Math 147 Midterm Exam Solutions

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Problem 1: Some Definitions

You can look up the definitions in the book. I won't bother to copy them.

(a) [2 points] Let $f : X \to Y$ be a smooth map of manifolds without boundary. State what it means for f to be a *submersion*. When f is a submersion, prove that $\dim(X) \geq \dim(Y)$.

Proof. Since T_xX is a vector space whose dimension is $\dim(X)$ (see problem 3), and $T_{f(x)}Y$ is a vector space of dimension $\dim(Y)$, then since $df_x : T_xX \to T_{f(x)}Y$ is surjective, this implies $\dim(X) \geq \dim(Y)$, since the dimension of the domain of a surjective linear map of vector spaces must be greater than the dimension of the range. □

(b) [3 points] Let X, Y, Z be manifolds without boundary, $Z \subset Y$ a submanifold, and let $f : X \to Y$ be a smooth map. State what it means for f to be *transverse* to Z. Prove that $f : \mathbf{R}^3 \to \mathbf{R}^2$ given by $f(x, y, z) = (x^2 + y^2 - 1, x + y + z)$ is transverse to $(0, 0) \in \mathbf{R}^2$.

Proof. Compute the derivative of f. It is a $2x3$ matrix whose rank is always 2 since $x = y = 0$ is impossible on the zeroes of the function f. Hence f is transverse to $(0, 0)$. □

Problem 2: Theorems

(a) [2 points] State the Inverse Function Theorem.

(b) [2 points] State Sard's Theorem for manifolds with boundary.

(c) [2 points] State the Local Immersion Theorem

These theorems are in the book, so I won't restate them here.

Problem 3: The Tangent Space

Let X be a k-dimensional manifold without boundary, and let $x \in X$. The tangent space $T_x X$ is defined using a parametrization $\phi : U \to V$ where $U \subset \mathbf{R}^k$ is open, and $V \subset X$ is open.

(a) [1 point] Define what it means for ϕ to be a *parametrization.*

(b) [1 point] Define the *tangent space* $T_x X$ in terms of the parametrization ϕ. These definitions are in the book.

(c) [3 points] Define the *tangent bundle* $T(X)$ for a k-dimensional manifold X. Prove that $T(X)$ is a manifold of dimension $2k$.

Proof. Let $x \in X$ and let $\phi : U \to V$ be a diffeomorphism of an open subset of $\mathbf{R}^k$ containing the origin with an open subset V of X such that $\phi(0) = x$. Now $U \times \mathbf{R}^k$ is an open subset of $\mathbf{R}^{2k} = \mathbf{R}^k \times \mathbf{R}^k$, and $V \times \mathbf{R}^k$ is an open subset of $T(X)$. I claim that the map $\Phi : U \times \mathbf{R}^k \to V \times \mathbf{R}^k$ defined by $\Phi(u, v) = (\phi(u), d\phi_u(v))$ is a diffeomorphism. Since ϕ is a diffeomorphism, $d\phi_u$ is an invertible linear map, and the map $\Psi : U \times \mathbf{R}^k \to V \times \mathbf{R}^k$ given by $\Psi(x, w) = (\phi^{-1}(x), (d\phi_{\phi(x)})^{-1}(w))$ is inverse to Φ, as both $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity. $\square$

Problem 4: The Transversality Theorem

(a) [2 points] State the Transversality Theorem.

In the book.

(b) [4 points] Let X and Y be manifolds without boundary, Y a submanifold of $\mathbf{R}^n$. Use the Transversality Theorem to prove that if $f : X \to \mathbf{R}^n$ is a smooth map, then there exists a smooth map $g : X \to \mathbf{R}^n$ such that g is transverse to Y and g is homotopic to f.

Proof. Let S denote the open unit ball in $\mathbf{R}^n$. Define $F : X \times S \to \mathbf{R}^n$ by $F(x, s) = f(x) + s$. For a fixed x, the map $S \to \mathbf{R}^n$ given by $s \mapsto f(x) + s$ is a submersion, since its derivative is the identity map from $\mathbf{R}^n$ to itself. Hence F itself is a submersion, and it follows easily that F is transverse to any boundaryless submanifold $Z \subset \mathbf{R}^n$. By the transversality theorem, for almost every $s \in S$, the restriction map $f_s = F|_{X \times \{x\}} : S \to \mathbf{R}^n$ is transverse to Z. Define a homotopy $H : X \times I \to \mathbf{R}^n$ by $H(x, t) = F(x, st)$. Then $H(x, 0) = f(x)$, and $H(x, 1) = f_s$, and H is a homotopy from f to f_s. □

Problem 5: Applications of Transversality

(a) [4 points] Prove that every smooth map $f : S^k \to S^n$ is homotopic to a constant map if $k < n$. You may use that stereographic projection is a diffeomorphism.

Proof. Let $x \in S^n$ be a regular value for f. Then $x \notin f(S^k)$ since f is transverse to $\{x\}$ and $k < n$. By choosing coordinates in $\mathbf{R}^{n+1}$ (the ambient space for S^n), we may assume that $x = (0, \ldots, 0, 1)$. Let $\pi : S^n - \{x\} \to \mathbf{R}^n$ be stereographic projection (a diffeomorphism). Hence we have the composed map $\pi \circ f : S^k \to \mathbf{R}^n$. Let $H : S^k \times I \to \mathbf{R}^n$ be $H(x,t) = t\pi \circ f(x)$. Then H is a homotopy from a constant map $H(x,0) = h_0$ to $\pi \circ f$. It follows that $\pi^{-1} \circ H : S^k \times I \to S^n - \{x\}$ is a homotopy from the constant map $\pi^{-1} \circ H(x,0)$ to $\pi^{-1} \circ \pi \circ f = f$. □

(b) [4 points] Use mod 2 degree to show that the identity map $id : S^n \to S^n$ is not homotopic to a constant map.

Proof. Note that S^n is both compact and connected. If $f : X \to Y$ is a smooth map from a compact manifold X to a connected manifold Y of the same dimension, then its mod 2 degree $I_2(f, \{y\})$ is independent of y. Also, if two maps $f, g : X \to Y$ are homotopic, then they have the same mod 2 degree. The mod 2 degree of a constant map is zero (all points but that constant value are regular values, and their inverse image is empty), whereas the mod 2 degree of the identity map is 1 (the identity map is a diffeomorphism, and the cardinality of the inverse image must be 1). Hence the identity and the constant map cannot be homotopic. □

Problem 6: The Whitney Immersion Theorem [4 points]

Prove the Whitney Immersion Theorem: For every k-dimensional manifold X, there exists an immersion $f : X \to \mathbf{R}^{2k}$. Hint: Consider a regular value of a map $g : T(X) \to \mathbf{R}^N$ given by $g(x, v) = df_x(v)$ for an immersion f.

Proof. Let $X \subset \mathbf{R}^N$. If $N \leq 2k$ we are done. If $N > 2k$, the following procedure produces an immersion of X into $\mathbf{R}^{N-1}$. So if $N = 2k + l$, repeat this procedure l times to produce an immersion of X in $\mathbf{R}^{2k}$. Let $f : X \to \mathbf{R}^N$ be an immersion (for instance, the inclusion of X in $\mathbf{R}^N$ is an immersion). Define $g : T(X) \to \mathbf{R}^N$ by $g(x, v) = df_x(v)$. Let $a \in \mathbf{R}^N$ be a regular value, which exists by Sard's Theorem. Since $\dim(T(X) = 2k < N$, this means a is not in the image of g. Note that $a \neq 0$ since 0 is in the image of g. Identify $\mathbf{R}^{N-1}$ as the orthogonal complement of the span of a, and let $\pi : \mathbf{R}^N \to \mathbf{R}^{N-1}$ be the orthogonal projection onto that subspace. We claim that $\pi \circ f : X \to \mathbf{R}^{N-1}$ is an immersion, that is, that its derivative is an injective linear map for each x. By the chain rule, $d(\pi \circ f)_x = d\pi_{f(x)} df_x$. If $d\pi_{f(x)} df_x(v) = 0$, then $df_x(v) \in \ker(d\pi_{f(x)}$ and hence $df_x(v) = ta$ for some nonzero $t \in \mathbf{R}$ (unless $v = 0$ in which case we are done). But this implies that $g(x, (1/t)v) = a$, which is impossible since a is a regular value for g. □