THINKING THE CALCULUS WAY
MATHEMATICAL INDUCTION

For WEDNESDAY, July 7:
You must use the definition of the derivative in the following questions. No formulas allowed!

Section 2.8: 5, 7, 13, 19, 20
Section 2.9: 1, 21, 22, 23, 26

Read the very short blurb about Mathematical Induction on page 81 of Stewart.

All of the following proofs should use mathematical induction.

1. Prove that
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$
for every integer n.

2. Prove that
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = n(n+1)(n+2)/3$$
for every integer n.

3. Prove that $n^2 < n!$ for $n \geq 4$.

4. Prove that $2^n < n!$ for $n \geq 4$.

5. The tower of Hanoi puzzle. This puzzle includes three pegs and eight rings of different sizes placed in order of size (largest on the bottom, smallest on the top) on the first peg. The goal is to move the rings, one at a time without ever placing a larger ring on top of a smaller ring, from the first peg to the second peg, using the third peg as an auxiliary peg. We are going to consider a generalized puzzle where there are n rings. You might try to think of the cases where there are 2, 3, and 4 rings to see if you understand how the game works. Use induction to show that the minimum number of moves to transfer n rings from one peg to another, according to the rules we described, is $2^n - 1$. Assuming it takes 1 second to move each ring, how many minutes would it take to move 10 rings? How many days to move 20? How many years to move 30?

6. What's wrong with the following proof that all horses are the same color? (We're not asking what's wrong with the clearly erroneous statement that all horses are the same color, but rather why the proof fails to work.) Clearly any set of 1 horse are all the same color. This completes the basis step. Now assume that all horses in any set of n horses are the same color. Consider a set of $n+1$ horses, labeled with integers $1, 2, \ldots, n, n+1$. By induction, horses $1, 2, \ldots, n$ are the same color, as are horses $2, 3, \ldots, n+1$ (because each of these sets consists of n horses). Since these two sets of horses have common members (namely $2, 3, \ldots, n$), all $n+1$ horses must be the same color.

EXTRA PROBLEM: Use mathematical induction to show that a $2^n \times 2^n$ chessboard with one square missing can be covered by L-shaped pieces, where each L-shaped piece covers three squares.