MATH 108 – HOMEWORK #1

Due by 5pm, April 17. Late work will not be accepted!

Note: $(*)$ = **required,** $(**)$ = **optional,** $(***)$ = **unsolved.**

(1) $(*)$ Show that a finite simple graph has at least two vertices of the same degree. (Hint: assume otherwise, derive a contradiction.)

(2) $(*)$ Show that the following are equivalent conditions for a finite simple graph G:
   (a) G is connected and has no cycles (i.e. G is a *tree*)
   (b) G is connected but removing any edge results in a disconnected graph.
   (c) G has no cycles, but adding any edge creates a cycle.
   (d) G is connected and $\#$ vertices $-\#$ edges $= 1$.

(3) The Petersen graph P is shown in Figure 1.
   (a) $(*)$ Show that P is non-Eulerian.
   (b) $(**)$ Show that P is non-Hamiltonian.

(4) $(*)$ Prove that the complete bipartite graph $K_{3,3}$ is non-planar.

Denote the *chromatic number* **of a graph** G **by** $\chi(G)$**.**

(5) $(*)$ Show that if C_n is a cycle on n vertices (i.e. as in the boundary of a polygon with n sides), $\chi(C_n) = 2$ or 3. Which values of n give 2 and which give 3?

(6) $(*)$ Show that if P is the Petersen graph, then $\chi(P) = 3$.

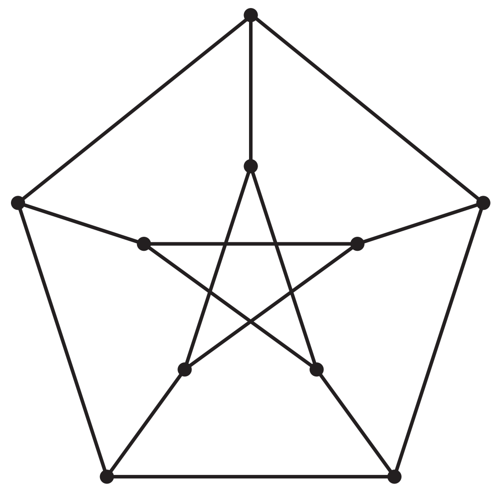

FIGURE 1. The Petersen graph.

(7) For positive integers n, k with $k < n/2$ define the *Kneser graph* $K(n,k)$ to have vertices for every subset of $[n] = \{1, 2, \ldots, n\}$ of size k. Connect a pair of vertices by an edge if the corresponding subsets are disjoint. (For example, it turns out that $K(5,2)$ is the Petersen graph.)
(a) $(**)$ Show that $\chi(K(n,k)) \leq n - 2k + 2$.
(b) $(**++)$ Show that $n - 2k + 2 \leq \chi(K(n,k))$.

(8) Define a graph G, where the vertices of G are the integers, with x and y connected by an edge whenever $|x - y| = n!$ for some integer n. So, for example, 0 is connected to $\pm 1, \pm 6, \pm 24, \ldots$.
(a) $(**)$ Show that $\chi(G)$ is finite.
(b) $(**+)$ Find $\chi(G)$ exactly.

(9) Define a graph G, where the vertices of G are the points in the Euclidean plane, with points p and q connected by an edge whenever $d(p,q) = 1$.
(a) $(**)$ Show that $4 \leq \chi(G) \leq 7$.
(b) $(***)$ Either show that $5 \leq \chi(G)$ or that $\chi(G) \leq 6$.