Math 115, Fall 2003
Convergence and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$

One thing we never got to talk about in class was

Proposition 1. *$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.*

Since we use these series all the time in the comparison test to prove convergence and divergence of other series, it's important to understand why this proposition is true. Our book uses the integral test to prove this, and I spent some time making fun of this in class, but there are serious objections to using it, even if the ideas behind it make sense. For one, we haven't developed the concept of integral over an interval $[a, b]$, let alone over an infinite interval $[a, \infty)$. Also, there is way to prove this which doesn't use such high-powered machinery. What I mean to say is that the integral test doesn't feel like it gives the right reason why these series converge and diverge, and I don't get any new insight into these series proving this proposition this way. So I'm going to show you how to do it using (essentially) the comparison test. The method I'll describe is a generalization of the same method Ross suggests in problem 14.14 to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and I learned it from Walter Rudin's *Principles of Mathematical Analysis*. I essentially copy his arguments, but I've tried to be a bit wordier. This may have the effect of being overbearing or making it more accessible; I guess that's up to your taste. At the end, I'll work out both this and the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ explicitly (actually, I didn't have enough time to do this, so I'll leave it up to you. Sorry, but this little project took a lot longer than I thought, and I had to stop before it got way out of hand. But I'm happy to talk to any of you about this if you're really interested). One other thing that is good about this method is that it works for most series whose convergence or divergence appears only obtainable by the integral test. We begin with a theorem characterizing those series of nonnegative terms which converge. There really isn't much to this one; it's just a matter of translating back so we can use theorems about sequence we've already proved.

Theorem 2. *Let $\sum a_n$ be a series of nonnegative terms. Then $\sum a_n$ converges if and only if the sequence of partial sums is a bounded sequence.*

Proof. Well, if $\sum a_n$ converges then the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ is a convergent sequence, and convergent sequences are bounded by theorem 9.1.

Now suppose that the sequence of partial sums (s_n) is bounded. Since the a_n are all nonnegative, the sequence (s_n) is monotone increasing, and all bounded monotone sequence converge by theorem 10.2. Hence (s_n) converges and so does the series. □

Now we state and prove the theorem which we use to prove the convergence and divergence of $\sum \frac{1}{n^p}$. We'll use the theorem we just proved to make life a little easier.

Theorem 3. *Suppose* (a_n) *is a decreasing sequence of nonnegative numbers* ($a_1 \geq a_2 \geq \cdots \geq 0$). *Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the series* $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$ *converges.*

Before embarking on the proof, let's make sure we understand what this is saying. The sequence a_{2^k} is a subsequence of a_n consisting of those terms for which n is a power of 2. That is, $(a_n) = (a_1, a_2, a_3, \ldots)$ and $(a_{2^k}) = (a_1, a_2, a_4, a_8, \ldots)$.

Proof. First we derive two useful inequalities. Note that since (a_n) is decreasing, we have $a_1+(a_2+a_3)+(a_4+a_5+a_6+a_7) \leq a_1+(a_2+a_2)+(a_4+a_4+a_4+a_4) = a_1+2a_3+4a_4$. So the partial sums of $\sum a_n$ are bounded above by the partial sums of $\sum 2^k a_{2^k}$. In fact, if we denote by $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ be the n^{th} partial sum of $\sum a_n$, and $t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$ be the $2^k th$ partial sum of $\sum 2^k a_{2^k}$, we have that for $n < 2^k$

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_k \\ &= t_k \end{aligned}$$

so $s_n \leq t_k$. Note that this is exactly the first observation we made when $k = 2$.

Similarly, notice that $s_5 = a_1+a_2+a_3+a_4+a_5 \geq a_1+a_2+(a_3+a_4) \geq \frac{1}{2}a_1+a_2+2a_4 = \frac{1}{2}t_4$. Generalizing this observation as before, we see that for $n > 2^k$, we have

$$\begin{aligned} s_n &\geq s_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2(a_4) \cdots + 2^{k-1} a_k \\ &= \frac{1}{2} t_k \end{aligned}$$

so $s_n \geq \frac{1}{2}t_k$. Now we're ready to be done quickly. If $\sum 2^k a_{2^k}$ converges, this means (t_k) converges and hence is a bounded sequence, so that there is some M such that $t_k \leq M$ for all k. Since for $n < 2^k$ we have $s_n < s_{2^k}$, the sequence (s_n) is bounded too (also by M) and hence $\sum a_n$ converges. Similarly if $\sum a_n$ converges, this means (s_n) is a bounded sequence. Since for $n > 2^k$ we have $s_n \geq \frac{1}{2}t_k$, this means that $(\frac{1}{2}t_k)$ must be a bounded sequence too, and hence so must (t_k) be bounded, and thus $\sum 2^k a_k$ converges.

□

Now let's see that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$. First of all, if $p \leq 0$, then $-p \geq 0$, and $\lim(\frac{1}{n^p}) = \lim(n^{-p}) = +\infty$. So suppose $p > 0$. Then if $a_n = \frac{1}{n^p}$, certainly (a_n) is a decreasing sequence of positive numbers, so the theorem applies. We see that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$$

converges. This latter series is

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} (2^{(1-p)})^k$$

which is a geometric series. We know that this converges when $2^{1-p} < 1$ and diverges otherwise. Clearly $2^{1-p} < 1$ if and only if $1 - p < 0$, so that

$$\sum_{k=0}^{\infty} (2^{(1-p)})^k$$

converges if and only if $p > 1$, and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

It would be a good exercise to see how the series you get when $p = 1$ is related to the series you are asked to use in exercise 14.14. If you follow through the proofs above with $p = 1$, you should be able to see it. It also might be helpful to see how this all works in the case $p = 2$.