MATH 108 – HOMEWORK #4

Due by 5pm, Wednesday, June 3.

Note: (*) = required, (**) = optional, (***) = unsolved.

(1) (*) What is the value of the sum
\[
\sum_{(i,j,k) \in S} \frac{8!}{i!j!k!},
\]
where \((i, j, k)\) varies over all ordered triples of non-negative integers \(S = \{(i, j, k) | i, j, k \geq 0, i + j + k = 8\}\)?

(2) (a) (*) What is the number of paths from \((0, 0, 0)\) to \((4, 3, 2)\) using only steps that increase the \(x, y,\) or \(z\) coordinate by one?

(b) (*) How many paths are there if they also are required to stay in the region \(x \geq y \geq z\) at each step?

(3) (*) (For this problem and the next, let \(F(n)\) denote the \(n\)th Fibonacci number, defined by \(F(0) = 0, F(1) = 1,\)
and \(F(n + 2) = F(n + 1) + F(n),\) for \(n \geq 0.\))

Show that for \(m, n \geq 0,\)
\[
F(n + m + 1) = F(m + 1)F(n + 1) + F(m)F(n).
\]

(4) (a) (**) Show that, with only finitely many exceptions, that if \(F(n)\) is prime, then \(n\) is prime.

(b) (***) Determine whether there are infinitely many Fibonacci primes.

(5) (*) Show that
\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}.
\]
(6) (*) Let $C_n$ denote the $n$th Catalan number. Show that
\[ \lim_{n \to \infty} \frac{C_n n^{3/2} \sqrt{\pi}}{4^n} = 1. \]

(7) (**) As shown in class, the characteristic polynomial for the Fibonacci numbers is $x^2 - x - 1$, and the generating function is $\frac{-1}{x^2 + x - 1}$. What is the relationship between the two quadratics $x^2 - x - 1$ and $x^2 + x - 1$? In particular, what is the relationship between their roots? Now explain this relationship.

(8) (a) (*) Determine which Fibonacci numbers are odd, and give a proof.

(b) (***) Similarly, but with Catalan numbers.

(9) (*) Show that $C_n$ counts the number of shortest lattice walks in from $(0, 0, \ldots, 0)$ to $(2, 2, \ldots, 2)$, so that at each step $x_1 \geq x_2 \geq \cdots \geq x_n$.

(10) (a) (*) How many ways are there to color the edges of a cube with two colors, if two colorings are the same if they agree after a rotation? (The cube has 24 rotations, including the identity.)

(b) (**) How many ways, if the two colorings agree after some combination of rotations and reflections? (The cube has 48 symmetries total.)

(11) (**) How many rotationally distinct ways are there to color the faces of an icosahedron with three colors? Which colors would you choose?

(12) (**) (a) How many different proper 3-colorings are there of the odd cycle $C_{2k+1}$? This is meant in the sense of graph coloring; i.e. adjacent vertices receive distinct colors.

(b) What is the answer for a 15-cycle, if two colorings are considered the same if they agree after a rotation?