MATH 108 – HOMEWORK #4

Due by 5pm, Wednesday, June 3.

Note: $(*)$ = **required,** $(**)$ = **optional,** $(***)$ = **unsolved.**

(1) (*) What is the value of the sum

$$\sum_{(i,j,k)\in S} \frac{8!}{i!j!k!},$$

where (i, j, k) varies over all ordered triples of non-negative integers $S = \{(i, j, k) \mid i, j, k \geq 0, i + j + k = 8\}$?

(2) (a) (*) What is the number of paths from $(0, 0, 0)$ to $(4, 3, 2)$ using only steps that increase the x, y, or z coordinate by one?

(b) (*) How many paths are there if they also are required to stay in the region $x \geq y \geq z$ at each step?

(3) (*) *(For this problem and the next, let $F(n)$ denote the nth Fibonacci number, defined by $F(0) = 0$, $F(1) = 1$, and $F(n + 2) = F(n + 1) + F(n)$, for $n \geq 0$.)*

Show that for $m, n \geq 0$,

$$F(n + m + 1) = F(m + 1)F(n + 1) + F(m)F(n).$$

(4) (a) (**) Show that, with only finitely many exceptions, that if $F(n)$ is prime, then n is prime.

(b) (***) Determine whether there are infinitely many Fibonacci primes.

(5) (*) Show that

$$\lim_{n\to\infty} F_{n+1}/F_n = \frac{1 + \sqrt{5}}{2}.$$

(6) (*) Let C_n denote the nth Catalan number. Show that

$$\lim_{n\to\infty} \frac{C_n n^{3/2}\sqrt{\pi}}{4^n} = 1.$$

(7) (**) As shown in class, the characteristic polynomial for the Fibonacci numbers is $x^2 - x - 1$, and the generating function is $\frac{-1}{x^2+x-1}$. What is the relationship between the two quadratics $x^2 - x - 1$ and $x^2 + x - 1$? In particular, what is the relationship between their roots? Now explain this relationship.

(8) (a) (*) Determine which Fibonacci numbers are odd, and give a proof.
   (b) (**) Similarly, but with Catalan numbers.

(9) (*) Show that C_n counts the number of shortest lattice walks in from $(0, 0, \dots, 0)$ to $(2, 2, \dots 2)$, so that at each step $x_1 \geq x_2 \geq \cdots \geq x_n$.

(10) (a) (*) How many ways are there to color the edges of a cube with two colors, if two colorings are the same if they agree after a rotation? (The cube has 24 rotations, including the identity.)
   (b) (**) How many ways, if the two colorings agree after some combination of rotations and reflections? (The cube has 48 symmetries total.)

(11) (**) How many rotationally distinct ways are there to color the faces of an icosahedron with three colors? Which colors would you choose?

(12) (**)
   (a) How many different proper 3-colorings are there of the odd cycle C_{2k+1}? This is meant in the sense of graph coloring; i.e. adjacent vertices receive distinct colors.
   (b) What is the answer for a 15-cycle, if two colorings are considered the same if they agree after a rotation?