1.4. (a) The formula is $1 + 3 + \cdots + (2n - 1) = n^2$ for all $n$.

(b) \ \begin{proof}
For $n = 1$, we have $1 = 1^2$. Assume that $1 + 3 + \cdots + (2n - 1) = n^2$. Then $1 + 3 + \cdots + (2n - 1) + (2(n + 1) - 1) = n^2 + 2n + 1$ by the induction hypothesis. The right side of this equality simplifies to $(n + 1)^2$. Hence, by mathematical induction, $1 + 3 + \cdots + (2n - 1) = n^2$ for all $n$.
\end{proof}

3.6. Statement (a) follows from (b), with $n = 3$. We prove by induction that if $a_i$ are real numbers, then $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$.

\begin{proof}
The case $n = 1$ states $|a_1| \leq |a_2|$, which is clearly true. Assume $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ for all real numbers $a_i$. Now $|a_1 + \cdots + a_n + a_{n+1}| \leq |a_1| + \cdots + |a_n + a_{n+1}|$ by the induction hypothesis (where we replace $a_n$ with the number $a_n + a_{n+1}$). Then the triangle inequality shows that $|a_1 + \cdots + a_n + a_{n+1}| \leq |a_1| + \cdots + |a_n| + |a_{n+1}|$. Hence, by induction, $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ for every $n$.
\end{proof}

3.8. Prove: If $a$ and $b$ are real numbers such that $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

\begin{proof}
We prove this by contradiction. Assume that $a$ and $b$ are real numbers such that $a \leq b_1$ for every $b_1 > b$, and yet $a > b$. Let $b_1 = (a + b)/2$. Then $b_1 > b$, and yet $b_1 = (a + b)/2 < a$, since $a > b$. This contradicts the fact that every number $b_1$ satisfying $b_1 > b$ also satisfies $b_1 \geq a$.
\end{proof}

Remark: As is often the case, it helps tremendously to draw a picture of the situation described in the proof. The way I understand that this proof is correct is in terms of the picture.

4.6. Let $S$ be a nonempty bounded subset of $\mathbb{R}$. (a) Prove $\inf(S) \leq \sup(S)$.

\begin{proof}
Certainly both numbers exist by the completeness axiom. By definition, we have the inequalities $\inf(S) \leq s \leq \sup(S)$ for every $s \in S$. Hence $\inf(S) \leq \sup(S)$.
\end{proof}

(b) If $\inf(S) = \sup(S)$, the $S$ contains exactly one element. This is clear from the inequality in part (a).

4.14. Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$. Denote by $S = \{a + b : a \in A, b \in B\}$. (a) Prove that $\sup(S) = \sup(A) + \sup(B)$.
Proof. We verify this by checking the definition. First we show that \( \text{sup}(A) + \text{sup}(B) \) is an upper bound for \( S \), and then we check that no number smaller than it can be an upper bound. I am using the characterization of supremum given near the bottom of page 21, which I mentioned in class.

For every \( s \in S \), we have \( s = a + b \) for some \( a \in A \) and \( b \in B \). Since \( a \leq \text{sup}(S) \) and \( b \leq \text{sup}(B) \), we must have \( s = a + b \leq \text{sup}(A) + \text{sup}(B) \) for every \( s \in S \). Suppose \( M < \text{sup}(A) + \text{sup}(B) \). Choose numbers \( x, y \) such that \( x < \text{sup}(A) \), and \( y < \text{sup}(B) \), and \( M = x + y \). For example, if \( \epsilon \) is the difference between \( M \) and \( \text{sup}(A) + \text{sup}(B) \), then we may choose \( x = \text{sup}(A) - \epsilon/2 \) and \( y = \text{sup}(B) - \epsilon/2 \). (You really should prove, as I have done, that such \( x \) and \( y \) exist.) Then by definition of supremum, there exists \( a_1 \in A \), \( b_1 \in B \) such that \( a_1 > x \) and \( b_1 > y \). Hence \( a_1 + b_1 \in S \), and \( a_1 + b_1 > x + y \), so that \( x + y = M \) cannot be an upper bound for \( S \). Hence \( \text{sup}(A) + \text{sup}(B) \) is the least upper bound of \( S \). \( \square \).