MATH 108 – HOMEWORK #2

Due by 5pm, Friday, May 1.

Note: $(*) =$ **required,** $(**) =$ **optional,** $(***) =$ **unsolved.**

(1) $(*)$ Prove that $\chi(G) \geq 3$ if and only if G has an odd cycle subgraph.

(2) $(**)$ Prove that every graph G has a bipartite subgraph H with $\#E(H) \geq \frac{1}{2}\#E(G)$.

(3) $(*)$ Prove that every connected graph has a spanning tree.

(4) (a) $(*)$ Show that an outer-planar graph has no $K_{2,3}$ or K_4 minors.
(b) $(**)$ Show that a graph with no $K_{2,3}$ or K_4 minors is outer-planar.

(5) (a) $(*)$ Show that K_7 embeds in the torus, without edges crossing.
(b) $(*)$ Prove a 7-color theorem for graphs embedded in the torus. You may (indeed, should) use the fact that $v - e + f = 0$ for triangulations of the torus.

(6) (a) $(**)$ Show that K_6 embeds in the projective plane, without edges crossing.
(b) $(**)$ Prove a 6-color theorem for graphs embedded in the projective plane. You may use the fact that $v - e + f = 1$ for triangulations of the projective plane.

(7) Let $R(m, n)$ denote the smallest number N such that if the edges of the complete graph K_n are colored red and

blue, there is either a red K_m or a blue K_n.

(a) $(*)$ Show that $8 \leq R(3,4) \leq 10$.
(b) $(**)$ Find the exact value of $R(3,4)$.
(c) $(**+)$ Similarly, $R(3,5)$.
(d) $(**++)$ $R(4,5)$.
(e) $(***+)$ $R(5,5)$. (It is known that $43 \leq R(5,5) \leq 49$.)
(f) $(***++)$ $R(6,6)$. (It is known that $102 \leq R(6,6) \leq 165$.)

(8) (a) $(**)$ Show that for large enough n,

$$2^{n/2} \leq R(n,n) \leq 4^n.$$

(b) $(****++)$ Find $\lim_{n\to\infty} R(n,n)^{1/n}$. (Even whether this limit exists is an open problem.)

(9) $(*)$ Prove that in every coloring of the edges of K_{2i+1} with i colors, there is a monochromatic cycle.