WRINKLING OF SMOOTH MAPPINGS-II

Wrinkling of embeddings and K. Igusa's theorem

Y. M. Eliashberg[*]
Stanford University,
Stanford, CA 94305 USA

N. M. Mishachev[†]
Lipetsk Technical University,
Lipetsk, 398055 Russia

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0 Introduction

This paper was promised to Jean Cerf almost ten years ago. We are happy to finally fulfill our promise.

This is the second paper in our series devoted to applications of the wrinkling process. This method, which was described in the first paper (see [EM]), allows us to construct families of mappings with simple singularities. As an application of this method we prove in the present paper an h-principle for functions without higher singularities (i. e. functions with only Morse and death-birth type singularities) or so-called *generalized Morse functions*. This result extends theorems of K. Igusa (see [Ig1]) and V. Vassiliev (see [Va]).

Let us say here a few words about the strategy of our proof. The singular locus of a generic family of functions $f_t : M \to \mathbb{R}$, $t \in Q$, is given on an n-dimensional manifold M parametrized by a k-dimensional manifold Q is a k-dimensional submanifold V of the product $W = M \times Q$. The higher singularities of this family coincide with the singularities of the projection $V \to Q$. The case of a family of functions with only Morse and death-birth type singularities corresponds to the case when the projection $V \to Q$ has only folds. Thus our first goal is to deform V via a C^0-small isotopy in W in order to get rid of higher singularities of the projection $V \to Q$. This is done in §1 using the results from [E]. Next we need to "integrate" this deformation to realize it by a local deformation of the original family of functions f_t in a neighborhood of the singularity V. The formal data necessary for this step are provided by a section of the 3-jet bundle $J^3(M)$. Thus the deformation of V should be done respecting these data which we incorporate in the notion of an *admissible splitting* defined in Section 1.4. Finally we extend the local deformation to the whole W using the Wrinkling Theorem which is the main result our paper [EM].

The paper has the following organization. In §1 we formulate and prove theorems about the wrinkling of individual embeddings. In §2 we apply these results together with the main result from [EM] to prove theorems about (fibered) mappings with moderate singularities. As applications we prove a theorem about the space of embeddings of closed k-dimensional manifolds into $\mathbb{R}^{k+1}$ (see Section 2.7) and a h-principle for functions without higher singularities (see Section 2.8), i.e. a generalized K. Igusa's theorem.

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1 Wrinkling of embeddings

1.1 Terminology and notation

Let W and Q be manifolds of dimension $n+k$ and k, respectively, and $p : W \to Q$ be a submersion. We denote by $\mathcal{F}$ the foliation formed by preimages $p^{-1}(q), q \in Q$, and by Vert, the vector subbundle of the tangent bundle TW formed by tangent spaces to the foliation $\mathcal{F}$.

Let $V \hookrightarrow W$ be a k-dimensional submanifold of W. We are going to deform it by an isotopy in order to simplify the singularities of its projection to Q. The restriction of the bundle Vert to V will be denoted by $\mathrm{Vert}(V)$. More generally, we denote by $\mathrm{Vert}(A)$ the restriction of Vert to any subset $A \subset W$. The normal bundle to V in W will be denoted by $\nu(V)$ (we fix a Riemannian metric on W). Notice that both bundles $\mathrm{Vert}(V)$ and $\nu(V)$ are n-dimensional. Let K be the projection $\mathrm{Vert}(V) \to \nu(V)$ along $T(V)$. Then K is a fiberwise isomorphism over $V \setminus \Sigma(p|_V)$ where $\Sigma(f)$ denotes the set of singular points of the map f. We will call K the *canonical homomorphism.*

Given an isotopy $h_t : W \to W$, $t \in [0,1]$, with $h_0 = \mathrm{Id}$ we denote the moving submanifold $h_t(V)$ by V_t.

1.2 Wrinkling of open submanifolds

Theorem 1.2A. *Let $L \subset V$ be a submanifold of codimension ≥ 1. Suppose that there exists an isomorphism $F : \mathrm{Vert}(L) \to \nu(V)|_L$. Then there exists a C^0-small isotopy $h_t : W \to W$, $t \in [0,1]$, with $h_0 = \mathrm{Id}$ and fixed outside of a neighborhood of L such that $p \circ h_1|_V$ is an immersion near L. Moreover, $h_t|_L$ can be covered by a family of isomorphisms F_t : $\mathrm{Vert}(L_t) \to \nu(V_t)|_L$ such that $F_0 = F$ and $F_1 = K|_{L_1}$ where $K : \mathrm{Vert}(V_1) \to \nu(V_1)$ is the canonical homomorphism.*

This proposition is a corollary of Gromov's theorem about directed embeddings (see [Gr], Section 2.4.5). He proves it by the method of convex integration. However, we will give here an independent proof as a corollary of our general results (see section 1.8 below). Proposition 1.2A is also true in the relative form, i.e. as an extension result.

1.3 Folded maps

If $V \hookrightarrow W$ is a closed submanifold then one cannot hope, in general, to deform V into V_1 such that $p|_{V_1}$ is an immersion. Hence our goal should be to get a map with the simplest possible, i.e. *fold* singularities.

Let us recall that a point $p \in V$ is a called a *fold* point of a map $f : V \to Q$ between two equidimensional manifolds, if the map f can be given by the formula

$$f(x_1, x_2, \ldots, x_n) = (x_1^2, x_2, \ldots, x_n)$$

in local coordinates centered at points p and $f(p)$, respectively. A map $f : M \to Q$ is called *folded* if it has only fold type singularities. In a neighborhood U of the *fold* $\Sigma = \Sigma(f)$, i.e. the singularity of f, there exists an involution $i : U \to U$, fixed on Σ, such that $f \circ i = f$.

If $p|_V : V \to Q$ is a folded map, then the line bundle $\mathrm{Ker} = \mathrm{Ker} d(p|_V) \subset \mathrm{Vert}(V)$ on the fold $\Sigma = \Sigma(p|_V) \subset V$ is tangent to V and transversal to Σ. If the fold $\Sigma \subset V$ is coorientable, then a choice of a coorientation of Σ in V is equivalent to a choice of a trivialization of the line bundle Ker. Let us denote by ν_{ext} vector field defined on the fold Σ which is normal to $T(V)|_\Sigma + \mathrm{Vert}(\Sigma)$ in $T(W)$. Notice that the normal bundle to $T(V)|_\Sigma + \mathrm{Vert}(\Sigma)$ in $T(W)$ is 1-dimensional. We choose the direction of ν_{ext} so that it is projected by dp to an outside normal to the fold.

1.4 Admissible splittings of the bundle $\mathrm{Vert}(V)$.

Let $V \hookrightarrow W$ and $\Sigma \subset V$ be a *cooriented* codimension one submanifold of V. We do not assume here that $p|_V$ is a folded map or that Σ is the singular locus of $p|_V$. Let U be a tubular neighborhood of Σ in V which is split into the product $\Sigma \times [-1, 1]$ according to the coorientation of Σ. We set $U_+ = \Sigma \times (0, 1]$ and $U_- = \Sigma \times [-1, 0)$.

Suppose we are given:

- a trivial (and trivialized) line bundle $\lambda \subset \mathrm{Vert}(\Sigma)$ and
- a splitting

  $$\mathrm{Vert}(V \setminus \Sigma) = \mathrm{Vert}_+ \oplus \mathrm{Vert}_-$$

  (dimensions of Vert_+ and Vert_- vary over different components of $V \setminus \Sigma$)

so that over the tubular neighborhood $U = \Sigma \times [-1, 1]$ the bundle Vert splits into the Whitney sum $\mathrm{Vert} = \mathrm{Ver}_+ \oplus \mathrm{Ver}_- \oplus \bar{\lambda}$, such that $\bar{\lambda}|_\Sigma = \lambda$ and

$$\mathrm{Vert}_+ = \begin{cases} \mathrm{Ver}_+ \oplus \bar{\lambda} & \text{on} \quad U_+ \\ \mathrm{Ver}_+ \quad \text{on} & U_-; \end{cases}$$

$$\mathrm{Vert}_- = \begin{cases} \mathrm{Ver}_- \quad \text{on} & U_+ \\ \mathrm{Ver}_+ \oplus \bar{\lambda} & \text{on} \quad U_-. \end{cases}$$

We say in this case that the data $\mathcal{L} = (\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ define an *admissible splitting* of the bundle $\mathrm{Vert}(V)$. Notice that according to the definition of admissible splitting the dimension of Vert_+ increases when we go through the Σ in the direction defined by the coorientation.

An isotopy $h_t : W \to W, h_0 = \mathrm{Id}$, can be canonically covered by a deformation of the splitting over the moving submanifold $V_t = h_t(V)$. Hence we can talk about the same splitting over isotopic submanifolds provided that the isotopy is given.

Given two admissible splittings $(\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ and $(\Sigma', \lambda', \mathrm{Vert}'_+, \mathrm{Vert}'_-)$ over the same manifold V we say that the second splitting is *subordinate* to the first one if

- $\Sigma' \supset \Sigma$;
- $\lambda'|_\Sigma = \lambda|_\Sigma$;
- for any component S of $V \setminus \Sigma'$ whose closure $\overline{S}$ in V intersects Σ we have $\mathrm{Vert}'_\pm|_S = \mathrm{Vert}_\pm|_S$;
- $\Sigma' \setminus \Sigma$ bounds in $V \setminus \Sigma$ a domain U such that for each component U' of U we have either $\mathrm{Vert}'_+ = \mathrm{Vert}_+$ and the trivialization of the line bundle $\lambda'|_{\partial U'}$ extends to U' as a

non-vanishing vector field in Vert_-, or $\mathrm{Vert}'_- = \mathrm{Vert}_-$ and the trivialization of the line bundle $\lambda'|_{\partial U'}$ extends to U' as a non-vanishing vector field in Vert_+.

Notice that *any splitting* $(\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ *is concordant to its subordinate splitting* $(\Sigma', \lambda', \mathrm{Vert}'_+, \mathrm{Vert}'_-)$, i.e there exists a splitting $(\widetilde{\Sigma}, \tilde{\lambda}, \widetilde{\mathrm{Vert}}_+, \widetilde{\mathrm{Vert}}_-)$ of the bundle $\mathrm{Vert} \times I$ over $V \times I$, which coincides with $(\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ over $V \times 0$, and with $(\Sigma', \lambda', \mathrm{Vert}'_+, \mathrm{Vert}'_-)$ over $V \times 1$.

Suppose now that $p|_V$ is a folded map and the fold $\Sigma = \Sigma(d(p|_V) \subset V$ is cooriented. We say that the splitting $(\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ *respects the fold* of V if $\lambda = \mathrm{Ker} d(p|_V)$ and the following orientation condition is satisfied: the trivialization of the line bundle λ defines the original coorientation of $\Sigma \subset V$.

1.5 Regularization of the canonical homomorphism

Suppose that $p|_V$ is a folded map. Let $\mathcal{L} = (\Sigma, \mathrm{Ker}, \mathrm{Vert}_+, \mathrm{Vert}_-)$ be an admissible splitting which respects the fold. We want to associate with $\mathcal{L}$ an *isomorphism*

$$\mathcal{R}K = \mathcal{R}_{\mathcal{L}}K : \mathrm{Vert}(V) \to \nu(V)$$

which will be called the *regularization* of the canonical homomorphism K according to the admissible splitting $\mathcal{L}$.

First we define $\mathcal{R}K$ over $V \setminus U$:

$$\mathcal{R}K = \begin{cases} K & \text{on } \mathrm{Vert}_+ \\ -K & \text{on } \mathrm{Vert}_-. \end{cases}$$

Then we set over $\Sigma \times t \subset U = \Sigma \times [-1, 1]$

$$\mathcal{R}K = \begin{cases} K & \text{on } \mathrm{Ver}_+ \\ -K & \text{on } \mathrm{Ver}_- \end{cases}$$

Finally we observe that $\mathcal{R}K$ can be extended to $\bar{\lambda}$ in a homotopically unique way in order to get a globally defined isomorphism $\mathrm{Vert} \to \nu$.

1.6 Wrinkling of closed submanifolds

Theorem 1.6A. *Let $\mathcal{L}$ be an admissible splitting. Suppose there exists an isomorphism*

$$F : \mathrm{Vert}(V) \to \nu(V)\,.$$

Then there exists a splitting $\mathcal{L}' = (\Sigma', \lambda', \mathrm{Vert}'_+, \mathrm{Vert}'_-)$ which is subordinate to $\mathcal{L}$, a C^0-small isotopy $h_t : W \to W$, $t \in [0,1]$, with $h_0 = \mathrm{Id}$, and a covering family of isomorphisms $F_t : \mathrm{Vert}(V_t) \to \nu(V_t)$ such that $F_0 = F$, $p|_{V_1}$ is a folded map, the splitting $\mathcal{L}'$ respects the fold of $p|_{V_1}$ and the isomorphism F_1 coincides with the regularization

$$\mathcal{R}_{\mathcal{L}'}K : \mathrm{Vert}(V_1) \to \nu(V_1).$$

This theorem also holds in the relative setting:

Theorem 1.6B. *Let F and $\mathcal{L}$ be as in Section 1.6A. Suppose that*

- *$p|_V$ is a folded map in a neighborhood Ω of a closed subset $A \subset V$;*
- *the splitting $\mathcal{L}|_\Omega$ already respects the fold and*
- *the isomorphism F coincides over Ω with the regularization $\mathcal{R}_{\mathcal{L}}K$ of the canonical homomorphism $K|_{\mathrm{Vert}(\Omega)}$.*

Then the splitting $\mathcal{L}'$ and the families h_t and F_t, $\mathcal{L}'_t$, $t \in [0,1]$, as in Theorem 1.6A, can be chosen in such a way that $\mathcal{L}' = \mathcal{L}$ near A and the deformations h_t and F_t are fixed on A.

Corollary 1.6C. *Suppose there exist an isomorphism*

$$F : \mathrm{Vert}(V) \to \nu(V)\,.$$

Then there exist a C^0-small isotopy $h_t : W \to W$, $t \in [0,1]$, with $h_0 = \mathrm{Id}$, such that $p|_{V_1}$ is a folded map.

Proof. Take $\mathcal{L} = (\emptyset, \emptyset, \mathrm{Vert}(V), \{0\})$ and apply Theorem 1.6A. ■

Remark 1.6D. The existence of a *stable* isomorphism $F : \mathrm{Vert}(V) \oplus \theta \to \nu(V) \oplus \theta$, where θ is a trivial line bundle over V is the necessary and sufficient condition for the existence of an isotopy $h_t : W \to W$ such that $p|_{V_1}$ is a folded map. The necessity is straightforward while the sufficiency can be deduced from 1.6A. However, we do not need this result for what follows.

1.7 The codimension one case

As the first step for proving Theorems 1.2A, 1.6A and 1.6B we consider here the case of submanifolds of codimension 1.

Let W and Q be manifolds of dimension $(k+1)$ and k, respectively, $p : W \to Q$ a submersion and $V, V \subset W$, a cooriented submanifold of codimension 1.

We assume that the one-dimensional bundle $\mathrm{Vert}(V)$ is trivialized by a vector field *vert* and that the non-vanishing normal vector field *norm* on V is chosen according to the coorientation of V.

A point $x \in V$ is called *positive* if it is non-singular for the projection $p|_V$ and if the coorientations defined at that point by the vectors *vert* and *norm* coincide.

Let Σ_+ and Σ_- be two disjoint, codimension one submanifolds of V. Suppose that the manifold $\Sigma = \Sigma_+ \cup \Sigma_-$ divides V into two parts V_+ and V_- so that $V = V_+ \cup V_-$ and $V_+ \cap V_- = \Sigma$. Let U, $U = \Sigma \times [-\varepsilon, \varepsilon]$, be a tubular neighborhood of Σ in V. Choose a vector field ν on Σ, normal to Σ in V, and pointed inside V_+ on Σ_+ and inside V_- on Σ_-. We then define a vector field

$$norm(\Sigma_+, \Sigma_-) = \begin{cases} norm & \text{on } V_+ \setminus U \\ -norm & \text{on } V_- \setminus U \,. \end{cases}$$

Along each fiber $\sigma \times [-\varepsilon, \varepsilon] \subset \Sigma \times [-\varepsilon, \varepsilon] = U$ the field $norm(\Sigma_+, \Sigma_-)$ rotates counterclockwise from $norm(\sigma)$ to $-norm(\sigma)$ in the plane generated by $(norm(\sigma), \nu(\sigma))$ according to the orientation defined by this basis.

Example 1.7A. Suppose that $p|_V$ is a folded map and $\Sigma = \Sigma(p|_V)$ is the fold. Let ν_{ext} be the vector field along Σ, normal to V in W, which was defined in Section 1.3 above. Let us divide Σ into two disjoint parts Σ_+ and Σ_- as follows. We say that a point $\sigma \in \Sigma$ belongs to Σ_+ if vectors $norm(\sigma)$ and $\nu_{ext}(\sigma)$ have the same directions, and to Σ_- otherwise. Σ divides V into two parts V_+ and V_-, so that $V = V_+ \cup V_-$ and $V_+ \cap V_- = \Sigma$. Notice that on V_+ the fields *norm* and *vert* define the same coorientation of V while on V_- the coorientations are opposite. One can check that *the vector fields* $norm(\Sigma_+,\Sigma_-)$ *and* $vert$ *are homotopic as sections of* $T(W)$.

The following Theorem 1.7B rephrases Theorem 4.4 from [E]. Let us note that V.I. Arnold provided in [Ar2] a proof of a lemma (see Lemma 3.2.B in [E]) about normal forms of certain singularities, which was formulated in [E] without a proof.

Theorem 1.7B. *Suppose that the manifolds* Σ_+ *and* Σ_- *are non-empty and that the vector fields* $norm(\Sigma_+,\Sigma_-)$ *and* $vert$ *are homotopic. Then there exists a* C^0*-small isotopy* $h_t : W \to W$ *and an isotopy* $g_t : V \to V, t \in [0,1]$, *such that* $p|_{V_1=h_1(V)}$ *is a folded map with* $\Sigma(p|_{V_1}) = \widetilde{\Sigma_+} \cup \widetilde{\Sigma_-}$ *where* $\widetilde{\Sigma_\pm} = h_1(g_1(\Sigma_\pm))$. *Moreover,* $\widetilde{\Sigma_+}$ *consists of points where directions of the vector fields* $norm$ *and* ν_{ext} *coincide and* $\widetilde{\Sigma_-}$ *consists of points where the directions are opposite. If the map* $p|_V$ *is an immersion in a neighborhood of a closed subset* $A \subset V \setminus (\Sigma_+ \cup \Sigma_-)$, Σ_+ *and* Σ_- *have non-empty intersections with each component of* $V \setminus A$, *and the vector fields* $norm(\Sigma_+,\Sigma_-)$ *and* $vert$ *are homotopic* rel A, *then the isotopy* h_t *can be chosen fixed near* A.

Notice that Example 1.7A shows that the conditions of Theorem 1.7B (except the non-emptiness of Σ_+ and Σ_-) are necessary for the existence of a deformation of V such that $p|_{V_1}$ is a folded map with the prescribed fold and the prescribed signs of the components of the fold.

As a corollary of Theorem 1.7B we get

Corollary 1.7C. *Let* $D \subset W$ *be an embedded* k*-disc, positive near* ∂D. *Then there exists a* C^0*-small isotopy* $h_t : D \to D, t \in [0,1]$, *fixed near* ∂D *and such that* $p|_{D_1}$ *is a folded map.*

Indeed, according to Theorem 1.7B we just need to arrange that the field $norm(\Sigma_+,\Sigma_-)$ is homotopic (rel. ∂D) to the field $vert$. It is sufficient to observe that one can arbitrarily change the homotopy class of the field $norm(\Sigma_+,\Sigma_-)$ by choosing Σ_+ and Σ_- consisting of certain numbers of disjoint $(k-1)$-spheres which bound k-discs in V. In the case when $norm$ and $vert$ are homotopic, no preliminary adjustment is necessary.

Another corollary of 1.7B is the following proposition

Corollary 1.7D. *Suppose that* $l < k$ *and* $h : D^l \times D^{k-l} \to W$ *is an embedding, positive near* $h(\partial D^l \times 0)$*. Then there exists a* C^0*-small isotopy* $h_t : D^l \times D^{k-l} \to W, t \in [0,1], h_0 = h$*, which is fixed near* $\partial D^l \times 0$ *and such that* $h_1(D^l \times D^{k-l})$ *is positive near* $h_1(D^l \times 0)$*.*

Proof. All sections of $T(W)$ over $h(D^l \times D^{k-l})$ are homotopic relative to $h(\partial D^l \times 0)$. Take a small triangulation of $D^l \times D^{k-l}$. Let us enumerate all simplices of the triangulation in order increasing dimension. Inductively applying Theorem 1.7B to each simplex of the triangulation we first deform the embedding h to make the projection $p \circ h$ a folded map on a neighborhood of the simplex with the fold on a sphere inside this neighborhood. By an additional isotopy inside this neighborhood we then push the simplex itself away from the fold. This isotopy is C^0-small because it is supported inside a small set. As the result of this process we construct an isotopy $h'_t : D^l \times D^{k-l} \to W$ such that $p \circ h'_1$ is a folded map with all components of the fold localized in disjoint small balls. Again by an additional C^0-small isotopy inside $h'_1(D^l \times D^{k-l})$ we can push a neighborhood of $h'(D^l \times 0)$ away from these balls. ■

Remark. Corollary 1.7D follows also from Theorem 2.4.5.C' in [Gr].

1.8 Proof of Theorem 1.2A

A subset A of W is called *small* if there exists a neighborhood U of A in W such that the projection $p|_U$ splits as $p|_U = \varphi \circ \pi_{k+1} \circ \ldots \circ \pi_{n+k} \circ \psi$,where $\psi : U \to \mathbb{R}^{n+k}$ and $\varphi : \mathbb{R}^k \to Q$ are immersions and π_s is the projection $\mathbb{R}^s \to \mathbb{R}^{s-1}, s = k+1, \ldots, n+k$.

Consider a triangulation of the manifold V by small simplices. We are going to construct the required isotopy extending it inductively from simplex to simplex. So it is sufficient to consider the situation when the manifold V is small and equal to $D^l \times D^{k-l}, l < k$, $L = D^l \times 0$, the projection $p|_V$ is an immersion near ∂L and the isomorphism F already coincides with the canonical homomorphism K near ∂L. Let U be the neighborhood of V in W and $p|_U = \varphi \circ \pi_{k+1} \circ \ldots \circ \pi_{n+k} \circ \psi$ be the splitting implied by the definition of a small set. Set $p_s = \pi_{n+k-s+1} \circ \ldots \circ \pi_{n+k} \circ \psi : U \to \mathbb{R}^{n+k-s}, s = 1, \ldots, n$. Then there exists a trivialization $\Theta = (\theta_1, \ldots, \theta_n)$ of the bundle $\operatorname{Vert}(V)$ such that the sections $\theta_1, \ldots \theta_i$ generate the kernel of dp_i over V for $i = 1, \dots, n$. Set $\alpha_i = F(\theta_i), i = 1 \dots, n$. There exists a neighborhood of V in W, still denoted by U, which admits a decomposition

$$U = V \times \underbrace{D^1 \dots \times D^1}_{n}$$

with the following property. Set

$$V^i = V \times \underbrace{D^1 \times \dots \times D^1}_{i}, \quad , i = 1, \dots, n,$$

so that we have $V^0 = V, V^n = U$. Then the vector field α_i is normal to V^{n-i} in $V^{n-i+1}, i = 1, \dots, n$.

We can consider L as a submanifold of each of $V^i, i = 1, \dots, n$. Now apply Corollary 1.7.D to the submanifold V^{n-1} to deform it (via an isotopy fixed near ∂L) into a submanifold , still denoted by V^{n-1}, such that the projection $p_1|_{V^{n-1}} : V^{n-1} \to \mathbb{R}^{n+k-1}$ is an immersion near L. We then continue the process by deforming the submanifold V^{n-2} inside V^{n-1} to arrange that the projection $p_2|_{V^{n-2}} : V^{n-2} \to \mathbb{R}^{n+k-2}$ is an immersion near L, etc. The composition of these successive isotopies gives us the deformation with the required properties. ■

1.9 Proof of Theorems 1.6A and 1.6B

We prove here only Theorem 1.6A. The proof of Theorem 1.6B is similar.

Let us begin the construction of the required isotopy near the manifold Σ which is supposed to become (a part of) the future fold. Roughly speaking, we are going to do the following. First we apply Theorem 1.2A to deform V in a neighborhood of the future fold Σ in such a way that the restriction of the projection p to this neighborhood is an immersion, and next we "bend" U along Σ to create the desired fold. Here is a more precise description.

Let us recall that near $\Sigma \subset V$ we have

$$\mathrm{Vert} = \bar{\lambda} \oplus \mathrm{Ver}_+ \oplus \mathrm{Ver}_-.$$

Denote by $\tilde{F}$ an isomorphism $\mathrm{Vert}|_U \to \nu(V)|_U$ which equals F on $\bar{\lambda} \oplus \mathrm{Ver}_+$ and equals $-F$ on Ver_-. According to Theorem 1.2A there exist an isotopy $g_t : W \to W$ and a family of isomorphisms $\Psi_t : \mathrm{Vert}(V_t) \to \nu(V_t), t \in [0,1]$, such that $\Psi_0|_U = \tilde{F}, g_0 = \mathrm{Id}$, $p|_{U_1 = g_1(U)}$ is an immersion, and Ψ_1 coincides with the canonical homomorphism K over U_1. Let $\tau \in \bar{\lambda}$ be a section which gives the prescribed trivialization of the line bundle $\bar{\lambda}$. Set $\tau_1 = \Psi_1(\tau)$. Thus τ_1 is normal to $T(U_1)|_{\Sigma_1}$. Let η_1 be a vector field normal to Σ_1 in U_1 which defines the given coorientation of Σ_1. Now we deform the inclusion $V_1 \hookrightarrow W$ near Σ_1 in the following way. Keeping Σ_1 fixed we rotate U_1 around Σ_1 so that the normal vector τ_1 rotates counterclockwise in the plane (τ_1, η_1) to the vector $-\eta_1$, thus moving η_1 to the position of τ_1. This rotation can be extended to an isotopy of the whole V_1. By a small additional C^1-perturbation we create a fold along Σ_1 in such a way that the vector field ν_{ext} for this fold coincides with the new position $\tilde{\tau}_1$ of the vector field τ. This additional perturbation can be canonically covered by a deformation of the isomorphism Ψ_1. We will continue to use the notation Ψ_1 for the perturbed isomorphism. The deformation of the isomorphism $\tilde{F}$ defines a deformation of the original isomorphism $F|_U$. Indeed, we can define F_t equal to Ψ_t on $\mathrm{Ver}^t_+ \oplus \bar{\lambda}^t$ and equal to $-\Psi_t$ on Ver^t_+. The family of isomorphisms $F_t|_{U_t}$ extends to the family of globally defined isomorphisms $F_t : \mathrm{Vert}(V_t) \to \nu(V_t)$. It remains to observe that the isomorphism $F_1 : \mathrm{Vert}(V_1) \to \nu(V_1)$ coincides over U_1 with the regularization $\mathcal{R}_{\mathcal{L}}K$ constructed in accordance with the splitting $\mathcal{L} = (\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$. This completes the part of the construction which makes Σ (a part of) the fold of the projection $p|_{V_1}$.

Now we need to take care of the remaining part of the manifold. To simplify the notation we assume that the manifold V itself already has properties which we achieved as the result of the first adjustment. Thus we assume that $p|_U$ is a folded map with the fold Σ and that the isomorphism F coincides with $\mathcal{R}_{\mathcal{L}}K$ on U.

We proceed now similarly to the proof of Theorem 1.2A in Section 1.8. Let P be a component of $V \setminus U$. Consider a triangulation of P by small (in the sense of Section 1.8) simplices. Let $\hat{F}$ be an isomorphism

$$\mathrm{Vert}(P) = \mathrm{Vert}_+ \oplus \mathrm{Vert}_- \to \nu(P)$$

which equals F on Vert_+ and equals $-F$ on Vert_-. Notice that over ∂P the isomorphism $\hat{F}$ coincides with the canonical homomorphism K. Successively applying Theorem 1.2A to simplices of the $(k-1)$-skeleton P^{k-1} of P we can achieve that the projection $p|_P$ is nonsingular near P^{k-1}, and that $\hat{F}$ coincides with K over there. Let $D, D \subset P$ be a k-simplex. Thus near ∂D the projection $p|_D$ is an immersion and we have the equality $\hat{F} = K$ which is equivalent to the equality $F = \mathcal{R}_{\mathcal{L}}K$. Now we continue similarly to Section 1.8. It can be arranged that the trivialization Θ of the bundle $\mathrm{Vert}(D)$ which is implied by the definition of the small set (see Section 1.8) is chosen in such a way that $\theta_1, \dots, \theta_l$ are sections of Vert_+ and $\theta_{l+1}, \dots, \theta_n$ belong to $\mathrm{Vert}_-, 0 \le l \le n$. Set $\alpha_i = \hat{F}(\theta_i), i = 1, \dots, n$. Suppose that the neighborhood $U \supset V = D$ and the splitting

$$U = V \times \underbrace{D^1 \dots \times D^1}_{n}$$

and

$$V^i = V \times \underbrace{D^1 \times \dots \times D^1}_{i}, i = 0, \dots, n$$

are as in Section 1.8. Now we sequentially apply Corollary 1.7D to each of the submanifolds $V^i, i = n-1, \dots, 1$ to deform V^i inside V^{i+1} via a fixed near ∂D isotopy into a submanifold, still denoted by V^i, such that the projection $p_{n-i}|_{V^i} : V^i \to \mathbb{R}^{k+i}$ is a positive immersion near D. We cannot apply Corollary 1.7D at the last step to deform $V^0 = D$ inside V^1. Instead, here we apply Corollary 1.7C to arrange that $p_n|_d : D \to \mathbb{R}^k$ is a folded map. Let us observe that the kernel Ker of the differential of the last projection π_{k+1} is generated by the section $\theta_n \in \mathrm{Vert}$ which is contained in Vert_-, unless $\dim(\mathrm{Vert}_-|_D) = 0$. Therefore, we can define a subordinate splitting $\mathcal{L}'$ with $\Sigma' \cap D = \Sigma(p_n)$ and $\lambda'|_{\Sigma' \cap D} = \mathrm{Ker}$, such that $\mathcal{L}'$ respect the fold of the map $p = \varphi \circ p_n$. It remains to observe that the isomorphism F had been deformed to $\mathcal{R}_{\mathcal{L}}K$ by the constructed isotopy. ■

2 Mappings with moderate singularities and K. Igusa's theorem

2.1 Isomorphism δ_Φ

We will continue using in this section the notation and terminology from Section 1. Thus by $p : W \to Q$ we denote a submersion of a $(n+k)$-dimensional manifold W to a k-dimensional manifold Q; the vector bundle Vert over W is formed by tangent spaces to the leaves of the foliation $\mathcal{F}$ by preimages $p^{-1}(q)$, $q \in Q$. The dual bundle to Vert will be denoted by Vert^*. A choice of a Riemannian metric allows us to identify Vert and Vert^*. Given a subset $A \hookrightarrow W$ we denote by $\mathrm{Vert}(A)$ and $\mathrm{Vert}^*(A)$ the restrictions of Vert and Vert^* to A and by $\nu(A)$ the normal bundle to A in W if A is a submanifold.

Let $\Phi : W \to \mathrm{Vert}^*$ be a section, transversal to the 0-section $W \subset \mathrm{Vert}^*$. Then the set of its zeroes $V = V(\Phi) = \Phi(W) \cap W$ is a k-dimensional submanifold. The tangent planes to $\Phi(W)$ along V give an isomorphism $\mathrm{Vert}^*(V) \to \nu(V)$. We will view it, via the identification $\mathrm{Vert}^*(V) \cong \mathrm{Vert}(V)$, as an isomorphism $\mathrm{Vert}(V) \to \nu(V)$ which will be denoted by δ_Φ.

2.2 Mappings with moderate singularities

Let

$$\begin{tikzcd}
W \arrow[rr, "f"] \arrow[dr, "p"'] & & Q \times \mathbb{R} \arrow[dl] \\
& Q &
\end{tikzcd}$$

be a fibered map. We say that f has *moderate singularities* if all its singular points are of fold or cusp type (see [EM]). Here we view f as usual (non-fibered) map. If f has moderate singularities than its restrictions to the leaves of the foliation $\mathcal{F}$ (and projected to $\mathbb{R}$) are functions with critical points of only Morse and "birth-death" (embryo) type (singularities A_1 and A_2 in Arnold's classification).

It is not true that a fibered map f which has fiberwise only A_1 and A_2 singularities has moderate singularities in our sense. However, *this is true generically.*

First, we observe

Proposition 2.2A. *Let $f : W \to Q \times \mathbb{R}$ be a fibered map with moderate singularities. Let $V = \Sigma^1(f)$ be the set of its singular points. Then V is a k-dimensional submanifold of W*

and $p|_V : V \to Q$ *is a folded map. The fold* $\Sigma = \Sigma(p|_V)$ *coincides with the cusp* $\Sigma^{1,1}(f)$ *of the map* f*.*

At each point $q \in V$ the tangent plane to the section $df(W)$ intersects $\mathrm{Vert} \oplus \mathrm{Vert}^*(q) \subset T(\mathrm{Vert}^*)$ along a Lagrangian plane, which is a graph of the (fiberwise) quadratic differential $d^2_Q f : \mathrm{Vert} \to \mathrm{Vert}^*$ of f. Via an identification $\mathrm{Vert}^*(V) \cong \mathrm{Vert}(V)$ we may view $d^2_Q f$ as a self-adjoint operator $\mathrm{Vert}(V) \to \mathrm{Vert}(V)$. In particular, we get a decomposition

$$\mathrm{Vert}(V \setminus \Sigma) = \mathrm{Vert}_+ \oplus \mathrm{Vert}_-$$

into its positive and negative eigenspaces. Similarly, the fiberwise cubic differential of f is defined and does not vanish on the line bundle $\mathrm{Ker} = \mathrm{Ker}(d^2_Q f)$ over Σ. Therefore, it allows us to define a trivialization of Ker given by the section ν_{int} of Ker such that $d^3_Q f(\nu_{int}) = 1$.

Our second observation is

Proposition 2.2B. *The quadruple* $\mathcal{L}_f = (\Sigma, \mathrm{Vert}_+, \mathrm{Vert}_-, \mathrm{Ker})$ *defines an admissible splitting which respects the fold of* $p|_V$*. The regularization* $\mathcal{R}_{\mathcal{L}}K : \mathrm{Vert}(V) \to \nu(V)$*, constructed in accordance with the splitting* $\mathcal{L}$*, is homotopic to* $\delta_{d_Q f}$*.*

Indeed, it follows from the definitions of the homomorphisms K, $d^2_Q f$ and δ that

$$K = \delta_{d_Q f} \circ d^2_Q f, \quad \text{or} \quad \delta^{-1}_{d_Q f} \circ K = d^2_Q f.$$

Taking into account the definition of the regularization $\mathcal{R}_{\mathcal{L}}K$ we see that the composition $\delta^{-1}_{d_Q f} \circ \mathcal{R}_{\mathcal{L}}K$ is a *positive definite* self-adjoint operator. Hence the existence of a homotopy between $\mathcal{R}_{\mathcal{L}}K$ and $\delta_{d_Q f}$ follows from the contractibility of the space of positive definite self-adjoint operators.

2.3 Construction of mappings with moderate singularities

Fibered wrinkled mappings

We recall here, for the convenience of the reader, the definition of a wrinkled map and the Wrinkling theorem (Theorem 1.6B from [EM]). See [EM] for more details.

Consider the map $w(n,q,s) : \mathbb{R}^{q-1} \times \mathbb{R}^{n-q} \times \mathbb{R}^1 \to \mathbb{R}^{q-1} \times \mathbb{R}^1$ given by the formula

$$(1) \qquad (y,z,x) \mapsto \left(y, z^3 + 3(|y|^2 - 1)z - \sum_1^s x_i^2 + \sum_{s+1}^{n-q} x_j^2 \right),$$

where $y \in \mathbb{R}^{q-1}$, $z \in \mathbb{R}$, $x \in \mathbb{R}^{n-q}$ and $|y|^2 = \sum\limits_1^{q-1} y_i^2$.

The singularity $\Sigma^1(w(n,q,s))$ is the $(q-1)$-dimensional sphere

$$S^{q-1} = S^{q-1} \times \subset \mathbb{R}^q \times \mathbb{R}^{n-q}.$$

whose equator $\{x = 0, z = 0, |y| = 1\} \subset \Sigma^1(w)$ consists of cusp points of index $s + \frac{1}{2}$. The upper hemisphere $\Sigma^1(w) \cap \{z > 0\}$ consists of folds of index s while the lower one $\Sigma^1(w) \cap \{z < 0\}$ consists of folds of index $s+1$.

A map $f : U \to N$ defined on an open subset $U \subset M$ is called a *wrinkle* of index $s + \frac{1}{2}$ if it is equivalent to the restriction of $w(n,q,s)$ to an open neighborhood W^n of the disk $D = D^q \times 0 \subset \mathbb{R}^q \times \mathbb{R}^{n-q}$. We use the term "wrinkle" also for the singularity $\Sigma(f)$ of a wrinkle f.

Notice that for $q = 1$ the wrinkle is a function with two nondegenerate critical points of indices s and $s+1$ given in a neighborhood of a gradient trajectory which connects the two critical points.

Although the differential $dw(n,q,s) : T(\mathbb{R}^n) \to T(\mathbb{R}^q)$ degenerates at points of $\Sigma(w)$, it can be canonically *regularized.* Namely, we can substitute the element $3(z^2 + |y|^2 - 1)$ in the Jacobi matrix of $w(n,q,s)$ by a function γ which coincides with $3(z^2 + |y|^2 - 1)$ outside an arbitrary small neighborhood V of the disc D and does not vanish along $V \cap \{x = 0\}$. The new bundle map $\mathcal{R}(dw) : T(\mathbb{R}^n) \to T(\mathbb{R}^q)$ provides a homotopically canonical extension of the map $dw : T(\mathbb{R}^n \setminus W^n) \to T(\mathbb{R}^q)$ to an epimorphism (fiberwise surjective bundle map) $T(\mathbb{R}^n) \to T(\mathbb{R}^q)$. We call $\mathcal{R}(dw)$ the *regularized differential* of the map $w(n,q,s)$.

A map $f : M \to N$ is called *wrinkled* if there exist disjoint open subsets $U_1, \dots, U_l \subset M$ such that $f|_{M \setminus U}$, $U = \bigcup_1^l U_i$, is a submersion (i.e. has rank equal q) and for each $i = 1, \dots, l$ the restriction $f|_{U_i}$ is a wrinkle.

The singular locus $\Sigma(f)$ of a wrinkled map f is a union of $(q-1)$-dimensional wrinkles $S_i = \Sigma^1(f|_{U_i}) \subset U_i$. Each S_i has a $(q-2)$-dimensional equator $T_i \subset S_i$ of cusps which divides

S_i into 2 hemispheres of folds of 2 neighboring indices. The differential $df : T(M) \to T(N)$ can be regularized to obtain an epimorphism $\mathcal{R}(df) : T(M) \to T(N)$. To get $\mathcal{R}(df)$ we regularize $df|_{U_i}$ for each wrinkle $f|_{U_i}$.

For any integer k, $0 \le k \le q-1$, the map $w(n,q,s)$ can be considered as a fibered map over $\mathbb{R}^k$. Namely, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^k \times \mathbb{R}^{q-1-k} \times \mathbb{R}^{n-q} \times \mathbb{R}^1 \arrow[rr, "w(n,q,s)"] \arrow[dr, "pr"'] & & \mathbb{R}^k \times \mathbb{R}^{q-1-k} \times \mathbb{R}^1 \arrow[dl, "pr"] \\
& \mathbb{R}^k &
\end{tikzcd}$$

where pr is the projection to the first factor. We refer to this fibered map as $w_k(n,q,s)$. A fibered map equivalent to the restriction of $w_k(n,q,s)$ to an open neighborhood $W^n \supset D$ is called a *fibered wrinkle.*

The regularized differential $\mathcal{R}(dw_k(n,q,s))$ is a fibered (over $\mathbb{R}^k$) epimorphism

$$\mathbb{R}^k \times T(\mathbb{R}^{q-1-k} \times \mathbb{R}^{n-q} \times \mathbb{R}^1) \xrightarrow{\mathcal{R}(dw_k(n,q,s))} \mathbb{R}^k \times T(\mathbb{R}^{q-1-k} \times \mathbb{R}^1)$$

A fibered (over B) map $f : M \to N$ is called a *fibered wrinkled map* if there exist disjoint open sets $U_1, U_2, \dots, U_l \subset M$ such that $f|_{M \setminus U}, U = \bigcup_1^l U_i$ is a fibered submersion and for each $i = 1, \dots, l$ the restriction $f|_{U_i}$ is a fibered wrinkle.

Similarly to the non-parametric case one can define regularized differential of a fibered over B wrinkled map $f : M \to Q$ which is a fibered epimorphism $\mathcal{R}(d_B f) : T_B M \to T_B Q$.

Wrinkling theorem. (Theorem 1.6B from [EM]) *Let $f : M \to N$ be a fibered over B map covered by a fibered epimorphism $F : T_B(M) \to T_B(N)$. Suppose that F coincides with df near a closed subset $K \subset M$ (in particular, f is a fibered submersion near K), then there exists a fibered wrinkled map $g : M \to N$ which extends f from a neighborhood of K, and such that the fibered epimorphisms $\mathcal{R}(dg)$ and F are homotopic rel. $T_B(M)|_K$.*

We will apply the Wrinkling theorem to fibered (over Q) maps of a $(n+k)$-dimensional manifold W to $Q \times \mathbb{R}$. Thus we will only need a partial case of the theorem, which corresponds to families of functions, i.e. the case $q = k + 1$.

Main theorems

Theorem 2.3A. *Let* $\Phi : W \to \mathrm{Vert}^*$ *be a section transversal to the* 0*-section and* $V = V(\Phi)$ *be its set of zeroes. Let* $\mathcal{L} = (\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ *be an admissible splitting and* $g : W \to Q \times \mathbb{R}$ *a fibered map. Then there exists a* C^0*-small isotopy* $h_t : W \to W$ *and a fibered map* $f : W \to Q \times \mathbb{R}$ *with moderate singularities such that:*

- f *is a fibered wrinkled map outside a neighborhood* $\Omega \supset V$*;*
- $\Sigma^1(f|_\Omega) = V_1\,(= h_1(V))$*;*
- $\mathcal{L}_{f|_\Omega}$ *is subordinate to* $\mathcal{L}$*;*
- f *is* C^0*-close to* g*.*

Let $\Psi : W \to \mathrm{Vert}^*$ *be the section which is equal to* $d_Q f$ *on* Ω *and to the regularized differential* $\mathcal{R}(d_Q f)$ *on* $W \setminus \Omega$*. Then* Ψ *is homotopic to* $\Phi \circ h_1$ *in the class of sections transversal to the* 0*-section and vanishing exactly on* V_1*.*

Theorem 2.3A also holds in the relative form:

Theorem 2.3B. *Suppose, in addition to the assumptions of Theorem* 2.3A, *that* Φ *coincides with* $d_Q g$ *and* $\mathcal{L}$ *coincides with* $\mathcal{L}_g$ *near a closed subset* $A \subset W$*. Then the map* f *can be chosen equal to* g *on* A*, and the homotopy between* Ψ *and* $\Phi \circ h_1$ *and the isotopy* $h_t, t \in [0,1]$*, can be chosen fixed on* A*.*

The following corollary of Theorem 2.3B is especially useful.

Theorem 2.3C. *Let* $g : W \to Q \times \mathbb{R}$ *be a fibered map which is non-singular near* $A \subset W$*. Then there exists a fibered map* $f : W \to Q \times \mathbb{R}$ *with moderate singularities which coincides with* g *on* A *and which is* C^0*-close to* g*.*

Proof. Set $\Phi = d_Q g$. The transversality theorem guarantees that for a generic g the map $\Phi : W \to \mathrm{Vert}^*$ is transverse to the 0-section. Now take the trivial splitting $\mathcal{L} = (\emptyset, \emptyset, \mathrm{Vert}(V), \{0\})$ on $V = V(\Phi)$ and apply Theorem 2.3B. ■

Remark 2.3D. The map f constructed in the proof of Theorem 2.3C has some additional important properties. For instance, all its singularities, other than wrinkles, have indices $0, 1$, or $\frac{1}{2}$.

The results formulated in this section are proven in Sections 2.4–2.8 below.

2.4 Deformation of the fold. Beginning of the proof of Theorems 2.3A and 2.3B.

We will construct the required map f in three steps. First we use Theorem 1.6A to deform the submanifold V by a C^0-small isotopy to arrange that $p|_{V_1}$ is a folded map. Second, we define f in a neighborhood of V_1 and finally apply Wrinkling theorem to extend f to the whole manifold W. Thus we begin with the following Lemma 2.4A which is a direct corollary of Theorems 1.6A and 1.6B.

Lemma 2.4A. *There exist an admissible splitting* $\mathcal{L}' = (\Sigma', \lambda', \mathrm{Vert}'_+, \mathrm{Vert}'_-)$ *subordinate to* $\mathcal{L}$*, a* C^0*-small isotopy* $h_t : W \to W, t \in [0,1]$*, and a covering family of isomorphisms* $\Lambda_t : \mathrm{Vert}(V) \to \nu(V)$ *such that:*

- $p|_{V_1}$ *is a folded map and the splitting* $\mathcal{L}'$ *respects the fold of* $p|_{V_1}$*;*
- $h_0 = \mathrm{Id}, \Lambda_0 = \delta_\Phi, \Lambda_1 = \mathcal{R}_{\mathcal{L}'}K$ *where* $\mathcal{R}_{\mathcal{L}'}K$ *is the regularization constructed in accordance with the splitting* $\mathcal{L}'$*.*

Additionally under the assumptions of Theorem 2.3B *one can make* h_t *and* Λ_t*,* $t \in [0,1]$*, to be fixed near* A*.*

2.5 Local integration. Continuation of the proof.

To simplify the notation we assume here that the manifold V and the splitting $\mathcal{L}$ (and not V_1 and $\mathcal{L}'$) already satisfy the conclusions of Lemma 2.4A.

Lemma 2.5A. *There exists a neighborhood* Ω *of* V *in* W *and a fibered map* $f : \Omega \to Q \times \mathbb{R}$ *such that:*

- *f is C^0-close to g;*
- *f has moderate singularities, $\Sigma^1(f) = V$ and $\Sigma^{1,1}(f) = \Sigma(p|_V) = \Sigma$;*
- *the isomorphism $\delta_{d_Q f}$ is homotopic to δ_Φ.*

Under the assumptions of Theorem 2.3B *one can additionally have $f = g$ near A and the homotopy between $\delta_{d_Q f}$ and δ_Φ to be fixed over A.*

Proof. Let us recall that

$$\mathrm{Vert}(\Sigma) = \mathrm{Ver}_+ \oplus \mathrm{Ver}_- \oplus \bar{\lambda}$$

over a neighborhood of Σ in W. Let $\nu_{int} \in \lambda = \mathrm{Ker}(d(p|_V))$ be the vector field which defines the trivialization of $\bar{\lambda}$ prescribed by the splitting $\mathcal{L}$. Set $I_\varepsilon = [-\varepsilon, \varepsilon]$ and take an embedding $\gamma : \Sigma \times I_\varepsilon \to W$ such that

- $\gamma|_{\Sigma \times 0}$ is the identity map $\Sigma \times 0 \to \Sigma$;
- $d\gamma\left(\frac{\partial}{\partial t}|_{t=0}\right) = \nu_{ext}$;
- $p \circ \gamma : \Sigma \times I_\varepsilon \to Q$ is an immersion.

Let N_ε be the ε-ball-subbundle of the bundle $\mathrm{Ver}_+ \oplus \mathrm{Ver}_-$ over Σ. Set $P_\varepsilon = N_\varepsilon \times I_\varepsilon \times I_\varepsilon$ and denote by t, u coordinates in P_ε which are given by the projections of P_ε to the two last factors. There exists an $\varepsilon > 0$, a neighborhood $\Omega_0 \supset \Sigma$ in W and a diffeomorphism $\mu : P_\varepsilon \to \Omega_0$ such that:

- $\mu|_{\Sigma \times I_\varepsilon \times 0} = \gamma$;
- $d\mu\left(\frac{\partial}{\partial u}\right) = \nu_{int}$;
- the manifold $\mu^{-1}(V) \subset P_\varepsilon$ is given by the equation $t = -u^2$;
- the diagram

$$\begin{tikzcd}
P_\varepsilon \arrow[rr, "\mu"] \arrow[dr, "\pi"'] & & W \arrow[dl, "p"] \\
& Q &
\end{tikzcd}$$

is commutative, i.e. the map $\mu : P_\varepsilon \to W$ is fibered over Q.

Here the projection π is defined by the formula $(\sigma, \zeta_+, \zeta_-, t, u) \mapsto p(\gamma(\sigma, t))$ for $\sigma \in \Sigma$, $\zeta_+ \in \mathrm{Ver}_+$, $\zeta_- \in \mathrm{Ver}_-$ and $t, u \in I_\varepsilon$.

Define a function $\alpha : P_\varepsilon \to \mathbb{R}$ by the formula

$$\alpha(\sigma, \zeta_+, \zeta_-, u, t) = u^3 + 3tu + \|\zeta_+\|^2 - \|\zeta_-\|^2$$

for $\sigma \in \Sigma$, $\zeta_+ \in \mathrm{Ver}_+$, $\zeta_- \in \mathrm{Ver}_-$ and $t, u \in I_\varepsilon$.

Set $\hat{\alpha} = \alpha \circ \mu^{-1} : \Omega_0 \to \mathbb{R}$. Let $\tilde{g} : W \to \mathbb{R}$ be the composition of the fibered map $g : W \to Q \times \mathbb{R}$ with the projection to the second factor. Define the function $\tilde{\tilde{g}} : P_\varepsilon \to \mathbb{R}$ as $\tilde{\tilde{g}}(\sigma, \zeta_+, \zeta_-, u, t) = \tilde{g}(\mu(\sigma, u))$. Set $\hat{g} = \tilde{\tilde{g}} \circ \mu^{-1} : \Omega_0 \to \mathbb{R}$. Now we define the fibered map f as $f = p \times (\hat{g} + \hat{\alpha})$. By its construction f is a map with moderate singularities, $\Sigma^1(f) = V \cap \Omega_0$ and the splitting $\mathcal{L}_f$ coincides with the restriction of the splitting $\mathcal{L}$ to $V \cap \Omega_0$. It remains to extend f to a neighborhood Ω of the whole manifold V. First extend f to $V \cup \Omega_0$ to be C^0-close to g. For a slightly smaller neighborhood $\tilde{\Omega}_0 \subset \Omega_0$ set $\tilde{V} = V \setminus \tilde{\Omega}_0$ and consider a fibered diffeomorphism $\tilde{\mu}$ of an ε-neighborhood M_ε of the 0-section $\tilde{V}$ in the bundle $\mathrm{Vert}(\tilde{V})$ onto a tubular neighborhood Ω' of $\tilde{V}$ in W.

Thus the diffeomorphism $\tilde{\mu}$ sends the 0-section in M_ε identically onto $\tilde{V}$ and the diagram

$$\begin{tikzcd}
M_\varepsilon \arrow[r, "\tilde{\mu}"] \arrow[d, "\pi"'] & W \arrow[d, "p"] \\
\tilde{V} \arrow[r, "p|_{\tilde{V}}"'] & Q
\end{tikzcd}$$

is commutative, where $\pi : M_\varepsilon \to \tilde{V}$ is the orthogonal projection of the tubular neighborhood to the submanifold.

Let the function $\tilde{f} : \Omega_0 \cup V \to \mathbb{R}$ be the composition of the map $\Omega_0 \cup V \xrightarrow{f} Q \times \mathbb{R}$ with the projection $Q \times \mathbb{R} \to \mathbb{R}$. Set $\tilde{\tilde{f}} = \tilde{f} \circ \tilde{\mu}$. On each fiber of the fibration $\pi : M_\varepsilon \to \tilde{V}$ over a point close to $\partial\tilde{V}$ the function $\tilde{\tilde{f}}$ has a non-degenerate critical point at 0, the positive and negative eigenspaces of its quadratic differential coincide with the fibers of Vert_+ and Vert_-. Thus perturbing, if necessary, $\tilde{\tilde{f}}$ we can think that $\tilde{\tilde{f}}(\sigma, \zeta_+, \zeta_-) = \|\zeta_+\|^2 - \|\zeta_-\|^2 + \tilde{\tilde{f}}(\sigma, 0, 0)$ near $\partial\tilde{V}$. Here $\sigma \in \Sigma, \zeta_+ \in \mathrm{Vert}_+, \zeta_- \in \mathrm{Vert}_-$. Therefore this formula gives an extension of $\tilde{f}$ to the whole M_ε, so that the desired extension of f to a neighborhood $\Omega_1 \subset \tilde{V}$ is given by the formula

$$f = p \times \left(\tilde{\tilde{f}} \circ \tilde{\mu}^{-1}\right)$$

Let us check that the homomorphism $\delta_{d_Q f}$ is homotopic to δ_Φ. Indeed, according to Proposition 2.2B. $\delta_{d_Q f}$ is homotopic to the regularization $\mathcal{R}_\mathcal{L} K$ associated with the splitting $\mathcal{L}$, but by Lemma 2.4A $\mathcal{R}_\mathcal{L} K$ is homotopic to δ_Φ. Hence $\delta_{d_Q f}$ is homotopic to δ_Φ. It is clear that the map g could be left unchanged where it already had the required properties. ■

2.6 End of the proof of Theorems 2.3A and 2.3B

We have already completed two first steps in the program of construction of the map f which was described in the beginning of Section 2.4.

Thus we already have a fibered map f with moderate singularities, defined in a neighborhood Ω of the manifold V. We have $\Sigma^1(f) = V_1$ and $\Sigma^{1,1}(f) = \Sigma(p|_{V_1})$. Besides, the isomorphism $\delta_{d_Q f}$ is homotopic to δ_Φ. The last condition guarantees that the homomorphism $d_Q f$ viewed as a section $\Omega \to \mathrm{Vert}^*$ extends to a section $\tilde{\Phi} : W \to \mathrm{Vert}^*$, non-vanishing outside V and which is homotopic to Φ in the class of sections having V as their transversal locus of zeroes. Hence we are in a position to apply the Wrinkling theorem to extend f as a fibered wrinkled map $f : W \to Q \times \mathbb{R}$ whose regularized differential $\mathcal{R}(d_Q f)$ is homotopic to $\tilde{\Phi}$ via a homotopy fixed near V_1. ■

2.7 Wrinkling of embeddings. Parametric version in the codimension 1

Theorem 1.6A (of this paper) shows that one can get rid of higher singularities of the projection of a k-dimensional submanifold $V \subset W$ onto a k-dimensional manifold Q. One would like to prove a similar result in the general case $\dim V \geq \dim Q$ and, what is even more important, for a family of submanifolds in W. Such a parametric version of Theorem 1.6A for the case $\operatorname{codim} V = 1$ and $Q = \mathbb{R}^1$ (see Theorem 2.7A bellow) will be deduced in this section from Theorem 2.3C.

Let P be an $(n+1)$-dimensional manifold split into the product $P = M \times \mathbb{R}$.

Let π_1 and π_2 be projections of P onto the first and the second factor. A diffeomorphism $g : P \to P$ is called *vertical* if it is a fibered map

$$\begin{tikzcd}
P \arrow[rr, "g"] \arrow[dr, "\pi_1"'] & & P \arrow[dl, "\pi_1"] \\
& M &
\end{tikzcd}$$

Let Q and V be a k-dimensional and an n-dimensional manifolds, respectively, and $\{f_x\}_{x \in Q}$ be a family of embeddings $f_x : V \to P$.

Theorem 2.7A. *There exists a family of C^0-small vertical diffeomorphisms $g_x : P \to P$, $x \in Q$ such that functions $\pi_2 \circ g_x \circ f_x : V \to \mathbb{R}$ do not have higher singularities.*

Proof. Consider three fibered maps:

- $f : Q \times V \to Q \times P$ given by $f(x, y) = (x, f_x(y))$;

- $f_1 : Q \times V \to Q \times M$ given by $f_1(x, y) = (x, \pi_1 \circ f_x(y))$ and

- $f_2 : Q \times V \to Q \times \mathbb{R}$ given by $f_2(x, y) = (x, \pi_2 \circ f_x(y))$

corresponding to the family $\{f_x\}_{x\in Q}$.

Our goal is to simplify the singularities of f_2. However, we first take f_1 and denote by $\triangle$, $\triangle \subset Q\times V$, its singular locus $\Sigma(f_1)$. Notice that if Ω is a sufficiently small neighborhood of $\triangle$ in $Q\times V$ then the map $f_2|_\Omega : \Omega \to Q\times\mathbb{R}$ is nonsingular. Therefore we need to deform the map f_2 only outside of $\triangle$. Set $W = (Q\times V)\setminus \Omega'$, where $\Omega' \subset \Omega$ is a slightly smaller neighborhood of $\triangle$.

Thus we have a fibered over Q map $f_2 : W \to Q\times\mathbb{R}$ which is nonsingular near ∂W. According to Theorem 2.3B there exists a fibered map $\tilde f_2 : W \to Q\times\mathbb{R}$ with moderate singularities which is C^0-close to f_2 and coincides with f_2 near ∂W. Let us define a map $\tilde f : W \to Q\times P = Q\times M\times\mathbb{R}$ as the direct sum of f_1 and $\tilde f_2$ over Q. In other words,

$$\tilde f(z) = (f_1(z), \pi_2\circ \tilde f_2(z)), \qquad z\in W.$$

If $\tilde f_2$ is sufficiently C^0-close to f_2 then $\tilde f$ is a fibered embedding which together with $f|_\triangle$ defines a fibered embedding $\hat f : Q\times V \to Q\times P$. For $(x,y)\in Q\times V$ we have $\hat f(x,y) = (x, \pi_1\circ f_x(y), \pi_2\circ \tilde f_x(y))$ where the functions $\pi_2\circ\tilde f_x, x\in Q$, do not have higher singularities. It is clear from the structure of the map that it can be written in the form $G\circ f$ where G is a fibered diffeomorphism

$$\begin{tikzcd}
Q\times P \arrow[rr, "G"] \arrow[dr] & & Q\times P \arrow[dl] \\
& Q\times M &
\end{tikzcd}$$

■

The important special case of the theorem is when $P=\mathbb{R}^{n+1}$ and we want to simplify critical points of the restriction of one of coordinate functions to a family of codimension one submanifolds of $\mathbb{R}^{n+1}$. Let us denote by $\mathrm{Emb}(M,\mathbb{R}^{n+1})$ the space of embeddings of a n-dimensional manifold M to $\mathbb{R}^{n+1}$, and by $\mathrm{Emb}_{\mathcal{H}}(M,\mathbb{R}^{n+1})$ the subspace of $\mathrm{Emb}(M,\mathbb{R}^{n+1})$ which consists of embeddings $f : M\to\mathbb{R}^{n+1}$ whose first coordinate function $f_1 : M\to\mathbb{R}$

has moderate singularities (i.e. f_1 is a "generalized Morse function" in Igusa's terminology).

Corollary 2.7B. *The homomorphism*

$$\pi_k(\mathrm{Emb}_{\mathcal{H}}(M, \mathbb{R}^{n+1})) \hookrightarrow \pi_k(\mathrm{Emb}(M, \mathbb{R}^{n+1})),$$

induced by the inclusion is surjective for all $k \geq 0$.

Remark 2.7C Notice that the inclusion map

$$\mathrm{Emb}_{\mathcal{H}}(M, \mathbb{R}^{n+1}) \hookrightarrow \mathrm{Emb}(M, \mathbb{R}^{n+1})$$

is not a weak homotopy equivalence.

2.8 Generalized K. Igusa's theorem.

We discuss in this section the relation of Theorems 2.3A and 2.3B with the theorem of K. Igusa from [Ig1].

Let M be a compact n-dimensional manifold, possibly with boundary and $g : M \to \mathbb{R}$ a function without critical points near ∂M. Let $\mathcal{H} = \mathcal{H}(M, g)$ be the space of generalized Morse functions on M, i. e. functions with only Morse and embryo type singularities, which coincide with g near ∂M. The condition of not having higher singularities defines a differential relation $\mathcal{R}_{\mathcal{H}}$ in the space $J^3(M, \mathbb{R})$ of 3-jets of functions on M. This $\mathcal{R}_{\mathcal{H}}$ is formed by 3-jets of functions from $\mathcal{H}$. In particular, any section of $\mathcal{R}_{\mathcal{H}}$ equals over ∂M to the 3-jet of the function g. Let $\Gamma_{\mathcal{H}}$ be the space of sections $M \to \mathcal{R}_{\mathcal{H}}$. A section $\gamma : M \to \mathcal{R}_{\mathcal{H}}$ can be thought of as a quadruple $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ where $\gamma_0 : M \to \mathbb{R}$ is a function, $\gamma_1 : T(M) \to \mathbb{R}$ is a 1-form vanishing on a subset $A_1 \subset M$, γ_2 is a quadratic function $T(M)|_{A_1} \to \mathbb{R}$ which has rank $\geq n-1$ and γ_3 is a non-vanishing cubic function $\lambda \to \mathbb{R}$. Here λ is the line bundle of kernels of the quadratic form γ_2 over the set $A_2 \subset A_1$ where γ_2 degenerates. It is required, in addition, that $\gamma_0|_{\partial M} = g$ and $\gamma_1|_{T(M)|_{\partial M}} = dg$.

Let us denote by J^3 the natural inclusion $\mathcal{H} \hookrightarrow \Gamma_{\mathcal{H}}$ which associates with a function its 3-jet.

Theorem 2.8A. *The map* $J^3 : \mathcal{H} \to \Gamma_{\mathcal{H}}$ *is a (weak) homotopy equivalence. In other words, the differential relation* $\mathcal{R}_{\mathcal{H}}$ *satisfies an h-principle.*

Remarks.

1. It was proven earlier by K. Igusa (see [Ig1]) that the map J^3 induces an isomorphism of homotopy groups up to dimension $k < \mathrm{dim} M$ (and an epimorphism for $k = \mathrm{dim} M$). His method is completely different from ours but, in some sense, close to the method of surgeries of singularities from [E].
2. The isomorphism in the critical case $k = 1 = \dim M$ was established by V.I. Arnold in [Ar1].
3. V. A. Vassiliev (see [Va]) proved that the map J^3 is a homological equivalence.
4. In fact, as will follow from our proof, the differential relation $\mathcal{R}_H$ satisfies the C^0-dense h-principle (see [Gr]).

Proof of Theorem 2.8A.

Let us prove the surjectivity of $J^3_* : \pi_k(\mathcal{H}) \to \pi_k(\Gamma_{\mathcal{H}}), k \geq 0$. The proof of injectivity is similar.

Let $\psi : S^k \to \Gamma_{\mathcal{H}}$ be a spheroid representing a homotopy class from $\pi_k(\Gamma_{\mathcal{H}})$. Set $Q = S^k, W = Q \times M$ and let p be the projection $W \to Q$. Then the spheroid ψ can be identified with the following data:

- a fibered map $\psi_0 : W \to Q \times \mathbb{R}$;
- a section $\psi_1 : W \to \mathrm{Vert}^*$ vanishing on a closed subset $V \subset W$;
- a quadratic map $\psi_2 : \mathrm{Vert}(V) \to \mathbb{R}$ with the fiberwise rank $\geq n-1$;
- a trivialization of the line bundle λ, the kernel of the quadratic map ψ_2.

For a generic ψ, the section ψ_1 is transversal to the $0-$section $W \subset \mathrm{Vert}^*$ and, therefore, V is a k-dimensional submanifold of W. Moreover, ψ_2 degenerates over a codimension 1 submanifold $\Sigma \subset V$. Let us denote by Vert_+ and Vert_- the subbundles of $\mathrm{Vert}(V)$ formed by positive and negative eigenspaces of the quadratic form $\psi_2 : \mathrm{Vert} \to \mathbb{R}$. Then it is clear that $\mathcal{L}_\psi = (\Sigma, \lambda, \mathrm{Vert}_+, \mathrm{Vert}_-)$ is an admissible splitting (see Section 1.4). Thus we can apply Theorem 2.3B to construct a C^0-small isotopy $h_t : W \to W$ fixed near $\partial M \times S^k$ and a fibered map $f : W \to Q \times \mathbb{R}$ with moderate singularities such that:

(i) f is a fibered wrinkled map outside a neighborhood $\Omega \supset V_1 = h_1(V)$;

(ii) $\Sigma^1(f|_\Omega) = V_1$;

(iii) $\mathcal{L}_{f|_\Omega}$ is subordinate to $\mathcal{L}$;

(iv) f is C^0-close to ψ_0 and near $\partial M \times S^k$ coincides with ψ_0;

(v) the section $H : W \to \mathrm{Vert}^*$, which is equal to $d_Q f$ on Ω and to the regularized differential of f on $W \setminus \Omega$ is homotopic to $\psi_1 \circ h_1$ in the space of sections transversal to the 0-section and vanishing on V_1.

To finish the proof we need to show that the 3-jet $J^3(f)$, viewed as a spheroid $S^k \to \Gamma_{\mathcal{H}}(M, g)$, is homotopic to ψ. The required homotopy can be constructed in three steps. First, we note that the fibered map f may have wrinkles outside of a neighborhood of $\Omega \supset V_1$. In other words, there are disjoint neighborhoods $U_1, \dots U_N \subset W \setminus \Omega$, such that over each of these neighborhoods the map f is equivalent to the map given by the formula (4) from Section 2.3. For each $i = 1, \dots, N$ consider a homotopy over U_i of the 3-jet $J^3(f)$ which deforms the fiberwise differential $d_Q f$ into its regularization $\mathcal{R}(d_Q f)$ while keeping fixed the second and third derivatives of f with respect to chosen coordinates in U_i. It is straightforward to check that this deformation is inside $\Gamma_{\mathcal{H}}$. The 1-jet part of the constructed fibered section φ equals the section $H : W \to \mathrm{Vert}^*$ defined in (v) above. Next, we use the property (v) to construct a homotopy between H and $\psi_1 \circ h_1$. This homotopy, together with $d^2_Q f$ on Vert, $d^3_Q f$ on $\lambda' = \mathrm{Ker} d^2_Q f$, and any fibered deformation between f and ψ_0 define a homotopy between φ and a new section φ' inside $\Gamma_{\mathcal{H}}$. Finally we use the fact that the splittings $\mathcal{L}$ and its subordinate splitting $\mathcal{L}_{f|_\Omega}$ are concordant (see Section 1.4) to deform φ' into $\psi \circ h_1$ (which is, of course, homotopic to ψ). At the last step we will deal only with the 2- and 3-jet parts of the section keeping the 1-jet part fixed. This completes the proof of surjectivity of the map J^3_*. ■

2.9 Applications to pseudoisotopy theory.

Let us formulate now some applications of Theorem 2.8A to the pseudoisotopy theory, according to J. Cerf [Ce] and K. Igusa [Ig1].

Let M be split as $M = N \times I$ and $g : N \times I \to I$ be the projection. We denote by $\mathcal{E} = \mathcal{E}(M)$ the subspace of $\mathcal{H}(M, g)$ of functions without critical points and by $\Gamma_{\mathcal{E}}$ the space of all non-vanishing 1-forms on M which coincide with dg near ∂M. The natural inclusion $\mathcal{E} \to \Gamma_{\mathcal{E}}$ is denoted by d.

Let us recall that a pseudoisotopy of N is a diffeomorphism $M \to M$ which is the identity on $N \times 0 \cup \partial N \times I$. As was observed by J. Cerf the space $\mathcal{P}(N)$ of pseudoisotopies of N is homotopy equivalent to the space $\mathcal{E}$.

Theorem 2.9A. *The inclusion $d : \mathcal{E} \to \mathcal{H}$ induces the trivial homomorphism on all homotopy groups.*

Remarks.

1. K. Igusa gave in [Ig1] a simple argument which shows that the space $\mathcal{E}$ is contractible in $\Gamma_{\mathcal{H}}$. Therefore, Theorem 2.9A is a corollary of Theorem 2.8A. Alternatively, we will see that Theorem 2.9.A is an immediate corollary of Theorem 2.3C.
2. For the particular case of this application the combination of Igusa's and Vassiliev's theorems is also sufficient for proving 2.8.B.

Proof of Theorem 2.9A.

Given a spheroid representing $\pi_{k-1}(\mathcal{E})$ we will view it, as above, as a fibered map $G : S^{k-1} \times M \to S^{k-1} \times \mathbb{R}$. Let us extend G to $D^k \times M \to D^k \times \mathbb{R}$ still denoted by G. Our goal is to C^0-approximate G by a fibered map $f : S^{k-1} \times M \to S^{k-1} \times \mathbb{R}$ with moderate singularities which coincides with G near $\partial W = S^{k-1} \times M \cup D^k \times \partial M$. But the existence of such f is just asserted by Theorem 2.3C. Notice that Remark 2.3.D gives a slightly stronger version of Theorem 2.9A. ■

As was observed by K. Igusa in [Ig1] the existence of the isomorphism $d_* : \pi_k(\mathcal{H}) \to \pi_k(\Gamma_{\mathcal{H}})$ easily implies the splitting $\pi_k(\mathcal{H}, \mathcal{E}) \cong \pi_k(\mathcal{H}) \oplus \pi_{k-1}(\mathcal{E})$ in the same range of dimensions and, therefore, he proved the latter isomorphism for $k \le \dim M$. Now his observation together with Theorem 2.8A gives

Theorem 2.9B. *For all* $k \ge 1$ *we have*

$$\pi_k(\mathcal{H}, \mathcal{E}) \simeq \pi_k(\mathcal{H}) \oplus \pi_{k-1}(\mathcal{E}).$$

In our paper [EM] we suggested an alternative approach for studying the pseudoisotopy spaces. Let us denote by $\mathcal{E}^w$ the space of wrinkled functions $M \to \mathbb{R}$ (see [EM], Section 1.3) which coincide with g near ∂M. This space can be interpreted as follows. Let f be a function from $\mathcal{H}$. An embedded arc $\gamma \subset M$ is called admissible if its ends p_0 and p_1 are Morse critical points of f such that the indices of p_0 and p_1 differ by 1, and if the restriction $f|_{\gamma \setminus \{p_0, p_1\}}$ has no critical points. A set of disjoint admissible arcs $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ is called *complete* if all *Morse* critical points of f are among the ends of arcs from Γ. Then the space $\mathcal{E}^w$ can be viewed as the space of pairs (f, Γ) where $f \in \mathcal{H}$ and Γ is a complete set of arcs admissible for f.

The additional structure Γ allows us, in particular, to regularize canonically the differential df and thus to obtain a map $\mathcal{R}d : \mathcal{E}^w \to \Gamma_{\mathcal{E}}$. As it is shown in [EM] (see Theorem 1.6A there), this map is a homotopy equivalence.

We suggested in [EM] that instead of using Cerf–Igusa scheme, i.e studying the inclusion $\mathcal{E} \hookrightarrow \mathcal{H}$ and then applying the homotopy equivalence $d : \mathcal{H} \to \Gamma_{\mathcal{H}}$, it is more convenient to consider the inclusion $\mathcal{E} \hookrightarrow \mathcal{E}^w$ and then use the homotopy equivalence $\mathcal{R}d : \mathcal{E}^w \to \Gamma_{\mathcal{E}}$. Thus we have an exact sequence

$$\dots \xrightarrow{i_*} \pi_{k+1}(\Gamma_{\mathcal{E}}) \xrightarrow{j_*} \pi_{k+1}(\mathcal{E}^w, \mathcal{E}) \xrightarrow{\partial} \pi_k(\mathcal{E}) \xrightarrow{i_*} \pi_k(\Gamma_{\mathcal{E}}) \xrightarrow{j_*} \dots \quad . \tag{$*$}$$

The proof of the next Proposition 2.9C we learned from K. Igusa [Ig3]. This argument goes back to F. Laudenbach and A. Douady (see [La]), and was independently used in Igusa's thesis.

Proposition 2.9C. (Douady-Laudenbach, Igusa) *The map* $d : \mathcal{E} \to \Gamma_{\mathcal{E}}$ *is contractible.*

Proposition 2.9C implies that the exact sequence $(*)$ splits and thus for all $k \geq 0$ there is an isomorphism

$$\pi_{k+1}(\mathcal{E}^w, \mathcal{E}) \simeq \pi_{k+1}(\Gamma_{\mathcal{E}}) \oplus \pi_k(\mathcal{E}),$$

which reduces the homotopical study of $\mathcal{E}$ to the study of the relative homotopy type of the pair $(\mathcal{E}^w, \mathcal{E})$. See [EM] for the further discussion of this approach.

Proof of Proposition 2.9C.

Let us fix a Riemannian metric on N and endow $M = N \times \mathbb{R}$ with the product-metric. The space $\Gamma_{\mathcal{E}}$ can be identified with the space of non-vanishing tangent vector fields on M which coincides with the vertical unit vector field $\frac{\partial}{\partial t}$ outside of $N \times [0,1]$ and near the boundary $\partial N \times \mathbb{R}$. The map d can be viewed as the inclusion $\mathcal{E} \hookrightarrow \Gamma_{\mathcal{E}}$ of the subspace of gradient vector field. For our purposes we can furthermore assume that all vector fields are normalized to have length 1. Let us fix an $\varepsilon > 0$, such that 2ε is smaller than the injectivity radius of N. If N has a non-empty boundary we add to N a collar U of width 2ε and extend all the vector fields to $\widehat{M} = (N \cup U) \times \mathbb{R}$ as equal to $\frac{\partial}{\partial t}$ on $U \times \mathbb{R}$. For each point $x \in N$ we denote by B_x and by $\tilde{B}_x$ balls in $N \cup U$ of radius ε and 2ε, respectively, centered at the point x. Set $G_x = B_x \times \mathbb{R}$, $\tilde{G}_x = \tilde{B}_x \times \mathbb{R}$. For each point $z = (x,t) \in M$ we identify, via the exponential map, the cylinder $\tilde{G}_x$ with the corresponding cylinder in the tangent space $T_z(M)$.

Let us denote by $v^u : M \to M, u \in \mathbb{R}$, the flow generated by the vector field v. Set $v^u(x,t) = (\bar{v}^u(x,t), \tau^u(x,t))$. For any gradient vector field $v \in \mathcal{E}$ its flow is diffeomorphic to the flow of the vertical vector field $\frac{\partial}{\partial t}$. Let us denote by $T(z,v)$ the time needed for the trajectory of v through a point $z = (x,t)$, $t \leq 1$ to reach the level $N \times 1$, i.e. $v^{T(z,v)} \in N \times 1$. Set $T(v) = \max\limits_{z \in N \times [0,1]} T(z,v)$. We also set

$$w_u(x,t) = \frac{v^u(x,t) - (x,t)}{||v^u(x,t) - (x,t)||},$$

$(x,t) \in M$, $u > 0$, provided that $v^u(x,t) \in \tilde{G}_x$.

Let us define now a family of vector fields v_u, $u \in [0, T(v)]$, as follows

$$v_u(x,t) = \begin{cases} v(x,t), & \text{if } t = 0; \\ w_u(x,t), & \text{if } t > 0 \text{ and } v^u(x,t) \in G_x; \\ (2 - \frac{||\bar{v}_u(x,t)||}{\varepsilon})w_u(x,t) + (\frac{||\bar{v}_u(x,t)|||}{\varepsilon} - 1)\frac{\partial}{\partial t}, & \text{if } v^u(x,t) \in \tilde{G}_x \setminus G_x; \\ \frac{\partial}{\partial t}, & \text{if } v^u(x,t) \notin \tilde{G}_x. \end{cases}$$

Notice that for all $u \geq 0$ the vector field v_u is continuous, nowhere vanishes, and for $u \geq T(v)$ and all $(x,t) \in M$, as well as for all $u \geq 0$ and $(x,t) \in (\partial M \times \mathbb{R}) \cup (M \times (\mathbb{R} \setminus (0,1)))$, the vectors $v_u(x,t) = (\bar{v}_u(x,t), \tau_u(x,t))$ and $\frac{\partial}{\partial t}$ are not orthogonal. Let us modify the family v_u to have $v_u = \frac{\partial}{\partial t}$ for these values of (x,t) and u. Namely, we first define

$$\tilde{v}_u(x,t) = \begin{cases} v_u(x,t), & \text{if } t > 0; \\ (1+t)v_u(x,t) - t\frac{\partial}{\partial t}, & \text{if } t \in [-1,0]; \\ \frac{\partial}{\partial t}, & \text{if } t \leq -1. \end{cases}$$

Next we modify $\tilde{v}_u$ further and set

$$\tilde{\tilde{v}}_u = \begin{cases} \tilde{v}_u, & \text{if } u \leq T(v); \\ (1 - u + T(v))\tilde{v}_u + (u - T(v))\frac{\partial}{\partial t}, & \text{if } T(v) \leq u \leq T(v) + 1; \\ \frac{\partial}{\partial t}, & \text{if } u > T(v) + 1. \end{cases}$$

The vector field $\tilde{\tilde{v}}_u$ is also continuous and nowhere vanishes for all $u \geq 0$. The normalized field, still denoted by $\tilde{\tilde{v}}_u$, equals to $\frac{\partial}{\partial t}$ near $\partial N \times \mathbb{R}$ and away from $N \times (-1,1)$.

Finally define the homothety $c_\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda \in (0,1]$, by the formula $c_\lambda(t) = 1 + \lambda(t - 1)$, $t \in \mathbb{R}$, and then define the required contraction $\gamma_s : \mathcal{E} \to \Gamma_{\mathcal{E}}$, $s \in [-1,1]$, as follows:

$$\gamma_s(v)(x,t) = \begin{cases} v(x, c_{\frac{1-s}{2}}(t)), & \text{if } s \in [-1,0]; \\ \tilde{\tilde{v}}_{s(T(v)+1)}(x, c_{\frac{1}{2}}(t)), & \text{if } s \in [0,1]. \end{cases}$$

It is straightforward to check that $\gamma_u(v) \in \Gamma_{\mathcal{E}}$ for all $u \geq 0$ and $v \in \mathcal{E}$, γ_{-1} coincides with the inclusion $d : \mathcal{E} \to \Gamma_{\mathcal{E}}$, while γ_1 is the constant map to the point $\frac{\partial}{\partial t} \in \Gamma_{\mathcal{E}}$. ■

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