Nathan Brown
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Partial Differential Equations Ma131

Practice Questions for Final Examination

Note: I am not familiar with all that you have or have not covered in the course, so these questions will likely not be entirely comprehensive and might require material you have not seen. I have not seen the final exam, nor do I know what will be asked on it. These practice problems are not meant to be representative of what will be on the final examination. If you are unsure as to whether you have the requisite background to answer a question, ask me, and I will tell you what you need to know to answer it. Good luck.

1. (Convergence of Fourier series)
   For each of the following functions $f : [0, \ell] \to \mathbb{R}$, state whether the Fourier cosine series on $[0, \ell]$ converges uniformily, pointwise, and/or in L^2. If it converges pointwise, state what it converges to for each $x \in [0, \ell]$.

   a. $f(x) = x \sin^2(x\pi/\ell)$
   b. $f(x) = 0$ for $0 \leq x \leq \ell/2$ and $f(x) = 1$ else.

2. (Variable Coefficient Equation)
   Solve the following questions concerning the variable coefficient equation:

   a. Find the general solution of the partial differential equation
   $$u_x + 2yu_y = 0.$$
   b. If we require $u(0, y) = y$, find the particular solution to part (a).
   c. Find the general solution to the partial differential equation
   $$u_x + 2yu_y = x.$$

3. Consider the equation
   $$u_t = Ku_{xx} - \alpha u,$$
   where $\alpha > 0$. This models a one-dimensional rod with heat loss through the lateral sides with zero outside temperature. Let the rod have length L, with
   $$u(0, t) = u(L, t) = 0.$$

a. The equilibrium temperatures are the functions u constant with respect to time, thus solving

$$u_{xx} - \alpha u = 0$$
$$u(0) = u(L) = 0.$$

Find the solutions for $u(x)$.

b. Solve the boundary problem given above (in the introduction to the problem) with initial condition $u(x,0) = f(x)$ using separation of variables.

c. Let $u(x,t)$ be your solution to part (b). Calculate $\lim_{t\to\infty} u(x,t)$. (Note that your answer to part (c) should agree with your answer to part (a)).

4. Use the coordinate method to solve

$$\begin{aligned} u_x + u_y &= u^2 \\ u(x,0) &= f(x) \end{aligned}$$

where $f(x)$ is an arbitrary function in x.

5. Solve the partial differential equation

$$u_{tt} - u_{xx} - \frac{2}{x} u_x = 0,$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 4x^2.$$

(Hint: Make the substitution $w = xu$.)

6. Consider the wave equation $u_{tt} = u_{xx}$ which satisfies the boundary conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

where $f(x)$ and $g(x)$ are continuous and differentiable functions which are identically zero for $|x| \geq R$ for some fixed $R > 0$.

a. Show there exists a strictly positive function $S(t)$ so that $u(x,t)$ is identically zero for $|x| \geq S(t)$.

b. Define the energy $E(t)$ of $u(x,t)$ to be

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u_x^2 + u_t^2)\, dx.$$

Show that the energy of the solution to the wave equation is constant with respect to time.

7. Recall that the fundamental solution of the diffusion equation $u_t - ku_{xx} = 0$ is

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \quad \text{for } t > 0.$$

   a. Show that $S(x,y,t) = S(x,t)S(y,t)$ solves the two-dimensional diffusion equation $u_t = k(u_{xx} + u_{yy})$ for $t > 0$.
   b. Use the above part to find a solution for the following problem

$$u_t = k(u_{xx} + u_{yy}) \qquad -\infty < x, y < \infty,\ t > 0$$

$$u(x,y,0) = \phi(x,y) \qquad -\infty < x, y < \infty.$$

8. Consider the initial value problem

$$\begin{aligned} &3u_{tt} + u_{xx} - 4u_{xt} = 0 \\ &u(x,0) = x \\ &u_t(x,0) = 0 \end{aligned}$$

   for $-\infty < x < \infty$ and $t > 0$.

   a. Classify the PDE as parabolic, hyperbolic, or elliptic.
   b. Find the solution to the above problem.

9. (Existence of Negative Eigenvalues) This problem uses Theorem 3 on page 118 in Strauss.

   a. Show that the condition $f(b)f'(b) - f(a)f'(a) \leq 0$ is valid for any function $f(x)$ that satisfies the Dirichlet, Neumann, or periodic boundary conditions.
   b. Show that it is also valid for the Robin boundary conditions provided that the constants a_0 and a_ℓ are positive.

10. (Calculation of Fourier series) Find the cosine series for x^4 and use it to calculate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

11. Find the full Fourier series in real and complex forms for $\sin(x)$ on $(-\ell, \ell)$, assuming ℓ is not an integer multiple of π.