Syllabus for Math 283: Calculus of Functors
Fall Quarter 2005
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For those interested in the short description, here is a list of topics that we will cover: polynomial approximation of a functor, k-excisive functors, cross effects, the Taylor tower of a functor, classification of homogeneous functors and derivatives, functors of spaces over a space, ρ-analyticity, k-cartesian and k-cocartesian cubical diagrams, Blakers-Massey theorem, disjunction for spaces of embeddings, the derivatives of the identity functor, and applications to spaces of embeddings and stable homotopy theory. For those interested in a more detailed description, read on.

In this course we will discuss a relatively new machine in algebraic topology known as the “calculus of functors” or “Goodwillie calculus”. It is a way of organizing and attacking certain kinds of questions in topology, which for us will mean differential topology and homotopy theory. It usually does not prove theorems for you, but it can help you to quickly identify the hard part of a given problem. The method is roughly this: approximate a functor F by a functor T_kF, its “k^{th} degree Taylor polynomial” which satisfies “k^{th} order excision”. Then one hopes the approximating functor is easier to deal with (it usually is, in a way), and one hopes that it says something interesting about the original functor (it usually does). It is very informative to study the difference between the k^{th} and the $(k-1)^{st}$ approximations, which is an example of a ”homogeneous functor of degree k”, because such functors can be classified.

There are three flavors of calculus, known as manifold calculus, homotopy calculus, and orthogonal calculus. They were invented in this order, and we will study them in this order, although the homotopy calculus has seen much wider use than the other two. The manifold calculus studies functors from the category of open subsets of a smooth manifold to spaces, the homotopy calculus studies functors from spaces or spectra to spaces or spectra, and the orthogonal calculus studies functors from the topological category whose objects are finite dimensional vector spaces with an inner product and whose morphisms are linear isometric inclusions to the category of spaces. Examples of functors of interest in the manifold calculus are functors which assign to an open set U of a smooth manifold M the space of embeddings of U in a smooth manifold N, or the space of diffeomorphisms of $M-U$, or the space of homotopy equivalences of $M-U$. Examples of functors in the homotopy calculus include the identity functor and the functor which assigns to X the space of maps of a complex K into X. Finally, some random examples of functors in the orthogonal calculus include those that assign the vector space V to $BO(V)$, S^V, and the space of embeddings of a smooth compact manifold M in $N \times V$. We will spend most of our time discussing the first two types of calculus, mostly because I know those the best, but I hope to be able to say something about orthogonal calculus near the end of the quarter.

What makes this machine effective is knowing that these polynomial approximations tell you something interesting about the original functor. The way you get such information is to prove that the transformation of functors $F \to T_k F$ has connectivity which increases with k. Proving statements like these is typically hard, although it doest not require a deep understanding of the $T_k F$. What it does require is some knowledge of cubical diagrams and the generalized Blakers-Massey theorem, and we will spend some time discussing these things.

There will be two main lines of inquiry: polynomial approximations and the classification of homogeneous functors in both the manifold and homotopy calculus, and cubical diagrams and the generalized Blakers-Massey theorem. We will weave these threads together and prove some interesting results about particular examples, such as those described above. I will try to emphasize examples whenever possible so that you have an idea of the kinds of theorems this machine can help you organize and prove.

There is no textbook for this course, but there are many papers which are worth taking a look at, some of which I've cited below. We will work directly from several of these papers throughout the quarter, especially [4], [5], [9], and [7]. I will distribute copies of these four papers.

References

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[2] G. Arone and M. Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743-788.

[3] T. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K-theory **4** (1990), 1-27.

[4] T. Goodwillie, *Calculus II: Analytic functors*, K-theory **5** (1992), 295-332.

[5] T. Goodwillie, *Calculus III: Taylor series*, Geom. Topol. **7** (2003), 645-711.

[6] T. Goodwillie, J. Klein and M. Weiss, *Spaces of smooth embeddings, disjunction and surgery*, Surveys on surgery theory, Vol. 2, 221-284, Ann. of Math. Stud., 149, *Princeton Univ. Press, Princeton, NJ*, 2001.

[7] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory, Part II*, Geometry and Topology **3** (1999), 103-118.

[8] B. Johnson, *The derivatives of homotopy theory*, Trans. Amer. Math. Soc. **347** (1995), no. 4, 1295-1321.

[9] M. Weiss, *Embeddings from the point of view of immersion theory, Part I*, Geometry and Topology **3** (1999), 67-101.

[10] M. Weiss, *Calculus of Embeddings*, Bull. Amer. Math. Soc. **33** (1996), 177-187.

[11] M.Weiss, *Orthogonal calculus*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 3743-3796.