Polya Seminar – November 30, 2009

1. (1985 - A1) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

   (a) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

   (b) $A_1 \cap A_2 \cap A_3 = \emptyset$,

   where $\emptyset$ denotes the empty set. Express your answer in the form $2^a 3^b 5^c 7^d$, where $a, , b, c, d$ are nonnegative integers.

2. (1986 - A3) Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2 + n + 1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

3. (1988 - A1) Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

4. (1988 - B1) A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$ with $x, y,$ and z positive integers.

5. (1989 - A1) How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

6. (1989 - A4) If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $1/2$. A game is finite if with probability 1 it must end in a finite number of moves.)

7. (1989 - B1) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$ where a, b, c, d are positive integers.

8. (1993 - B3) Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

9. (1994 - A1) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$. Prove that the sum $\sum_{n=1}^{\infty} a_n$ diverges.

10. (1996 - A1) Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangle.

11. (1998 - A1) A right circular cone has base radius 1 and height 3. A cube is inscribed in the cone so that the cube is contained in the base of the cone. What is the side-length of the cube?

12. (1998 - A6) Let A, B, C denote distinct points with integer coordinates in the plane. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

13. (1998 - B1) Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

14. (1999 - B6) Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

Master class

1. (Melanie Wood) If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum
$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$
of binomial coefficients is divisible by p^2. (Putnam 1996 - A5)

2. (Minh Thao Nguyen) Prove:
$$\sum_{k=1}^{p-1} \frac{(y-x)(p(y+x)+2k)}{((px+k)(py+k))^2}$$
is divisible by p^2 for all prime $p > 5$ and x, y positive integers.

3. (Pak Hin Lee) Let $d_r(n)$ be the number of positive divisors of n of the form $3k + r$. Prove that $d_1(n) \geq d_{-1}(n)$ for all positive integers n.

4. (Temple He) A restless lion roamed about his circular cage, of radius 10 meters, along a polygonal path of total length 30 km. Prove that, counting both left turns and right turns (considering each angle to be positive), he must have turned through a total of at least 2998 radians.

5. (Nathaniel Shar) In the game of Lion vs. Man, the Lion and Man begin at points in the first quadrant of the plane. They take turns moving a distance of up to 1 in any direction, always remaining in the first quadrant. If the Lion ever moves to the same point as the Man, the Lion eats the Man and wins. If the Man can prevent the Lion from doing this within any finite number of moves, the Man wins. Determine exactly when each player has a winning strategy. (This problem is by David Gale.)

6. (Elina Robeva) Find all integers n for which there exists a permutation $\sigma \in \mathcal{S}_n$ such that
$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\sigma(3) + \cdots + \sqrt{\sigma(n-1) + \sqrt{\sigma(n)}}}}}$$
is a rational number. (Balkan Mathamatical Olympiad 2007)