Math 147 Final Exam

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This exam may be found at http://math.stanford.edu/∼munson/finalexam.pdf

Rules: This exam is to be turned in to my office (382-H) by 12:00 PM on Monday June 12th. If you wish to turn it in earlier, please send me an e-mail. You must work by yourself, and you may only discuss this exam with me. You may use only your textbook, notes from class, homework, midterm exam, and the solutions to the midterm exam. And remember the honor code.

Guidelines for writing solutions: If you use a theorem as part of a proof, you must carefully check that its hypotheses are satisfied. You do not need to copy the statement of the theorem in your solutions (although this is acceptable), but you should give an explicit reference to the theorem used (e.g., "by Sard's Theorem on page 39 of the text . . ."). You may also cite homework problems that I have assigned, but only those. Thus if you find the result of a problem from the text useful for a solution, but it is not a problem I assigned during the quarter, in order to use the result you must complete the exercise. It is very important that your solutions be carefully cited and complete in this way, because I am allowing you the use of your book, notes, and homework. You should justify every step of your solutions. Only hand in your neatly written complete solutions.

Format: There are six problems. Problem 5 has two parts, and problem 6 has three parts. There are a total of 31 possible points, distributed unevenly. The total number possible for each part is indicated in brackets. Good luck!

Questions: If you have any questions, please send me an e-mail. I check it very regularly and will get back to you very soon. If you need to meet with me, please send me an e-mail to make an appointment. If you are unsure about whether or not you need to justify something, please ask. Many of the points awarded for correct solutions are specifically tied to particular observations and their justification, so if you fail to supply proof at certain times, you will lose points. Err on the side of saying too much. If anyone brings up a point that I think needs clarification, I will notify the entire class of this via e-mail.

Problem 1:[6 points] Let X be a compact k-dimensional manifold, where $0 < k < n$, and suppose $f : X \to S^n$ is a smooth map, and that $Y \subset S^n$ is a submanifold of dimension $(n-k)$. Prove that there is a map f_1 which is homotopic to f such that $f_1 : X \to S^n - Y$ (that is, f is homotopic to a map which misses Y).

Problem 2:[6 points] Suppose $f : S^1 \to \mathbf{R}^4$ is a smooth embedding, so that $f(S^1)$ is a compact submanifold of $\mathbf{R}^4$. Prove that the complement $\mathbf{R}^4 - f(S^1)$ is simply-connected.

Problem 3:[4 points] Let X, Y and Z be k-dimensional oriented manifolds without boundary. Suppose X and Y are compact, and that Y and Z are connected. Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Prove that $\deg(g \circ f)$=$\deg(g) \cdot \deg(f)$. By degree here I mean oriented degree.

Problem 4:[2 points] Let $f : S^2 \to S^1$ be a smooth map. Represent S^2 as the unit sphere in $\mathbf{R}^3$. Prove that the restriction of f to the subset $\{(x, y, z) \in S^2 \subset \mathbf{R}^3 : z = 0\}$ has (oriented) degree zero. S^2 and S^1 are respectively oriented as the boundaries of the closed unit balls in $\mathbf{R}^3$ and $\mathbf{R}^2$.

Problem 5: For a pair of maps $f_1, f_2 : S^1 \to S^3$ such that $f_1(S^1) \cap f_2(S^1) = \emptyset$, define their (oriented) linking number $L(f_1, f_2)$ as the degree of the map $F : S^1 \times S^1 \to S^2$ given by $F(x, y) = (f_1(x) - f_2(y))/|f_1(x) - f_2(y)|$. That is, $L(f_1, f_2) = \deg(F)$.

(a)[2 points] Prove that $L(f_1, f_2) = \pm L(f_2, f_1)$ (and determine whether $L(f_1, f_2) = L(f_2, f_1)$ or $L(f_1, f_2) = -L(f_2, f_1)$).

(b)[3 points] Prove that if $f_1(S^1)$ is the boundary of an oriented submanifold W of $\mathbf{R}^3$ that meets $f_2(S^1)$ transversely, then $L(f_1, f_2) =$ the number of signed intersections of $f_2(S^1)$ with W. S^2 and S^1 are respectively oriented as the boundaries of the closed unit balls in $\mathbf{R}^3$ and $\mathbf{R}^2$, and $\mathbf{R}^n$ always is assumed to have the standard orientation. Part of this problem is to make careful sense of "signed intersection", and your book can certainly help you here.

Problem 6: Let $\mathbf{C}P^1$ be the complex projective plane. It is defined to be a quotient of $\mathbf{C}^2 - (0, 0)$ by the equivalence relation $(z, w) \sim (\lambda z, \lambda w)$ for any complex number $\lambda \neq 0$. Denote the equivalence class of (z, w) by $[z, w]$. $\mathbf{C}P^1$ is a manifold.

(a)[2 points] Find a diffeomorphism $g : \mathbf{C}P^1 \to S^2$. (Hint: For $z \neq 0$ send $[z, w]$ to $\frac{w}{z} \in \mathbf{C}$; then use the inverse of stereographic projection. Now figure out where to send $[0, 1]$.)

(b)[3 points] Consider S^3 as the set of unit vectors in $\mathbf{C}^2 = \mathbf{R}^4$ and define $f : S^3 \to S^2$ by $f(z, w) = g([z, w])$. Prove that for any $a \in S^2$, the pre-image

$f^{-1}(a)$ is diffeomorphic to a circle.

(c)[3 points] Prove that $L(f^{-1}(a), f^{-1}(b)) = 1$ (the linking number is defined in the previous problem), where $a = g([1,0])$ and $b = g([1,1])$ and the linking number is computed in $\mathbf{R}^3 = S^3 - c$ for $c = (0, \sqrt{-1}) \in \mathbf{C}^2$ (viewing S^3 as a subset of $\mathbf{C}^2$). We orient S^3 and S^2 as the boundaries of the closed unit balls in $\mathbf{R}^4$ and $\mathbf{R}^3$, respectively, and give the pre-images the pre-image orientation. Part of this problem is to figure out what I mean by "pre-image orientation": you should figure out how the various given orientations induce orientations on $f^{-1}(a)$ and $f^{-1}(b)$.