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06 December 2005

Partial Differential Equations Ma131

Practice Questions for Final Examination

I will add to these solutions as I have time.

1. (Convergence of Fourier series)
   For each of the following functions $f : [0, \ell] \to \mathbb{R}$, state whether the Fourier cosine series on $[0, \ell]$ converges uniformily, pointwise, and/or in L^2. If it converges pointwise, state what it converges to for each $x \in [0, \ell]$.

   a. $f(x) = x \sin^2(x\pi/\ell)$
   b. $f(x) = 0$ for $0 \leq x \leq \ell/2$ and $f(x) = 1$ else.

   Solution. We use theorems 2 through 4 in Section 5.4.

   a. Let $f(x) = x \sin^2(x\pi/\ell)$.
      1. The fourier series is uniformly convergent.
         The conditions are satisfied. The functions $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous on the interval $[0, \ell]$. We have $f(0) = f(\ell) = 0$, satisfying the usual boundary conditions.
      2. The fourier series is pointwise convergent.
         Clearly, $f(x)$ and $f'(x)$ are both continuous, so the series is pointwise convergent and converges to $f(x)$.
      3. The fourier series is L^2 convergent.
         Note that

$$\begin{aligned}\int_0^\ell \left(x \sin^2(\pi x/\ell)\right)^2 dx &= \int_0^\ell x^2 \sin^4(\pi x/\ell)\, dx \\ &\leq \int_0^\ell x^2\, dx \\ &= \frac{1}{3}\ell^2,\end{aligned}$$

         which is of course finite. Thus, the fourier series converges in the L^2 sense.

   b. Let $f(x) = 0$ for $0 \leq x \leq \ell/2$ and $f(x) = 1$ else.

1. The fourier series is not uniformly convergent.
   The function $f(x)$ is not continuous itself, and thus the series does not converge uniformly.
2. The fourier series does converge pointwise.
   The functions $f(x)$ and $f'(x)$ are both piecewise continuous. Therefore, the series is pointwise convergent, and it converges to

$$g(x) = \begin{cases} 0 & 0 \le x < \ell/2 \\ 1/2 & x = \ell/2 \\ 1 & \ell/2 < x \le \ell \end{cases}$$

3. The series does converge in L^2.
   We just need to calculate the integral. We get

$$\int_0^\ell f^2 \, dx = \int_{\ell/2}^\ell 1 \, dx = \frac{\ell}{2},$$

which is finite, so the series converges in L^2.

□

2. (Variable Coefficient Equation)
   Solve the following questions concerning the variable coefficient equation:

   a. Find the general solution of the partial differential equation

$$u_x + 2yu_y = 0.$$

   b. If we require $u(0, y) = y$, find the particular solution to part (a).
   c. Find the general solution to the partial differential equation

$$u_x + 2yu_y = x.$$

Solution.

a. Using the geometric method, we find the characteristic lines solve $\frac{dy}{dx} = \frac{2y}{1}$, so that $y = Ce^{2x}$. Because u is constant along the characteristic curves, we have $u(x, y) = f(e^{-2x}y)$ for some function $f \in C^2$.
b. If we impose $u(0, y) = y$, then we have $f(e^0 y) = f(y) = y$. Therefore, $u(x, y) = e^{-2x}y$.

c. One could make a coordinate transform to solve this problem, but it is much easier to simply look for a particular solution. Notice that the right hand side is dependent only on x. Because the u_y coefficient only depends on y and the u_x coefficient is constant, we can look for a function $u_p(x,y) = g(x)$ for some function g of x only. Plugging this in, we see $g'(x) = x$, so that $g(x) = \frac{1}{2}x^2$. Adding this to our answer to part (a), we see $u(x,y) = \frac{1}{2}x^2 + f(e^{-2x}y)$ for $f \in C^2$.

□

3. Consider the equation

$$u_t = Ku_{xx} - \alpha u,$$

where $\alpha > 0$. This models a one-dimensional rod with heat loss through the lateral sides with zero outside temperature. Let the rod have length L, with

$$u(0,t) = u(L,t) = 0.$$

a. The equilibrium temperatures are the functions u constant with respect to time, thus solving

$$u_{xx} - \alpha u = 0$$
$$u(0) = u(L) = 0.$$

Find the solutions for $u(x)$.

b. Solve the boundary problem given above (in the introduction to the problem) with initial condition $u(x,0) = f(x)$ using separation of variables.

c. Let $u(x,t)$ be your solution to part (b). Calculate $\lim_{t\to\infty} u(x,t)$. (Note that your answer to part (c) should agree with your answer to part (a)).

Solution.

a. We look for $u(x) = Ae^{\beta x}$. Then $u_{xx} - \alpha u = \beta^2 u - \alpha u = 0$, so $\beta^2 = \alpha$. Therefore, $u(x) = Ae^{\sqrt{\alpha}x}$. From the boundary conditions, we have $u(0) = A$ and $u(L) = Ae^{\alpha L} = 0$, so $A = 0$. Therefore, $u(x) = 0$ for all x.

b. Using separation of variables, we obtain the two differential equations

$$X''(x) = -\frac{\lambda - \alpha}{k} X(x)$$
$$T'(t) = -\lambda T(t).$$

Solving these in the usual way and imposing the boundary conditions, we have $\lambda_n = \alpha + k\left(\frac{n\pi}{L}\right)^2$, so that

$$X_n(x) = \sin(n\pi x/L),$$

$$T(t) = e^{-(\alpha+k(n\pi/L)^2)t}.$$

Putting this together, we have our solution for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\alpha+k(n\pi/L)^2)t} \sin(n\pi x/L),$$

where A_n are determined by the fourier sine series for $f(x)$.

c. We see $\lim_{t\to\infty} u(x,t) = 0$, as $\lim_{t\to\infty} e^{-(\alpha+k(n\pi/L)^2)t} = 0$. This describes the steady-state solution and does indeed match our solution to part (a).

☐

4. Use the coordinate method to solve

$$\begin{aligned} u_x + u_y &= u^2 \\ u(x,0) &= f(x) \end{aligned}$$

where $f(x)$ is an arbitrary function in x.

Solution. Let $x' = x + y$ and $y' = x - y$. Following the usual procedure, we see $u_{x'} = u^2/2$, so that $-1/u = \frac{1}{2}x' + C(y')$. That is,

$$u(x', y') = -\frac{2}{x' + 2C(y')}.$$

Converting this back to the normal variables, we have

$$u(x,y) = -\frac{2}{x + y + 2C(x-y)}.$$

But we are given $u(x,0) = f(x)$. It follows that $C(x) = -1/f(x) - x/2$, so that

$$\begin{aligned} u(x,y) &= -\frac{2}{x + y - 2\left(\frac{1}{f(x-y)} + \frac{x-y}{2}\right)} \\ &= \frac{f(x-y)}{1 - yf(x-y)}. \end{aligned}$$

☐

5. Solve the partial differential equation

$$u_{tt} - u_{xx} - \frac{2}{x}u_x = 0,$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 4x^2.$$

(Hint: Make the substitution $w = xu$.)

Solution. Making the substitution, $u = w/x$, we have

$$w_{tt}/x - w_{xx}/x + 2w_x/x^2 - 2w/x^3 - 2w_x/x^2 + 2w/x^3 = 0,$$

or equivalently,

$$w_{tt}/x - w_{xx}/x = 0,$$

which we know how to solve (just the wave equation, when we multiply through by x). The new initial conditions are $w(x,0) = 0$ and $w_t(x,0) = 4x^3$. This gives us the solution

$$w(x,t) = \frac{1}{2}\int_{x-t}^{x+t} 4s^3\,ds = \frac{1}{2}\left((x+t)^4 - (x-t)^4\right) = 4x^3t + 4xt^3,$$

so that $u(x,t) = 4x^2t + 4t^3$. □

6. Consider the wave equation $u_{tt} = u_{xx}$ which satisfies the boundary conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

where $f(x)$ and $g(x)$ are continuous and differentiable functions which are identically zero for $|x| \geq R$ for some fixed $R > 0$.

   a. Show there exists a strictly positive function $S(t)$ so that $u(x,t)$ is identically zero for $|x| \geq S(t)$.
   b. Define the energy $E(t)$ of $u(x,t)$ to be

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_x^2 + u_t^2)\,dx.$$

   Show that the energy of the solution to the wave equation is constant with respect to time.

Solution.

   a. The solution to the aforementioned wave equation is

$$u(x,t) = \frac{1}{2}\left(f(x+t) - f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds,$$

   which is d'Alembert's solution. Therefore, we need the interval $(x-t, x+t)$ to be contained entirely outside $(-R, R)$ for the solution to be zero. To do so, we can take $S(t) = t + R$. Then when $x \geq t + R$, our solution is determined in the region $(R, R+2t)$ or further to the right. When $x \leq -t - R$, then our solution is determined from $(-2t - R, -R)$ or further to the left. In either case, the functions $f(x)$ and $g(x)$ are strictly zero, so $u(x,t) \equiv 0$. As we assume $t \geq 0$, then $S(t)$ is strictly positive.

b. We check that $\frac{d}{dt}E(t) = 0$. This is straightforward.

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(u_x^2 + u_t^2)\,dx \\
&= \frac{1}{2}\int_{-t-R}^{t+R}\frac{d}{dt}\left(u_x^2 + u_t^2\right)\,dx \\
&= \int_{-t-R}^{t+R} u_x u_{xt} + u_t u_{tt}\,dx \\
&= \int_{-t-R}^{t+R} u_x u_{xt} + u_t u_{xx}\,dx \\
&= \int_{-t-R}^{t+R} (u_t u_x)_x\,dx \\
&= u_t(t+R,t)u_x(t+R,t) - u_t(-t-R,t)u_x(-t-R,t) \\
&= 0,
\end{aligned}$$

as we showed above that $u \equiv 0$ for $|x| \geq t + R$, so that the derivative in those regions must be zero. Hence, $E(t)$ is constant.

$\square$

7. Recall that the fundamental solution of the diffusion equation $u_t - ku_{xx} = 0$ is

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}} \quad \text{for } t > 0.$$

a. Show that $S(x,y,t) = S(x,t)S(y,t)$ solves the two-dimensional diffusion equation $u_t = k(u_{xx} + u_{yy})$ for $t > 0$.

b. Use the above part to find a solution for the following problem

$$u_t = k(u_{xx} + u_{yy}) \qquad -\infty < x, y < \infty,\ t > 0$$

$$u(x,y,0) = \phi(x,y) \qquad -\infty < x, y < \infty.$$

Solution.

a. We have $S(x,y,t) = \frac{1}{4k\pi t}e^{-\frac{x^2+y^2}{4kt}}$. It is a straightforward calculation to show

$$\begin{aligned}
S_t &= -\frac{1}{4k\pi t}e^{-\frac{x^2+y^2}{4kt}} + \frac{x^2+y^2}{16k^2\pi t^3}e^{-\frac{x^2+y^2}{4kt}} \\
S_{xx} &= \frac{x^2}{16k^3\pi t^3}e^{-\frac{x^2+y^2}{4kt}} - \frac{2}{16k^2\pi t^2}e^{-\frac{x^2+y^2}{4kt}} \\
S_{yy} &= \frac{y^2}{16k^3\pi t^3}e^{-\frac{x^2+y^2}{4kt}} - \frac{2}{16k^2\pi t^2}e^{-\frac{x^2+y^2}{4kt}}.
\end{aligned}$$

It is easy to check that $k(S_{xx} + S_{yy}) = S_t$, which is what we sought to prove.

b. Using part (a), we have

$$u(x,y,t) = \frac{1}{4\pi kt} \iint e^{-\frac{(x-s)^2+(y-t)^2}{4kt}} \phi(s,t)\, ds\, dt.$$

$\square$

8. Consider the initial value problem

$$\begin{aligned} &3u_{tt} + u_{xx} - 4u_{xt} = 0 \\ &u(x,0) = x \\ &u_t(x,0) = 0 \end{aligned}$$

for $-\infty < x < \infty$ and $t > 0$.

a. Classify the PDE as parabolic, hyperbolic, or elliptic.
b. Find the solution to the above problem.

Solution.

a. We have $a_{11} = 1$, $a_{22} = 3$ and $a_{12} = -2$, so that

$$a_{12}^2 = 4 > a_{11}a_{22} = 3;$$

hence, the equation is hyperbolic.

b. It isn't too difficult to see the answer through inspection, but let's proceed as normal. Factoring the operator, we have

$$\left(3\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0.$$

From this information, we obtain $u(x,t) = f(x+t) + g(3x+t)$ (see solution to homework problem 2.1.9 for technique to obtain this result). Using our initial data, we have

$$\begin{aligned} u(x,0) &= f(x) + g(3x) = x \\ u_t(x,0) &= f'(x) + g'(3x) = 0. \end{aligned}$$

Differentiating the first with respect to x gives us the system

$$\begin{aligned} f'(x) + 3g'(3x) &= 1 \\ f'(x) + g'(3x) &= 0. \end{aligned}$$

Solving this system, we have $f(x) = -x/2 + C_1$ and $g(x) = x/2 + C_2$. Therefore,

$$u(x,t) = -(x+t)/2 + C_1 + (3x+t)/2 + C_2.$$

Applying the initial data, we see $C_1 + C_2 = 0$, so that $u(x,t) = x$.

$\square$

9. (Existence of Negative Eigenvalues) This problem uses Theorem 3 on page 118 in Strauss.

   a. Show that the condition $f(b)f'(b) - f(a)f'(a) \leq 0$ is valid for any function $f(x)$ that satisfies the Dirichlet, Neumann, or periodic boundary conditions.

   b. Show that it is also valid for the Robin boundary conditions provided that the constants a_0 and a_ℓ are positive.

   Solution.

   a. Under the Dirichlet boundary condition, we have $f(a) = f(b) = 0$, so that $f(b)f'(b) - f(a)f'(a) = 0$. Similarly, under the Neumann condition, $f'(a) = f'(b) = 0$, so that again, we satisfy the inequality trivially. Under the period boundary condition, $f(b) = f(a)$ and $f'(b) = f'(a)$ so that $f(b)f'(b) - f(a)f'(a) = 0$, again satisfying the inequality.

   b. If $a_0 > 0$ and $a_\ell > 0$, then we have $f'(a) = a_0 f(a)$ and $f'(b) = -a_\ell f(b)$. Plugging this into our expression, we have

   $$f(b)f'(b) - f(a)f'(a) = -a_\ell f(b)^2 - a_0 f(a)^2.$$

   As both variables are positive, this expression must be non-positive, completing the proof.

$\square$

10. (Calculation of Fourier series) Find the cosine series for x^4 and use it to calculate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

Solution. After some fun integration, we obtain

$$A_n = \frac{8\ell^4(-1)^n(n^2\pi^2 - 6)}{n^4\pi^4},$$

for $n > 0$, and $A_0 = 2\ell^4/5$. Putting this together,

$$x^4 = \frac{\ell^4}{5} + \frac{8\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n(n^2\pi^2-6)}{n^4}\cos\left(\frac{n\pi x}{\ell}\right).$$

Note that this could also be obtained by integrating our Fourier cosine series for x^2 twice.

Now, plugging in $x = 0$, we have

$$\begin{aligned} 0 &= \frac{1}{5} + \frac{8}{\pi^4}\left(\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} - 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right) \\ &= \frac{1}{5} + \frac{8}{\pi^4}\left(-\frac{\pi^4}{12} - 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right), \end{aligned}$$

where we have used the previous result that $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2 = \pi^2/12$. Solving for our sum, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = -\frac{7\pi^4}{720}.$$

□

11. Find the full Fourier series in real and complex forms for $\sin(x)$ on $(-\ell, \ell)$, assuming ℓ is not an integer multiple of π.

Solution. Note that $\sin(x) = (e^{ix} - e^{-ix})/2i$. Therefore, let us first calculate the Fourier complex coefficients a_n for e^{ix}. Then

$$\begin{aligned} a_n &= \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{ix}e^{-in\pi x/\ell}\,dx \\ &= \frac{1}{2\ell}\left(e^{xi-\frac{n\pi xi}{\ell}}/(i - n\pi i/\ell)\right)\Big|_{x=-\ell}^{x=\ell} \\ &= \frac{(-1)^n\sin(\ell)}{\ell - n\pi}. \end{aligned}$$

Calculating the coefficient b_n for e^{-ix} is similar. After doing so, we have

$$\begin{aligned} c_n &= (a_n - b_n)/2i \\ &= \frac{(-1)^{n+1}in\pi\sin(\ell)}{\ell^2 - n^2\pi^2}. \end{aligned}$$

Therefore, our complex series is

$$\sin(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} i n\pi \sin(\ell)}{\ell^2 - n^2\pi^2} e^{inx\pi/\ell}.$$

Upon applying the trigonometric expansion of the exponential, and noting that $n\cos(n\pi x/\ell)$ is odd with respect to n, we have

$$\sin(x) = 2\pi \sin(\ell) \sum_{n=1}^{\infty} \frac{(-1)^n n}{\ell^2 - n^2\pi^2} \sin\left(\frac{n\pi x}{\ell}\right).$$

Note: If you do not see how the $\cos(\cdot)$ term cancels, write out the expansion of the complex form:

$$\begin{aligned}\sin(x) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} i n\pi \sin(\ell)}{\ell^2 - n^2\pi^2} e^{inx\pi/\ell} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} i n\pi \sin(\ell)}{\ell^2 - n^2\pi^2} \left(\cos(n\pi x/\ell) + i\sin(n\pi x/\ell)\right) \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n} \pi \sin(\ell)}{\ell^2 - n^2\pi^2} \left(-n\cos(n\pi x/\ell) i + n\sin(n\pi x/\ell)\right),\end{aligned}$$

but because the sum ranges from $-\infty$ to ∞, for every positive n, we are adding $-n$ to the same sum. As $\cos(\cdot)$ is an even function, $n\cos(n\pi x/\ell)$ is odd with respect to n. Hence, each of the positive n terms cancels with the negative n term. Furthermore, as $n\sin(n\pi x/\ell)$ is even, we can re-write the sum from 0 to ∞ after multiplying by two to account for the $-n$ term which is equal to n by evenness. The result follows. $\square$