Homework 1, Math 263A: Lie Groups and Lie Algebras

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1. Use the determinant map to show that $SL(n, \mathbb{R})$ is a smooth submanifold of $GL(n, \mathbb{R})$, compute its dimension, and identify the tangent space at the identity with the space of trace zero $n \times n$ matrices. Follow the technique used in class to compute the same thing for $O(n)$, where we used the map $A \mapsto AA^t$.

2. Prove that if G is a Lie group of dimension d, then its tangent bundle is trivial. That is, prove that $TG \cong G \times \mathbb{R}^d$.

3. Check that the bracket $[X, Y]$ of two vector fields on a smooth manifold M is a derivation and hence a vector field on M.

4. Let $\phi : M \to N$ be a smooth map, and let X and Y be smooth vector fields on M and N respectively. Recall that X and Y are *ϕ-related* if $D\phi(X) = Y(\phi)$; that is, if for all smooth functions $f : N \to \mathbb{R}$, we have $D\phi(X)(f) = Y(f \circ \phi)$. For $i = 1, 2$, let X_i and Y_i be smooth vector fields on M and N respectively. Prove that if X_i is ϕ-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is ϕ-related to $[Y_1, Y_2]$.

5. Let G be a Lie group, T_eG its tangent space at the identity, and $\mathfrak{g}$ its Lie algebra of left-invariant vector fields. We defined maps $\alpha : \mathfrak{g} \to T_eG$ and $\beta : T_eG \to \mathfrak{g}$ and given by $\alpha(X) = X|_e$, and $\beta(v) = X_v$, where $X_v|_a = D_eL_a(v)$, where L_a is left-multiplication by a. Check that the composed maps $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity on T_eG and $\mathfrak{g}$ respectively.

6. Show that if G is a connected topological group, then a discrete normal subgroup is always in the center.

7. Show that every 2-dimensional Lie algebra has a basis $\{x, y\}$ such that $[x, y] = y$.

8. Show that $GL(n, \mathbf{C})$ is connected.

9. Show that a Lie algebra representation π of $\mathfrak{g}$, when tensored with the trivial representation and regarded as a representation of $\mathfrak{g}$, is isomorphic to π.

10. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Decompose the adjoint representation

$$\text{ad: } \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g})$$

as a direct sum of irreducible representations.

11. Prove that given any finite dimensional complex representation Π of a Lie group G, there is a unique complex representation π of $\mathfrak{g}$ such that

$$\Pi(e^X) = e^{\pi(X)} \quad \text{for all } X \in \mathfrak{g}$$

and that

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\bigg|_{t=0}.$$

12. If (G, Π) is a Lie group and representation with associated Lie algebra and representation $(\mathfrak{g}, \pi)$ as above, show that π is irreducible if and only if Π is irreducible.

13. Show that the adjoint representation and the standard representation are equivalent on the Lie algebra $\mathfrak{so}(3)$.

14. Some exercises from Bump: 10.1, 10.2, 11.1