Supplemental notes on Section 2.3

These comments are meant to fill in the details for things I thought were slightly vague in our text. I hope you find them helpful.

(1) What does the book mean when it says that exponentiation and multiplication of sets works in a similar way to numbers? Here is one way to make sense of this comment.

The point is that the sets $C^{A \times B}$ and $(C^B)^A$ are closely related. What do we mean by closely related? Well, even if two sets are not exactly the same set, perhaps they are in one-to-one correspondence, in a particularly nice way. Can you find a one-to-one correspondence between these two sets? Try working through the following two examples, and then coming back to this.

(2) Perhaps an easier example of two sets that are “closely related” is the power set $\mathcal{P}(X)$ and the exponential set $\{0, 1\}^X$. How are they related?

Suppose $S \in \mathcal{P}(X)$. How can I find a function $f_S : X \mapsto \{0, 1\}$? What I mean is: given a subset of $X$, is there a function I can naturally associate with the subset? The answer is yes, and it is sometimes called the characteristic function of the subset, defined as follows: $f_S(x) = 1$ if $x \in S$, and $f_S(x) = 0$ if $x \notin S$.

Similarly, given a function $f : X \mapsto \{0, 1\}$, can we find a nice subset $S_f$ somehow corresponding to this function? We define $S_f = \{x | f(x) = 1\}$.

Define a function $T_1 : \{0, 1\}^X \mapsto \mathcal{P}(X)$ defined by $T_1(f) = S_f$. Check that $T_1$ is one-to-one and onto. (Hint: define $T_2(S) = f_S$ and show that $T_1$ and $T_2$ are inverses.)

(3) What is the relation between the two different kinds of products of sets we have seen? In particular, we have seen one definition $A_1 \times A_2 = \{(a_1, a_2) | a_1 \in A_1 \text{ and } a_2 \in A_2\}$. Another definition is $\prod_{i \in \{1, 2\}} A = \{f : \{1, 2\} \mapsto A_1 \cup A_2 \text{ such that } f(1) \in A_1 \text{ and } f(2) \in A_2\}$.

Can we give a nice bijection between these two sets? Sure. Given $x \in A_1 \times A_2$, write $x = (a_1, a_2)$, and define $f_x$ by $f_x(0) = a_1$, and $f_x(1) = a_2$. Similarly, given $g \in \prod_{i \in \{1, 2\}} A$, define $a_g \in A_1 \times A_2$ by $a_g = (g(0), g(1))$. Check that this correspondence is bijective.