1. Suppose that \( f : [0, 1] \to [0, 1] \) is continuous. Prove that there exists a number \( c \in [0, 1] \) such that \( f(c) = c \). (Note: this is the one-dimensional case of the celebrated Brouwer fixed point theorem.)

2. For a nonzero real number \( x \) prove that \( e^x > x + 1 \).

3. A real-valued continuous function satisfies for all real \( x \) and \( y \) the functional equation
   \[
   f(\sqrt{x^2 + y^2}) = f(x)f(y).
   \]
   If \( f(1) = 1 \) show that \( f(x) = 1 \) for all \( x \).

4. (Putnam 1988 – A2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that \((fg)' = f'g'\). If \( f(x) = e^{x^2} \), determine, with proof, whether there exists an open interval \((a, b)\) and a nonzero function \( g \) defined on \((a, b)\) such that this wrong product rule is true for \( x \) in \((a, b)\).

5. (Putnam 1987 – B1) Evaluate
   \[
   \int_{2}^{4} \frac{\sqrt{\log(9-x)}dx}{\sqrt{\log(9-x)} + \sqrt{\log(x+3)}}.
   \]

6. (Putnam 1995 – A2) For what pairs \((a, b)\) of positive real numbers does the improper integral
   \[
   \int_{b}^{\infty} \left( \sqrt{x + a} - \sqrt{x} - \sqrt{x - \sqrt{x - b}} \right) dx
   \]
   converge?
7. (Putnam 2000 – A4) Show that the improper integral

\[
\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2) \, dx
\]

converges.

8. (Putnam 1998 – A3) Let \( f \) be a real function on the real line with continuous third derivative. Prove that there exists a point \( a \) such that

\[
f(a) f'(a) f''(a) f'''(a) \geq 0.
\]

9. Evaluate

\[
\lim_{n \to \infty} \frac{1}{n^2} \prod_{i=1}^{2n} \left( n^2 + i^2 \right)^{1/n}.
\]

10. (Putnam 1996 – A6) Let \( c \geq 0 \) be a constant. Give a complete description, with proof, of the set of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = f(x^2 + c) \) for all \( x \in \mathbb{R} \).