Problems for Polya Seminar – October 12, 2009

(1) (a) Given \( n + 1 \) numbers chosen from \( \{1, 2, \ldots, 2n\} \), show that some two of them are relatively prime.
(b) Given \( n + 1 \) numbers chosen from \( \{1, 2, \ldots, 2n\} \), show that one of them divides another.

(2) Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements.

(3) Given nine points in a unit square, show that some three of them form a triangle whose area does not exceed \( 1/8 \).

(4) If the plane is colored with finitely many colors, show that one can find a rectangle with sides parallel to the axes and four vertices of the same color.

(5) Prove that there is a Fibonacci number evenly divisible by 1000.

(6) Prove that any convex polyhedron has two faces with the same number of edges.

(7) Let \( x_1, x_2, \ldots, x_k \) be real numbers such that the set
\[
A = \{ \cos (n\pi x_1) + \cos (n\pi x_2) + \cdots + \cos (n\pi x_k) \mid n \geq 1 \}
\]
is finite. Prove that the \( x_i \) are all rational numbers.

(8) Given five points in a 3–4–5 triangle, prove that some pair of the points is within distance 2.