

Problems on Linking Number and the Borsuk-Ulam Theorem

Math 147

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In addition to exercises 2 and 3 of chapter 2, section 6, please complete the following problems. As indicated on the web, this assignment is due Friday, May 26.

Problem 1: Prove the generalized bisection theorem: Given $(n + 1)$ open sets in \mathbf{R}^{n+1} , prove that there is some n -plane which simultaneously bisects them. You should mimic the proof I gave in class in the case $n = 1$. Some people refer to the $n = 2$ case as the “Ham Sandwich Theorem” (one open set is the ham, and the other two are slices of bread (or maybe one is bread and the other cheese, I don’t remember), and the bisecting plane is a knife).

Problem 2: Prove that the Hopf link has mod 2 linking number equal to 1 according to the following outline (although you can choose to not follow the outline). Let S^1 be represented by the unit circle in the plane. The pair of maps $f_1, f_2 : S^1 \rightarrow \mathbf{R}^3$, where $f_1(x, y) = (x, y, 0)$ and $f_2(z, w) = (0, z - 1, w)$ constitute the Hopf link.

- (a) Draw a picture of the images of f_1 and f_2 in \mathbf{R}^3 . Prove that $f_1(S^1) \cap f_2(S^1) = \emptyset$.
- (b) Form the map $F : S^1 \times S^1 \rightarrow S^2$ as we did in class, $F(x, y, z, w) = (f_1(x, y) - f_2(z, w)) / |f_1(x, y) - f_2(z, w)|$. Find all solutions to $F(x, y, z, w) = (0, 0, -1)$.
- (c) Prove that $(0, 0, 1)$ is a regular value for F , and use your result from part (b) to conclude that the mod 2 linking number of f_1 and f_2 is 1. You will need to locally parametrize both $S^1 \times S^1$ and S^2 in order to compute the derivative of F .