx f(y, x) \). From here we easily see that for any terms \( f', g' \) of a bounded width reduct of \( \mathbb{A}_2 \), we have \( f' \approx f \) and \( g \approx g'(\alpha(x, y, z)) \).

We are left with checking that \( \mathbb{A}_3 \) is minimal. Suppose there are terms \( f', g' \) of a bounded width reduct of \( \mathbb{A}_3 \), with \( f' \) chosen as in Corollary 4. Since \( (a, b)(c, d) \) is an automorphism of \( \mathbb{A}_3 \), we either have \( f'(a, b) = a, f'(b, a) = b \) or \( f'(a, b) = c, f'(b, a) = d \) (we can’t have \( f'(a, b) = d \) since \( d \) is not reachable from \( a \)). If \( f'(a, b) = c, f'(b, a) = d \), then \( f' \approx f \) and \( g \approx g'(\alpha(x, y, z)) \), so we only have to prove that there is no term \( g' \) which acts as a majority operation on \( \{ a, b \} \). In other words, we must show that

\[
(a, a, a) \notin S_{g_{b,c}} \{ (a, a, b), (a, b, a), (b, a, a) \}.
\]

Let \( S \) be the set of triples \((u, v, w)\) such that at least two of \( u, v, w \) are in \( \{ a, c \} \), and such that if none of \( u, v, w \) is \( c \) then \((u, v, w)\) is a cyclic permutation of \((a, a, b)\). We claim that \( S \) is closed under \( g \) (in fact, it turns out that \( S = S_{g_{b,c}} \{ (a, a, b), (a, b, a), (b, a, a) \} \)). The key observation needed to check this claim is that if \( x, y, z \in \mathbb{A}_3 \) have at least two of \( x, y, z \) in \( \{ a, c \} \) and at least one of \( x, y, z \) equal to \( c \), then \( g(x, y, z) = c \). We leave the details to the reader. \( \qed \)

We also give two examples of larger algebras which are generated by two elements in Figure 4. For brevity, we only describe the function \( f \) and the subalgebras which are generated by pairs of elements.

12. Conjectures

**Conjecture 1.** In every bounded width algebra, there are terms \( w, s \) satisfying the identities

\[
w(x, x, y) \approx w(x, y, x) \approx w(y, x, x) \approx s(x, y)
\]

and

\[
s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y).
\]
By Theorem 13 Conjecture 1 holds in every spiral, and Conjecture 1 clearly holds in any majority algebra. Although it looks innocent, if true it would immediately imply Bulatov’s yellow connectivity property.

**Conjecture 2.** If \( A \) is a minimal bounded width algebra and \( a, b \in A \) have \((b, b) \in Sg_A^2\{(a, a), (a, b), (b, a)\}\), then \(a \rightarrow b\).

Conjecture 2 implies, in particular, that a minimal bounded width algebra \( A \) has no minority pairs, that is, pairs \(a \neq b \in A\) such that there is a ternary term \( p \) with

\[
p(a, a, b) = p(b, a, a) = b
\]

and

\[
p(a, b, b) = p(b, b, a) = a.
\]

Again, Conjecture 2 holds in every majority algebra and every spiral.

**Conjecture 3.** If \( A \) is a minimal bounded width algebra and \( a, b \in A \), then \((Sg\{a, b\}) \setminus \{a\}\) is a subalgebra of \( A \).

Conjecture 3 is an analogue of Theorem 12 for general bounded width algebras. If true, it shows that any minimal bounded width algebra which is generated by two elements \( a, b \) can be built out of the smaller minimal bounded width algebras \((Sg\{a, b\}) \setminus \{a\}\) and \((Sg\{a, b\}) \setminus \{b\}\), together with values for \( f(a, b), f(b, a) \), and \( g(x, y, z) \) where \(\{a, b\} \subset \{x, y, z\}\). Since an algebra has bounded width if and only if every subalgebra which is generated by two elements has bounded width, this would be an enormous help to the classification of minimal bounded width algebras of small cardinality.

**Conjecture 4.** A minimal bounded width algebra \( A \) is determined up to term equivalence by \( Inv_2(A) \), the collection of subalgebras of \( A \times A \).

A general algebra \( A \) is determined up to term equivalence by \( Inv(A) \), the collection of all subalgebras of powers of \( A \). Since any subpower of a majority algebra is determined by its projections onto pairs of coordinates, Conjecture 4 holds for majority algebras. \( Inv_2(A) \) contains a large amount of
the structural information of $A$: every automorphism of $A$ and every congruence of $A$ is a subalgebra of $A \times A$, and similarly we can read off the automorphism groups and congruence lattices of all quotients of subalgebras of $A$ from $\text{Inv}_2(A)$.

Since every minimal bounded width algebra has a ternary term as its basic operation, in order to describe $\text{Inv}_2(A)$ one only has to describe the collection of subalgebras of $A \times A$ which are generated by at most three elements. Thus, if true Conjecture 4 would give an efficient method to test whether two minimal bounded width algebras are term equivalent.

References


Appendix A. Strengthening of the Yellow Connectivity Property

The argument in this section follows Bulatov’s argument from [7]. Although the logical structure of the argument has been violently rearranged, the ideas used to prove the intermediate lemmas can all be found in Bulatov’s work.

Definition 14. Let $A_f = (A, f)$ be an idempotent algebra which has been prepared as in Lemma 2. We say that $A_f$ is yellow connected if for any pair of upwards closed subsets $A, B$ of $A_f$, there are $a \in A$ and $b \in B$ such that $f$ acts as first projection on $\{a, b\}$, that is, $f(a, b) = a$ and $f(b, a) = b$. We say that a pair of maximal elements $a, b$ of $A_f$ are yellow connected if there are $a', b'$ in the strongly connected components of $a, b$, respectively, such that $f$ acts as first projection on $\{a', b'\}$. We say that $A_f$ is hereditarily yellow connected if every subalgebra of $A_f$ is yellow connected.

We will show that if $A = (A, f, g)$ is a minimal bounded width algebra such that $f, g$ are chosen as in Theorem 4 then every subalgebra $B$ of $A_f = (A, f)$ is yellow connected. We will prove this...
Lemma 16. Let \( \mathbb{A} = (A, f, g) \) be a minimal bounded width algebra with \( f, g \) chosen as in Theorem 4, let \( \mathbb{A}_f = (A, f) \), and let \( \mathbb{B} = \text{Sg}_{\mathbb{A}_f}\{a, b\} \). Suppose that every proper subalgebra of \( \mathbb{B} \) is yellow connected, and that \( \mathbb{B} \) has a surjective homomorphism to a two element set. Then \( \mathbb{B} \) is yellow connected.

Proof. We may assume without loss of generality that \( (a, b) \) is a maximal element of \( \text{Sg}_{\mathbb{B}}\{(a, b), (b, a)\} \). Since \( \mathbb{B} \) has a homomorphism to a two element set and \( f \) satisfies the identity \( f(f(x, y), f(y, x)) \approx f(x, y) \), \( \mathbb{B} \) has a congruence \( \theta \) with two congruence classes \( a/\theta \) and \( b/\theta \) such that if \( x \in a/\theta \) and \( y \in b/\theta \), then \( f(x, y) \in a/\theta \) and \( f(y, x) \in b/\theta \). Let \( \mathbb{B}_2 = \text{Sg}_{\mathbb{B}}\{(a, b), (b, a)\} \), let \( a_2 = (a, b) \), and let \( b_2 = (b, a) \), and extend \( \theta \) to \( \mathbb{B}_2 \) in the natural way. Let \( \mathbb{R} \) be

\[
\text{Sg}_{\mathbb{A}_f}\left\{\begin{array}{ccc}
a & b \\
b & a & b \\
a & b & a
\end{array}\right\} \cap \mathbb{B}_2^3,
\]

so that

\[
\text{Sg}_{\mathbb{B}}\{(a_2, a_2, b_2), (a_2, b_2, a_2), (b_2, a_2, a_2)\} \subseteq \mathbb{R}
\]

and \( \mathbb{R} \) is closed under \( f \). Then

\[
(f(a_2, b_2), f(a_2, b_2), f(a_2, b_2)) \in \mathbb{R} \cap (a_2/\theta)^3,
\]

and since \( a_2/\theta \) is a yellow connected algebra, there is some maximal element \( (f_1, f_2, f_3) \in \mathbb{R} \cap (a_2/\theta)^3 \) which is reachable from \( (f(a_2, b_2), f(a_2, b_2), f(a_2, b_2)) \) and some \( a' \) in the strongly connected component of \( \mathbb{B}_2 \) containing \( a_2 \) such that \( f \) acts on \( \{a', f_1\} \) as first projection. Then since \( a', f_2, f_3 \) are maximal in \( \mathbb{B}_2 \), by Lemma 9 there is some \( u \in \mathbb{B}_2 \) and some \( v \in \mathbb{B}_2 \) such that \( (a', f_2, u) \in \mathbb{R} \) and \( (a', v, f_3) \in \mathbb{R} \). Thus,

\[
(a', f(f_2, v), f(f_3, u)) = (g(f_1, a', a'), g(f_2, f_2, v), g(f_3, u, f_3)) = g((f_1, f_2, f_3), (a', f_2, u), (a', v, f_3)) \in \mathbb{R} \cap (a_2/\theta)^3.
\]

Letting \( A \) be the strongly connected component of \( \mathbb{B}_2 \) which contains \( a_2 \), we see that there are strongly connected components \( C, D \) of \( \mathbb{B}_2 \) such that \( C \) is reachable from \( f(f_2, v) \) and \( D \) is reachable from \( f(f_3, u) \), with \( \mathbb{R} \cap A \times C \times D \neq \emptyset \). Note that we must then have \( A, C, D \subseteq a_2/\theta \). By Lemma 9 we have \( A \times C \times D \subseteq \mathbb{R} \). Since \( a_2/\theta \) is yellow connected, there are \( a'' \in A \) and \( d \in D \) such that \( f \) acts on \( \{a''\} \) as first projection. Letting \( c \) be any element of \( C \), we have \( (a'', c, d), (d, c, a'') ) \in \mathbb{R} \cap (a_2/\theta)^3 \). From \( (a_2, b_2, a_2) \in \mathbb{R} \) and Lemma 9 there is some \( v \in \mathbb{B}_2 \) with \( (a'', w, a'') \in \mathbb{R} \). Thus, we have

\[
(a'', f(c, w), a'') = (g(a'', d, a''), g(c, c, w), g(d, a'', a'')) = g((a'', c, d), (d, c, a''), (a'', w, a'')) \in \mathbb{R} \cap (a_2/\theta)^3.
\]

Applying Lemma 9 again, we have \( A \times C \times A \subseteq \mathbb{R} \cap (a_2/\theta)^3 \), and similarly \( C \times A \times A \subseteq \mathbb{R} \cap (a_2/\theta)^3 \). To finish, we argue as in Lemma 10 to see that \( (a_2, a_2, a_2) \in \mathbb{R} \), so by Theorem 7 \( \{a, b\} \) is a majority subalgebra of \( \mathbb{A} \).

Corollary 10. Let \( \mathbb{A} = (A, f, g) \) be a minimal bounded width algebra with \( f, g \) chosen as in Theorem 4, let \( \mathbb{A}_f = (A, f) \), and let \( \mathbb{B} = \text{Sg}_{\mathbb{A}_f}\{a, b\} \). Suppose that every proper subalgebra of \( \mathbb{B} \) is yellow connected, and that \( \mathbb{B} \) is not simple. Then \( \mathbb{B} \) is yellow connected.

Lemma 17. Let \( \mathbb{A} = (A, f, g) \) be a minimal bounded width algebra with \( f, g \) chosen as in Theorem 4, let \( \mathbb{A}_f = (A, f) \), and let \( \mathbb{B} = \text{Sg}_{\mathbb{A}_f}\{a, b\} \). Suppose that every proper subalgebra of \( \mathbb{B} \) is yellow connected, and that \( (a, b) \) is a maximal element of \( \text{Sg}_{\mathbb{B}}\{(a, b), (b, a)\} \). Suppose also that there is
Lemma 18. Let $A = (a,f,g)$ be a minimal bounded width algebra with $f,g$ chosen as in Theorem 4, let $A_f = (A,f)$, and let $B = Sg_{A_f}(a,b)$. Suppose that every proper subalgebra of $B$ is yellow connected, and that $(a,b)$ is a maximal element of $Sg_{B_2}\{a,b\}$. Suppose that there is some $e \in B$ such that $\{e\} \times B \subseteq Sg_{B_2}\{a,b\}$. Then $\{a,b\}$ is a majority subalgebra of $A$. \qed

Proof. It isn’t hard to see that if $e \to e'$ then $\{e'\} \times B \subseteq Sg_{B_2}\{a,b\}$, so we may assume without loss of generality that $e$ is maximal in $B$. If $b \in Sg_{B_2}\{a,e\}$, then $(b,b) \in Sg_{B_2}\{a,b\}$, so $a \to b$, which contradicts the assumption that $(a,b)$ is maximal. Thus we must have $Sg_{B_2}\{a,e\} \neq B$, and similarly $Sg_{B_2}\{b,e\} \neq B$, so both $Sg_{B_2}\{a,e\}, Sg_{B_2}\{b,e\}$ are yellow connected. Thus there are $e',e''$ in the strongly connected component of $e$ and $a',b'$ in the strongly connected components of $a,b$, respectively, such that $f$ acts on $\{a',e'\}$ and $\{b',e''\}$ as first projection. Now we can apply Lemma 17 with $(c,d) = (e',e'')$ and $(c',d') = (e'',e')$ to finish. \qed

Lemma 19. Let $A = (a,f,g)$ be a minimal bounded width algebra with $f,g$ chosen as in Theorem 4, let $A_f = (A,f)$, and let $B = Sg_{A_f}(a,b)$. Suppose that every proper subalgebra of $B$ is yellow connected, and that $(a,b)$ is a maximal element of $Sg_{B_2}\{a,b\}$. Suppose that there is some $e \in B$ and some $b'$ in the strongly connected component of $b$ such that $\{b',e\}$ is a majority subalgebra of $A$, and such that for all $d \in B$ at least one of $(b',d), (e,d)$ is an element of $Sg_{B_2}\{a,b\}$. Then $\{a,b\}$ is a majority subalgebra of $A$.

Proof. Let $E = Sg_{B_2}\{(a,b),(b,a)\} \cap \{(b',e) \times B\}$. By previous results, we may assume without loss of generality that $B$ is simple and has no homomorphism to $\{b',e\}$, so $E$ is linked when considered as a subdirect product of $\{b',e\}$ and $B$. By the Lemma 18, we may assume that $\{e'\} \times B \not\subseteq E$. Let $A$ be the strongly connected component of $a$ and let $B$ be the strongly connected component of $b$. By Proposition 8, if $(b',b') \in E$, then $\{b'\} \times B \not\subseteq E$, from which we easily conclude that $(b,b) \in Sg_{B_2}\{a,b\}$, so $a \to b$, a contradiction. Thus we must have $\{e,b'\} \in E$. Since there is some $a' \in A$ with $(b',a') \in E$, by Proposition 8 we have $\{b'\} \times A \subseteq E$, from which we can conclude that $B \times A \subseteq Sg_{B_2}\{a,b\}$. Also, by Proposition 8 we have $\{e\} \times B \subseteq E$.

Since $E$ is linked, there is some $d \in B$ such that both $\{e,d\}$ and $\{b',d\}$ are in $E$. By Proposition 8 we may assume without loss of generality that $d$ is a maximal element of $B$. Let $D$ be the strongly connected component containing $d$, and let $E$ be a maximal strongly connected component which is reachable from $e$. By Proposition 8, $\{e'\} \times D \subseteq E$, so $E \times D \subseteq Sg_{B_2}\{(a,b),(b,a)\}$. Since $\{b'\} \times (A \cup D) \subseteq Sg_{B_2}\{(a,b),(b,a)\}$, $A \cup D$ generate a proper subalgebra of $B$, so there are $a' \in A$ and $d' \in D$ with $\{a',d'\}$ a majority subalgebra of $A$. Similarly, since $\{e\} \times (B \cup D) \subseteq Sg_{B_2}\{(a,b),(b,a)\}$, there are $b'' \in B, d'' \in D$ with $\{b'',d''\}$ a majority subalgebra of $A$. Also, by Proposition 8...
Let \( A = (f, g) \) be a minimal bounded width algebra with \( f, g \) chosen as in Theorem 17. Let \( A = (f, g) \), and let \( B = S_{\mathcal{A}_f} \{a, b\} \). Suppose that every proper subalgebra of \( B \) is yellow connected, and that \( (a, b) \) is a maximal element of \( S_{\mathcal{A}_f} \{a, b\} \). If \( S_{\mathcal{A}_f} \{a, b\} \) is linked, then there are \( a', b' \) in the strongly connected components of \( a, b \), respectively, such that \( \{a', b'\} \) is a major subalgebra of \( A \).

Proof. Since \( S_{\mathcal{A}_f} \{a, b\} \) is linked, there is a sequence \( a = d_1, e_1, d_2, e_2, ..., d_k, e_k, d_{k+1} = b' \) with \( (d_i, e_i), (d_{i+1}, e_i) \in S_{\mathcal{A}_f} \{a, b\} \), with each \( d_i, e_i \) a maximal element of \( B \) and with \( b' \) in the strongly connected component of \( b \). Let \( D_i \) be the strongly connected component containing \( d_i \), and let \( E_i \) be the strongly connected component containing \( e_i \), and let \( A = D_1 \) be the strongly connected component containing \( a \), and \( B = D_{k+1} \) be the strongly connected component containing \( b \). We can assume without loss of generality that \( S_{\mathcal{A}_f} \{a', b'\} \subseteq B \) for any \( a' \in A \), \( b' \in B \), and we have \( S_{\mathcal{A}_f} \{a', b'\} = \{a\} \). Then there is either some \( i \) such that for all \( a' \in A \), \( d'_{i+1} \in D_i+1 \) we have \( S_{\mathcal{A}_f} \{a', d'_{i+1}\} = \mathbb{B} \) and for some \( b' \in B \), \( e'_i \in E_i \) we have \( S_{\mathcal{A}_f} \{b', e'_i\} \neq \mathbb{B} \), or else there is some \( i \) such that for all \( b' \in B \), \( e'_i \in E_i \) we have \( S_{\mathcal{A}_f} \{b', e'_i\} = \mathbb{B} \) and for some \( a' \in A \), \( d'_{i} \in D_i \) we have \( S_{\mathcal{A}_f} \{a', d'_i\} \neq \mathbb{B} \). Assume without loss of generality that there is an \( i \) such that for all \( a' \in A \), \( d'_{i+1} \in D_i+1 \) we have \( S_{\mathcal{A}_f} \{a', d'_{i+1}\} = \mathbb{B} \) and for some \( b' \in B \), \( e'_i \in E_i \) we have \( S_{\mathcal{A}_f} \{b', e'_i\} \neq \mathbb{B} \). Since \( b', e'_i \) generate a proper subalgebra of \( B \), we may assume without loss of generality that \( \{b', e'_i\} \) is a major subalgebra of \( A \). Then there is some \( a' \in A \) such that \( (b', a') \in S_{\mathcal{A}_f} \{a', b, (a, b)\} \) and there is a \( d'_{i+1} \in D_i+1 \) such that \( (e'_i, d'_{i+1}) \in S_{\mathcal{A}_f} \{a, b, (a, b)\} \). Then \( E = S_{\mathcal{A}_f} \{a, b, (a, b)\} \cap \{b', e'_i\} \times B \) has \( S_{\mathcal{A}_f} \{a', d'_{i+1}\} \subseteq \pi_2 E \), so \( \pi_2 E = B \) and we can finish by using the Lemma 19.

Lemma 21. Let \( A = (f, g) \) be a minimal bounded width algebra with \( f, g \) chosen as in Theorem 17. Let \( A = (f, g) \), and let \( B = S_{\mathcal{A}_f} \{a, b\} \). Suppose that every proper subalgebra of \( B \) is yellow connected, and that \( a, b \) are maximal elements of \( B \). If there is some maximal element \( c \in \mathbb{B} \) such that \( a \) and \( c \) are yellow connected and such that \( c \) and \( b \) are yellow connected, then \( a \) and \( b \) are yellow connected.

Proof. We may assume without loss of generality that \( (a, b) \) is maximal in \( S_{\mathcal{A}_f} \{a, b\} \). By Corollary 10 and Lemma 20, we may assume that \( B \) is simple, and that \( S_{\mathcal{A}_f} \{a, b\} \) is not linked, so \( S_{\mathcal{A}_f} \{a, b\} \) is the graph of an automorphism which interchanges \( a \) and \( b \). Thus there is a maximal \( d \in B \) with \( (c, d) \in S_{\mathcal{A}_f} \{a, b\} \) such that \( (a, b) \) is yellow connected to \( (c, d) \) and \( (c, d) \) is yellow connected to \( (a, b) \), so we are done by Lemma 17.

Lemma 22. Let \( A = (f, g) \) be a minimal bounded width algebra with \( f, g \) chosen as in Theorem 17. Let \( A = (f, g) \), and let \( B = S_{\mathcal{A}_f} \{a, b\} \). Suppose that every proper subalgebra of \( B \) is yellow connected, and that \( (a, b) \) is a maximal element of \( S_{\mathcal{A}_f} \{a, b\} \). If there is some element \( c \in \mathbb{B} \) such that \( c \rightarrow a \) and \( b \) is reachable from \( c \), then \( \{a, b\} \) is a majority subalgebra of \( A \).

Proof. By Corollary 10 and Lemma 20, we may assume that \( B \) is simple, and that \( S_{\mathcal{A}_f} \{a, b\} \) is not linked, so \( S_{\mathcal{A}_f} \{a, b, (a, b)\} \) is the graph of an automorphism which interchanges \( a \) and \( b \). Thus there is a \( d \in B \) with \( (c, d) \in S_{\mathcal{A}_f} \{a, b\} \) such that \( (c, d) \rightarrow (a, b) \). Let \( R = S_{\mathcal{A}_f} \{a, b\} \cap B^3 \). Since \( c \in S_{\mathcal{A}_f} \{a, b\} \), we have \( (a, c, d), (c, a, d) \in R \). Applying \( f \) to \( (a, c, d), (c, a, d) \), we see that \( (a, a, d) \in R \). Since \( b \) is reachable from \( c \), \( a \) is reachable from \( d \), so if \( A \) is the strongly connected component of \( B \) containing \( a \), then \( R \cap A^3 \neq \emptyset \). Thus \( (a, a, a) \in R \) by Lemma 9 and so by Theorem 7 we are done.
Lemma 23. Let $\mathcal{A} = (A, f, g)$ be a minimal bounded width algebra with $f, g$ chosen as in Theorem 4, let $\mathcal{A}_f = (A, f)$, and let $\mathcal{B} = Sg_{\mathcal{A}_f} \{a, b\}$. Suppose that every proper subalgebra of $\mathcal{B}$ is yellow connected, and that $(a, b)$ is a maximal element of $Sg_{\mathcal{B}} \{(a, b), (b, a)\}$. If there is some element $c \in \mathcal{B}$ such that $c \to a$ and $c$ is not maximal in $\mathcal{B}$, then $a, b$ are yellow connected.

Proof. If $Sg_{\mathcal{B}} \{b, c\} \neq \mathcal{B}$ then we are done by Lemmas 21 and 22. Otherwise, since there is clearly no automorphism of $\mathcal{B}$ which interchanges $b$ and $c$, $Sg_{\mathcal{B}} \{(b, c), (c, b)\}$ must be linked. Thus there is a sequence $a = a_1, a_2, ..., a_k$ with $a_k$ in the strongly connected component of $b$ and with each $a_i$ maximal, such that for each $i$ there is a maximal $c_i$ with $(a_i, e_i), (a_{i+1}, e_i) \in Sg_{\mathcal{B}} \{(b, c), (c, b)\}$. If $Sg_{\mathcal{B}} \{a_i, a_{i+1}\} \neq \mathcal{B}$ for all $i$, then we easily finish using Lemma 19. Thus we may assume that there is some maximal $e$ such that $\{e\} \times \mathcal{B} \subseteq Sg_{\mathcal{B}} \{(b, c), (c, b)\}$. If $Sg_{\mathcal{B}} \{b, e\} = \mathcal{B}$, then since $(b, c), (e, c)$ are in $Sg_{\mathcal{B}} \{(b, c), (c, b)\}$, we have $(c, c) \in Sg_{\mathcal{B}} \{(b, c), (c, b)\}$, so $b \to c$, a contradiction. Similarly, if $Sg_{\mathcal{B}} \{c, e\} = \mathcal{B}$ then $c \to b$ and we finish using Lemma 22. Thus we may assume that $Sg_{\mathcal{B}} \{b, e\}, Sg_{\mathcal{B}} \{c, e\}$ are proper subalgebras of $\mathcal{B}$. Thus $b, e$ are yellow connected. Similarly, there is some $c'$ which is reachable from $c$ and some $e'$ in the strongly connected component of $e$ such that $(c', e')$ is a majority subalgebra of $\mathcal{A}$. Letting $c''$ be a maximal element of $\mathcal{B}$ which is reachable from $c'$, we can use Lemmas 17, 20, 21, and 22 to see that $a, c''$ are yellow connected and that $c'', e'$ are yellow connected, so $a, b$ are yellow connected.

Theorem 17. Let $\mathcal{A} = (A, f, g)$ be a minimal bounded width algebra with $f, g$ chosen as in Theorem 4, and let $\mathcal{A}_f = (A, f)$. Then $\mathcal{A}_f$ is hereditarily yellow connected.

Proof. Suppose for a contradiction that there is a subalgebra $\mathcal{B} \leq \mathcal{A}_f$ which is not yellow connected, and take such a $\mathcal{B}$ of minimal size. Then for any pair of maximal elements $a, b \in \mathcal{B}$ which are not yellow connected we must have $\mathcal{B} = Sg_{\mathcal{A}_f} \{a, b\}$. By Corollary 10, $\mathcal{B}$ must be simple. By Theorem 4, there is a sequence $a = p_1, p_2, ..., p_n = b$ of elements of $\mathcal{B}$ such that $\{p_i, p_{i+1}\}$ is a two element subalgebra of $\mathcal{B}$ for each $i$. For each $i$, let $p_i'$ be a maximal element of $\mathcal{B}$ which is reachable from $p_i$, with $p_i' = p_i$ if $p_i$ is already maximal.

By Lemma 21 there must be some $i$ such that $p_i'$ is not yellow connected to $p_{i+1}'$. Pick $p_i''$, $p_i'''$ which are reachable from $p_i', p_{i+1}'$, respectively, such that $(p_i'', p_{i+1}''')$ is maximal in $Sg_{\mathcal{B}} \{(p_i'', p_{i+1}'''), (p_i'''', p_{i+1}''')\}$. By Lemma 20 there is some automorphism $\iota$ of $\mathcal{B}$ which interchanges $p_i''$ and $p_i'''$. Then since $p_i''$ is reachable from $p_i'''$, $\iota(p_i''')$ must be reachable from $\iota(p_i''') = p_{i+1}'''$, and $(p_{i+1}, \iota(p_{i+1}'))$ is thus maximal in $Sg_{\mathcal{B}} \{(p_{i+1}, \iota(p_{i+1}')), (p_{i+1}', \iota(p_{i+1}'))\} = Sg_{\mathcal{B}} \{(p_i, \iota(p_i)), (\iota(p_i'), p_{i+1}')\}$. Therefore we may apply Lemma 23 to see that if $p_i = p_i'$ then $p_i'$ is yellow connected to $p_{i+1}'$. Thus we must have $p_i' = p_i$, and similarly $p_{i+1}' = p_{i+1}$, so $\{p_i', p_{i+1}'\}$ is a two element subalgebra of $\mathcal{B}$, contradicting our assumption that $p_i'$ and $p_{i+1}'$ are not yellow connected.

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