Sec 3.2, Ex 19  Suppose $M$ is another subgroup of $G$ of order $|N|$. To show that $M = N$, it suffices to show $|N : M \cap N| = 1$, for then $M \cap N = N$, which means $M = N$ since they have the same order.

Since $N$ is normal in $G$, $M$ is contained in the normaliser of $N$. Hence $MN$ is a subgroup of $G$, and we have the lattice of subgroups of $G$:

\[
\begin{array}{c}
MN \\
M \\
N \\
M \cap N
\end{array}
\]

From this we have $|N : M \cap N| = |MN : M| = |MN : N|$, which must divide the index $|G : N|$ (we have the inclusions $N \subset MN \subset G$).

On the other hand, $|N : M \cap N|$ must also divide $|N|$. So the positive integer $|N : M \cap N|$ divides both $|N|$ and $|G : N|$, which are coprime, forcing $|N : M \cap N|$ to be 1.

sec 3.2, Ex 21  Suppose $Q$ has a proper subgroup $H$ of finite index. Say $n = |Q : H| = |Q/H|$ (since $Q$ is abelian, any subgroup is normal, and so we can quotient by it). Then for any $q \in Q$, we have $n(q + Q) = 0 + Q$, the identity element in $Q/H$. That is, for any $q \in Q$, we have $nq \in H$. But for any $q \in Q$, $q/n$ is also in $Q$ ($n$ is a positive integer), and so $n(q/n) = q$ is in $H$. We have shown $Q \subset H$, contradicting $H$ being a proper subgroup. Therefore, $Q$ does not have a proper subgroup with finite index.

Consider the quotient homomorphism $\phi : Q \to Q/Z$. If $Q/Z$ had a proper subgroup $H$ of finite index, then $\phi^{-1}(H)$ would also be proper subgroup of finite index (note that $|Q : \phi^{-1}(H)| = |Q/Z : H|$). So $Q/Z$ cannot have a proper subgroup of finite index by the previous paragraph.

Sec 3.3, Ex 10  (The techniques here are similar to those used in Exercise 19 in Section 3.2.) Since $N$ is normal in $G$, $HN$ is a subgroup of $G$, and we can consider the lattice of subgroups of $G$:

\[
\begin{array}{c}
HN \\
H \\
N \\
H \cap N
\end{array}
\]
Since $H$ is a Hall subgroup of $G$, it is also a Hall subgroup of $HN$. But $|HN : H| = |N : H \cap N|$, so $1 = ([H], |HN : H|) = ([H], |N : H \cap N|)$, and so $1 = ([H \cap N], |N : H \cap N|)$ since $|H \cap N|$ divides $H$. Therefore, $H \cap N$ is a Hall subgroup of $N$.

As for $HN/N$, we have:

- $|HN/N| = |HN : N| = |H : H \cap N|$, which divides $|H|;


Since $|H|$ and $|G : H|$ are coprime, $|HN/N|$ and $|G/N : HN/N|$ are coprime, and so $HN/N$ is a Hall subgroup of $G/N$.

**Sec 3.4, Ex 4**  We can do induction on the order of the finite abelian group. Let $P(m)$ be the statement ‘for any finite abelian group $A$ of order $m$, and any positive integer $n|m$, there is a subgroup of $A$ of order $n’$. Clearly $P(1)$ is true. Suppose $P(k)$ is true for all $1 \leq k < m$, for some fixed integer $m$. Let us show that $P(m)$ holds:

Take any finite abelian group $A$ of order $m$. Let $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime factorisation of $m$, where the $p_i$ are distinct primes, and $\alpha_i > 0$ for all $i$. Take any positive integer $n$ dividing $m$, and let $n = p_1^{\beta_1} \cdots p_k^{\beta_k}$ be the prime factorisation of $n$. Note that $0 \leq \beta_i \leq \alpha_i$ for all $i$.

If $\beta_i = 0$ for all $i$, then $n = 1$, and clearly $A$ has a subgroup of order 1, the trivial subgroup. So we might as well assume that there is an index $j$ such that $\beta_j > 0$. Then $p_j$ is a prime dividing $|A|$, and by Cauchy’s Theorem, $A$ has an element of order $p_j$; let $H$ be the subgroup of $A$ of order $p_j$ generated by this element. Consider the quotient homomorphism $\phi : A \to A/H$. Then $n/p_j$ is an integer dividing $|A/H|$, and $A/H$ is a finite abelian group of order strictly less than $m$. By our induction hypothesis, $A/H$ has a subgroup $B$ of order $n/p_j$. Then $\phi^{-1}(B)$ is a subgroup of $A$ of order $(n/p_j) \cdot p_j = n$, as desired.

**Sec 3.4, Ex 5**  Suppose $G$ is a solvable group, with a chain of subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_t = G$$

(1)

where each quotient $G_{i+1}/G_i$ is abelian.

Let $H$ be any subgroup of $G$. Then we have a chain of subgroups of $H$

$$1 = H \cap G_0 \leq H \cap G_1 \leq \cdots \leq H \cap G_t = H \cap G = H.$$  

(2)

Since $G_i \triangleleft G_{i+1}$ for each $i$, we have $H \cap G_i \triangleleft H \cap G_{i+1}$ for each $i$. To show that $H \cap G_{i+1}/H \cap G_i$ is abelian, consider the group homomorphism $\psi$ that is the composition of group homomorphisms

$$H \cap G_{i+1} \hookrightarrow G_{i+1} \rightarrow G_{i+1}/G_i.$$  

The kernel of $\psi$ is exactly $H \cap G_i$ (think about it - which elements in $H \cap G_{i+1}$ go to the identity element under the composite?), and by the first isomorphism theorem, $H \cap G_{i+1}/H \cap G_i = H \cap G_{i+1}/\ker(\psi)$ is isomorphic to $\im(\psi)$, a subgroup of $G_{i+1}/G_i$.  

2
which is abelian. So \( H \cap G_{i+1}/H \cap G_i \) itself is abelian, and so the series ((2)) satisfies the requirements for \( H \cap G \) to be a solvable subgroup.

Now, let \( Q \) be any quotient of \( G \). To say \( Q \) is a quotient of \( G \) is to say there is a surjective homomorphism \( \phi : G \to Q \). Consider the chain of subgroups of \( Q \) obtained by mapping the subgroups in ((1)) into \( Q \):

\[
1 = \phi(G_0) \leq \phi(G_1) \leq \cdots \leq \phi(G_s) = \phi(G) = Q. \tag{3}
\]

We show that this chain of subgroup make \( Q \) a solvable group. Fix any index \( i \). We need to show that \( \phi(G_i) \) is normal in \( \phi(G_{i+1}) \) and \( \phi(G_{i+1})/\phi(G_i) \) is abelian:

- \( \phi(G_i) \trianglelefteq \phi(G_{i+1}) \): take any \( \tilde{g} \in \phi(G_i) \) and \( \tilde{h} \in \phi(G_{i+1}) \). Then \( \tilde{g} = \phi(g) \) and \( \tilde{h} = \phi(h) \) for some \( g \in G_i \) and \( h \in G_{i+1} \). Then \( hgh^{-1} = \phi(h)\phi(g)\phi(h)^{-1} = \phi(hgh^{-1}) \), where \( hgh^{-1} \in G_i \) because \( G_i \) is normal in \( G_{i+1} \). This means \( \phi(hgh^{-1}) \in \phi(G_i) \), and so we have normality.

- \( \phi(G_{i+1})/\phi(G_i) \) is abelian: let \( \psi \) be the homomorphism \( G_{i+1} \to \phi(G_{i+1}) : x \mapsto \phi(x) \) (that is, \( \psi \) is obtained by restricting the domain of \( \phi \) to \( G_{i+1} \), and the image of \( \phi \) to \( \phi(G_{i+1}) \)). Then the subgroup \( \phi(G_i) \) of \( \phi(G_{i+1}) \) corresponds to a subgroup \( H \) of \( G_{i+1} \) (see the fourth isomorphism theorem), and \( H \) contains \( G_i \). By part (5) of the fourth isomorphism theorem, \( H \) is normal in \( G_{i+1} \) because \( \phi(G_i) \) is normal in \( \phi(G_{i+1}) \). Besides, we have a surjective homomorphism

\[
G_{i+1}/G_i \to G_{i+1}/H : xG_i \mapsto xH.
\]

Since \( G_{i+1}/G_i \) is abelian, so is the quotient \( G_{i+1}/H \cong \phi(G_{i+1})/\phi(G_i) \).

Therefore, \( Q \) is solvable.

**Sec 3.4, Ex 7** Take any normal subgroup \( H \trianglelefteq G \). If \( H = 1 \) or \( G \), then any composition series of \( G \) has \( H \) as one of the terms. So suppose \( H \) is strictly bigger than \( 1 \) and strictly smaller than \( G \). Take any composition series of \( H \) and \( G/H \), say

\[
1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_s = H \quad \text{and} \quad 1_{G/H} = I_0 \triangleleft I_1 \triangleleft \cdots \triangleleft I_t = G/H,
\]

where all the \( H_{i+1}/H_i \) and \( I_{j+1}/I_j \) are simple groups.

Let \( \phi : G \to G/H \) be the quotient homomorphism. Then we have a chain of subgroups

\[
\phi^{-1}(I_0) \leq \phi^{-1}(I_1) \leq \cdots \leq \phi^{-1}(I_t)
\]

where \( \phi^{-1}(I_i) \) is normal in \( \phi^{-1}(I_{i+1}) \) by part (5) of the fourth isomorphism theorem, and \( \phi^{-1}(I_{i+1})/\phi^{-1}(I_i) \cong I_{i+1}/I_i \) is simple. Besides, each \( \phi^{-1}(I_i) \) contains \( \ker(\phi) = H \).

Now we obtain a chain of subgroups of \( G \)

\[
1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_s = H \leq \phi^{-1}(I_0) \leq \phi^{-1}(I_1) \leq \cdots \leq \phi^{-1}(I_t).
\]

Since \( \phi^{-1}(I_0) = H \) and \( \phi^{-1}(I_t) = G \), this is a composition series for \( G \), and \( H \) is one of the terms.
Sec 3.4, Ex 12  (i) implies (ii): suppose (i) holds. Take any simple group \( G \) whose order is odd and greater than 1. By hypothesis, \( G \) is solvable, and so must be abelian. So \( G \) is now an abelian simple group. If \( |G| \) is not a prime number, then there is a prime \( p \) dividing \( |G| \), and \( p < |G| \). By Cauchy’s Theorem, there is a subgroup \( H \) of order \( p \) in \( G \). Then \( H \) is a proper subgroup of \( G \), contradicting \( G \) being simple. So \( |G| \) must be prime, and (ii) holds.

(ii) implies (i): suppose (ii) holds. Take any finite group \( G \) of odd order. Take any composition series \( \cdots \triangleleft G_i \triangleleft G_{i+1} \triangleleft \cdots \) of \( G \). Each quotient \( G_{i+1}/G_i \) is a simple group by the definition of a composition series. Besides, for each \( i \), \( |G_{i+1} : G_i| \) divides \( |G_{i+1}| \), which divides \( |G| \), and so \( |G_{i+1} : G_i| \) is odd. By assumption, the order of \( G_{i+1}/G_i \) is prime, and so \( G_{i+1}/G_i \) is cyclic, and in particular, abelian. Therefore, \( G \) is solvable.