UNIQUE CONTINUATION AT INFINITY AND EMBEDDED
EIGENVALUES FOR ASYMPTOTICALLY HYPERBOLIC
MANIFOLDS

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I. Introduction. In this note we derive Carleman estimates at in-
finity for the Laplace operator on certain Riemannian manifolds. These
estimates are applied to prove a unique continuation theorem for the
Laplacians on these manifolds and also for a class of second order elliptic
equations which degenerate along a hypersurface, much as the Laplacian
on hyperbolic space does on the sphere at infinity, see Corollary (11)
and Theorem (14). We also study eigenvalues embedded in the contin-
uous spectrum for the Hodge-Laplacian, acting on differential forms,
on geometrically finite hyperbolic manifolds of infinite volume and on
the conformally compact manifolds introduced in [Ma 1], see also
[Ma 2], [Ma-Me]. By proving that the corresponding eigenforms must
decay rapidly, the unique continuation result mentioned above implies
that there are at most a finite number of "trivial" embedded eigenvalues.
See Theorem (16) for the precise statement.

An operator $L$ satisfies the unique continuation property at a point
$p$ (which might, for example, be the point at infinity in $\mathbb{R}^n$) if a solution
$v$ of $Lv = 0$ which vanishes to infinite order at $p$ must vanish identically
in a neighborhood of $p$. A standard technique to verify this property
for a particular operator and point is to prove an estimate of Carleman
type. This takes the form

$$C \int F(x)|u|^2dx \leq \int F(x)|Lu|^2dx,$$

where $F$ is a positive function defined in a neighborhood $U$ of $p$, assuming
a strict maximum at $p$, $u \in C^0(U \setminus p)$, and $C$ is a constant independent
of the (large) parameter $t$. If $u = \phi v$ is set into this inequality, where

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$L\nu = 0$, $\nu$ vanishes to infinite order at $p$, and $\phi \in C_0^\infty(U)$, a contradiction results as $t \to \infty$. The proof of Corollary (11) contains the details of this argument. There are many variations in this method; [A-B], [A-K-S], and [H\ö 1] contain some of the many important uniqueness theorems. A completely different approach is to be found in [G-L].

The Carleman estimate and unique continuation theorem presented here are not really new. The estimate is achieved in the manner used by Hörmander, [H\ö 2], Vol. III, Theorem 17.2.6, and the uniqueness result is almost certainly known to Alinhac and Baouendi, see Theorem 4' of [A-B]. It also bears a resemblance to Theorem 1.4 of [G-L]. Our only claim to originality here is to recast these theorems geometrically and to show how their proofs seem quite natural in this context.

In the problem of embedded eigenvalues the use of unique continuation theorems is well known, see [H\ö 2], Vol. II, Chapter XIV. When $M$ is a complete negatively curved manifold, related results have been obtained with very different methods. If $M$ has constant negative curvature, hence is represented as $\mathbb{H}^n/\Gamma$, has infinite volume and is geometrically finite, a fundamental domain for $\Gamma$ may be chosen which contains a half-ball $U$ at infinity in the upper half-space model. Lax and Phillips [L-P 1] prove that there are no solutions of $(\Delta - \lambda)\nu = 0$ in $L^2(U)$ for $\lambda \geq (n - 1)^2/4$. Since the continuous spectrum of $\Delta$ on $M$ extends from $(n - 1)^2/4$ to $\infty$, there can be no embedded eigenvalues. Their proof of the local result involves taking noneuclidean Fourier transform of $\nu$ along each of a one parameter family of hyperplanes foliating $U$. The transformed function $\tilde{\nu}$ satisfies an ordinary differential equation. By well-known asymptotic integration techniques for ODE's $\tilde{\nu}$ cannot decay rapidly enough to lie in $L^2$.

If $M = \mathbb{H}^n/\Gamma$ has finite volume, the Laplacian may well have embedded eigenvalues in its continuous spectrum, see the beautiful monograph [L-P 2]. It is easy to see that an eigenfunction must decay rapidly along each cusp, so a uniqueness theorem would need to assume far more rapid decay than that in Theorem (7) below.

On the other hand, in [D 1] Donnelly proves that there are no embedded eigenvalues for $\Delta$ acting on functions when $M$ is simply connected and has curvature approaching $-1$ at a certain rate. His proof is global; differential inequalities for the spherical averages of a putative eigenfunction are analyzed to get a contradiction as above. This is related to a theorem of Kato's for Schrödinger operators.

Our theorem here is local in nature. In Theorem (16) below it is
shown in particular that there is no $L^2$ differential $k$-form $\omega$ defined on a neighborhood $U$ near infinity in $\mathbb{H}^n$ which satisfies $(\Delta - \lambda)\omega = 0$ for $\lambda \geq \max (n - 2k \pm 1)^{3/4}$. The continuous spectrum of the Laplacian for $k$-forms on the geometrically finite quotient $M = \mathbb{H}^n/\Gamma$ consists of two sheets, each of infinite multiplicity: one extends from $(n - 2k - 1)^{3/4}$ to $\infty$ and corresponds to $\Delta$ restricted to the range of $\delta$ and the other, corresponding to $\Delta$ restricted to the range of $d$, extends from $(n - 2k + 1)^{3/4}$ to $\infty$. (In fact, apart from an eigenvalue of infinite multiplicity at 0 when $k = n/2$, this is the full spectrum on $\mathbb{H}^n$, see [D 2].) There is therefore an interval in the continuous spectrum not covered in the assertion above. Indeed, when $k < n/2$ for example, there may exist eigenvalues $\lambda \in [(n - 2k + 1)^{3/4}, (n - 2k - 1)^{3/4}]$. These, we shall show, must arise as differentials of $(k - 1)$-eigenforms, where now $\lambda$ lies outside the continuous spectrum for $(k - 1)$-forms. Explicitly, we prove the

**Theorem.** Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite infinite volume quotient of hyperbolic space. Then the Laplacian acting on $L^2$ differential forms of degree $k$ has no embedded eigenvalues $\lambda \in [(n - 2k + 1)^{3/4}, \infty)$ when $k \leq n/2$ and $\lambda \in [(n - 2k - 1)^{3/4}, \infty)$ when $k \geq n/2$. Any eigenvalue $\lambda \in [(n - 2k - 1)^{3/4}, (n - 2k + 1)^{3/4}]$ when $k \leq n/2$ must have its corresponding eigenform $\omega$ exact, $\omega = d\eta$. The form $\eta$ is an eigenform of degree $k - 1$ with eigenvalue below the essential spectrum for $(k - 1)$-forms. Similarly, if $\lambda \in [(n - 2k + 1)^{3/4}, (n - 2k - 1)^{3/4}]$ and $k \geq n/2$, then the eigenform $\omega$ is coexact, $\omega = \delta\eta$. The only exception is when $k = (n \pm 1)/2$: the eigenvalue $\lambda = 0$ may occur with multiplicity given by a purely topological quantity.

Eigenvalues below the continuous spectrum exist for many groups $\Gamma$; examples for functions are constructed in [P-S], but presumably the same arguments can be made to work for forms. A somewhat more general result than the one above, allowing certain variably curved manifolds, is the content of Theorem (16).

The proof again proceeds by demonstrating the rapid decay of an eigenform and then using the unique continuation theorem. This rapid decrease is attained by virtue of a calculus of degenerate pseudo-differential operators introduced in [Ma-Me], [Ma 1], [Ma 2]; for an expository treatment see [Me]. Many of the ideas are due to Richard Melrose, and the approach is closely related to Melrose and Mendoza’s approach to elliptic theory on spaces with conic singularities [Me]. In
essence, an eigenform has an asymptotic expansion near infinity with leading growth rate determined solely by $n$, $k$, $\lambda$ and the limiting curvature. For $\lambda$ sufficiently large, this leading term cannot lie in $L^2$, hence the expansion must be trivial and the eigenform rapidly decaying.

The natural geometric setting for these results is on a manifold $M$ with a metric which smoothly conformally compactifies. This means that $M$ is the interior of a compact manifold with boundary and the metric $g$ can be expressed as $\rho^{-2}h$ where $h$ is a metric smooth and nondegenerate up to the boundary and $\rho$ is a defining function for $\partial M$. Although this class seems quite restrictive, it is a reasonable setting for the development of a full scattering theory. This was initiated in [Ma-Me], and is the subject of recent work by Agmon. Therefore, the results in this paper are not presented in the fullest generality. They have been tailored so as to fit into this program and to include the geometrically important result stated in the theorem above. It would be quite interesting to extend them to metrics with far less regularity at infinity.

The rest of this paper is organized as follows: In Section II, the Carleman estimate and unique continuation theorem are proved for metrics of a special form in Fermi coordinates off a hypersurface. Section III contains a reinterpretation of these results which apply to a family of degenerate elliptic second order operators. Finally, in Section IV, the embedded eigenvalue problem is analyzed.

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**II. Unique continuation—geometric version.** The geometric version of the main Carleman estimate and unique continuation theorem of this paper will now be presented. In the next section these same results will be reinterpreted so as to apply to a family of degenerate elliptic operators.

Let $X$ be a half-space in the complete Riemannian manifold $(M, g)$. By this we mean that $X$ is bounded by a complete hypersurface $N \subset M$ and is diffeomorphic to its positive normal bundle under the exponential map. Thus $X$ is topologically a product: $X = \mathbb{R}^+ \times N$, and the metric assumes the form

$$
g = ds^2 + k(s)$$

where $s \in \mathbb{R}^+$ and $k(s)$ is a family of complete metrics on $N$. 
Suppose $E$ is a Riemannian vector bundle over $X$ with metric-compatible connection $\nabla$. There is a natural second order elliptic operator on sections of $E$, often called the rough Laplacian, which is given in terms of local coordinates $x^1, \ldots, x^{n-1}$ on $N$, $s = x^0 \in \mathbb{R}^+$, by

$$\nabla^* \nabla = -\sum_{i,j=0}^{n-1} g^{ij} \nabla_i \nabla_j$$

(2)

where $\nabla_i$ denotes $\nabla_{\partial x^i}$. It is nonnegative and self-adjoint with respect to the natural $L^2$ structure. We shall prove a Carleman estimate and unique continuation theorem for it under certain hypotheses on the growth of $g$ and the curvature of $\nabla$. This in turn will imply similar results for the other geometric second order elliptic operators on $E$. For simplicity, and because this is all we require for the application in Section 4, $E$ will be taken as a bundle of differential forms on $X$ and $\nabla$ as the Levi-Civita connection. The necessary modifications in the hypotheses and proof below for the more general situation are quite obvious, and left to the reader.

It is convenient to introduce the auxiliary family of metrics

$$h(s) = e^{-2rk(s)}$$

(3)

on $N$ where $r$ is some positive constant. We shall assume that

$$k' - 2rk = o(k)$$

(4)

or equivalently

$$h' = o(h)$$

(4')

as $s \to \infty$. The prime denotes derivative with respect to $s$ and the $o$ notation for quadratic forms has the usual meaning. We also demand that

$$\text{the sectional curvatures of } (X, g)$$

(5)

are bounded in magnitude

and
Here $II$ is the family of second fundamental forms for the hypersurfaces $\{s\} \times N$; note that these already appear implicitly in (4) because of the relationship $II = (1/2)k'(s)$. Also, the condition (6) may be replaced by $|\nabla_s II| \leq C$ in the application of Theorem (7) to scalar operators.

The model upon which these hypotheses are based is the metric $ds^2 + (\cosh^2 s)(s_0$ where $(N, g_0)$ is the hyperbolic space of one lower dimension. As discussed in the introduction, for a certain family of metrics generalizing this it is possible to prove that an $L^2$ solution of $\Delta u = \lambda u$ has a well behaved asymptotic expansion near infinity. In terms of these coordinates $u$ looks to leading order like an exponential $u_0 e^{-\alpha s}$ where $u_0$ is a smooth function on $N$, and $\alpha$ is determined by $\lambda$, the dimension, and the rank of the bundle $E$. If $\lambda$ is too large, this term cannot lie in $L^2$ and it is then possible to deduce that $u$ vanishes faster than any $e^{-\alpha s}$ as $s \to \infty$. This accounts for the remaining hypothesis in the theorem below. The method of proof is due to Hörmander [Hö 1], but it is quite simple and natural in this context. All norms and inner products below are taken with respect to $g$.

(7) Theorem. Let $u \in C^\infty(X, E)$ satisfy $|u| = O(e^{-\alpha s})$ for any $c > 0$. Suppose also that $u$ vanishes when $s \leq s_0$ and is in the $L^2$ domain of $\nabla^* \nabla$. If the metric $g$ satisfies (4), (5), and (6), then for $s_0$ and $t$ sufficiently large

$$t^3 \int e^{2s}|u|^2dV + t \int e^{2s}|\nabla u|^2dV \leq C \int e^{2s}|\nabla^* \nabla u|^2dV$$

for some constant $C$ which is independent of $t$.

Proof. Denote by $D$ the Levi-Civita connection on $(N, k(s))$ and set

$$L = -\nabla_i^2 + D^*D$$

$$= -\nabla_i^2 - e^{-2s} h^i D_j$$

(in terms of coordinates $x^1, \ldots, x^{n-1}$ on $N$). Each $D_i$ differs from $\nabla_i$ by a tensorial term which is off-diagonal in the sense that it interchanges ‘tangential’ and ‘normal’ parts of a differential form $u = \alpha + ds \wedge \beta$. 
In fact, this difference \( A_i = \nabla_i - D_i \) is expressible purely in terms of the second fundamental form. Because of this, (4) and (6) imply that \( L = \nabla^* \nabla + Q \), where the error \( Q \) is an operator of order less than or equal to one which satisfies

\[
|Qu| \leq C(|u| + |\nabla u|).
\]

For this one also needs to use the fact that \( |u|^2 = |\alpha|^2 e^{-2krs} + |\beta|^2 e^{-2(k-1)r\nu} \).

From these remarks it is clear that we need only prove (8) with \( \nabla^* \nabla \) replaced by \( L \). Also, since \( dV = e^{(n-1)r\nu} dsdV_h \) we may replace \( t \) by \( t + (n - 1)r/2 \) and \( dV \) by \( dsdV_h \) in all norms below. If \( v = e^\omega u \) then (8) is equivalent to

\[
(9) \quad t^3\|v\|^2 + t(\|\nabla v\|^2 + \|Dv\|^2) \leq C\|L_v\|^2,
\]

where \( L_{-\nu} = - (\nabla_s - t)^2 + \nu \mathcal{D} D_i D_j \).

(9) will follow from estimates for \( \|L_{\nu}v\|^2 \pm \|L_{-\nu}v\|^2 \). Indeed, since \( L_{-\nu} = L_{-\nu} - 4t\nabla_s \)

\[
\|L_{\nu}v\|^2 - \|L_{-\nu}v\|^2 = -8t\langle \nabla_s v, \nabla_s^2 v \rangle - 8t^3\langle \nabla_s v, v \rangle - 8t\langle \nabla_s v, e^{-2s\nu} D_i D_j v \rangle.
\]

Each of the last three terms can be estimated using (4') and (5), and integration by parts. The curvature condition arises in bounding expressions which result from commuting covariant derivatives in the last term. Because of (6), the error \([\nabla_s, \nabla] - [\nabla_s, D_i] \) is again controlled. Altogether, we arrive at the estimate

\[
(10) \quad 4rt\|Dv\|^2 + \|L_{-\nu}v\|^2 \leq \|L_{\nu}v\|^2 + \epsilon t^3\|v\|^2 + \epsilon t\|\nabla_s v\|^2,
\]

where \( \epsilon \) is arbitrarily small provided \( s_0 \) is large enough. Next

\[
\|L_{\nu}v\|^2 + \|L_{-\nu}v\|^2
\]

\[
= 2\|\nabla_s^2 v\|^2 + 8t^2\|\nabla_s v\|^2 + 2t^4\|v\|^2 + 2\|e^{-2s\nu} D_i D_j v\|^2
\]

\[
+ 4t^2\langle \nabla_s^2 v, v \rangle + 4t^2\langle v, e^{-2s\nu} D_i D_j v \rangle + 4\langle \nabla_s^2 v, e^{-2s\nu} D_i D_j v \rangle
\]
can be estimated in the same way, repeatedly integrating by parts, commuting covariant derivatives and applying the Cauchy-Schwarz inequality. The result is that

\[ t^4 \|v\|^2 + t^2 (\|\nabla v\|^2 + \|Dv\|^2) \leq C (\|L_+v\|^2 + \|L_-v\|^2 + t^2 \|Dv\|^2), \]

again for \( s_0 \) large enough. Upon dividing by \( t/4r \) and estimating the right side of this inequality by (10) we conclude that

\[ t^3 \|v\|^2 + t (\|\nabla v\|^2 + \|Dv\|^2) \leq C (\|L_+v\|^2 + \epsilon t^3 \|v\|^2 + \epsilon t \|\nabla v\|^2). \]

When \( \epsilon \) is sufficiently small, (9) follows.

(11) Corollary. Suppose \( w \) satisfies the differential inequality

\[ |\nabla^* \nabla w| \leq C (|w| + |\nabla w|) \]

on \( X \). If \( |w| = O(e^{-c}) \) for any \( c > 0 \) and is in the domain of \( \nabla^* \nabla \) then \( w = 0 \).

Proof. We only need to prove that \( w \) vanishes for \( s \) sufficiently large, for then a standard uniqueness theorem applies. The following argument is routine and is included for completeness. Set \( u = \phi w \) where \( \phi(s) \) vanishes for \( s \leq s_0 \) and \( \phi = 1 \) for \( s \geq s_0 + 1 \). Applying (8) we find, when \( t \gg 0 \), that

\[ t^3 \int_{s_0 + 1}^{s_0 + 1 + 1} e^{2u} |u|^2 dV \leq C \int_{s_0}^{s_0 + 1} e^{2u} |\nabla^* \nabla u|^2 dV. \]

As \( t \to \infty \) the left side increases faster than the right unless \( u = 0 \) for \( s \geq s_0 + 1 \).

Remark. As mentioned earlier, all geometric elliptic second order operators admit a Weitzenböck decomposition \( P = \nabla^* \nabla + R \), where \( R \) is a tensorial term of order zero which is determined solely by the curvature. In particular, if \( R \) is bounded then we may apply Corollary (11) to rapidly decreasing solution of \( Pu = \lambda u \). For example, if \( P \) is the Laplacian on differential forms, then \( R \) is bounded provided (5) holds, and we can conclude that rapidly decreasing solutions of \( \Delta u = \lambda u \) vanish identically.
There is one further extension of these results we shall require.

(12) **Corollary.** Suppose $\hat{g} = a^2 g$, where the metric $g$ satisfies (4), (5), and (6). If $a, |\nabla g|, |\nabla^2 g| \leq C$, then the estimate (8), and hence also Corollary (11), apply to $\nabla^* \nabla$, the rough Laplacian for $\hat{g}$.

**Proof.** Let $A$ be the difference tensor $\hat{\nabla} - \nabla$. $A$ can be expressed as a sum of covariant derivatives of $a$ with respect to $\nabla$. Thus $\nabla^* \nabla = a^{-2} g \nabla (A_i + A_j) (\nabla_i + A_i) = a^{-2} \nabla^* \nabla + Q$, where $Q$ is a first order operator satisfying $|Qu| \leq C(|u| + |\nabla u|)$. Since $a^{-2}$ is bounded above, it is clear that the estimate (8) implies an analogous one for $\nabla^* \nabla$.

**III. Unique continuation—analytic version.** There is a class of second order degenerate elliptic operators, intimately associated with asymptotically hyperbolic metrics, to which the results of the last section apply. These operators are studied extensively [Ma 1], [Ma 2], and [Ma-Me], and there is a good expository treatment in [Me].

Given a manifold with boundary, $M$, the Lie algebra of $C^\infty$ vector fields contains the natural subalgebra $\mathcal{V}_0$ of vector fields which vanish on $\partial M$. Alternately, $V \in \mathcal{V}_0$ iff $Vf|_{\partial M} = 0$ for all $f \in C^\infty(M)$. Associated to $\mathcal{V}_0$ are the differential operators, $\text{Diff}_0^\infty(M)$, which locally may be written as $C^\infty$ combinations of products of vector field in $\mathcal{V}_0$. In local coordinates $(x, y)$ near $\partial M$, where $x = 0$ on the boundary, $\mathcal{V}_0$ is generated over $C^\infty(M)$ by

$$\{x \partial_x, x \partial_y\},$$

so that any $P \in \text{Diff}_0^\infty(M)$ may be written

$$P = \sum_{j+[\alpha]=k} a_{j,\alpha}(x, y)(x \partial_x)^j (x \partial_y)^\alpha.$$

$P$ is elliptic if

$$^0 \sigma(P) = \sum_{j+[\alpha]=k} a_{j,\alpha}(x, y) \xi^j \eta^\alpha \neq 0 \quad \text{for} \quad (\xi, \eta) \neq 0.$$

This symbol has an invariant meaning.

In the references above it is shown that when $M$ is compact, elliptic elements of $\text{Diff}_0^\infty(M)$ which satisfy an additional ellipticity condition
on the boundary are Fredholm on natural weighted \( L^2 \) spaces. The basic idea is to construct a pseudodifferential operator with similar degeneracies to serve as a parametrix. This is discussed further in the next section.

Also associated to \( \mathcal{V}_0 \) are the metrics in the interior of \( M \) which are nondegenerate on \( \mathcal{V}_0 \) even up to the boundary. Such a metric has the form \( g = \rho^{-2}h \) where \( \rho \) is a defining function for \( \partial M \), i.e., \( \rho^{-1}(0) = \partial M \), \( \partial \rho \neq 0 \) there, and \( h \) is a smooth metric on the closed manifold. These metrics are obviously complete, and it is shown in [Ma 1] that they are asymptotically hyperbolic in the following sense. The geodesics which tend to infinity (the boundary) actually approach \( \partial M \) orthogonally (with respect to \( h \)). Furthermore, the curvature tensor becomes increasingly isotropic, with sectional curvatures approaching \( -(\partial \rho/\partial v)^2 \), the negative of the \( h \)-normal derivative of \( \rho \) squared, evaluated at the limiting point.

We now show that any "\( \mathcal{V}_0 \)-metric" for which \( \partial \rho/\partial v \) is constant on \( \partial M \) may be written in the form (1) with \( g \) satisfying (4), (5), and (6). A general \( \mathcal{V}_0 \)-metric is smoothly conformally related to one of this special type by a factor \( a^2 \) which satisfies the hypotheses of Corollary (12) of the last section; also, a general elliptic element of \( \text{Diff}_0^3(M) \) differs from the Laplacian of some \( \mathcal{V}_0 \) metric only in lower order terms. Thus, to prove unique continuation for these general operators, it suffices to verify that the Laplacians of special \( \mathcal{V}_0 \)-metrics satisfy the Carleman estimate (8) in some coordinate system.

Let us suppose now, for simplicity, that \( \partial \rho/\partial v = 1 \) on \( \partial M \). Introduce coordinates \((x, y)\) near the boundary, with \( x = 0 \) on the boundary, and let

\[
N = \{x^2 + |y|^2 = \epsilon, \ x > 0\}
\]

and

\[
X = \{x^2 + |y|^2 \leq \epsilon, \ x > 0\}.
\]

Provided \( \epsilon \) is small enough, all curvatures of the metric \( g = \rho^{-2}h \) are very close to \(-1\) on all of \( X \) so that (5) is trivially satisfied. Furthermore, the exponential map from the "inward pointing" normal bundle of \( N \) is a diffeomorphism onto \( X : X = N \times \mathbb{R}^+ \), and if \( s \) denotes distance to \( N \), then the metric \( g \) can be written in the form (1). To verify the other
conditions, we first note that the metric $k(s)$ in (1) is determined by the Jacobi fields emanating orthogonally from $N$. Indeed, let $J_w(s)$ be the Jacobi field along $\gamma(s) = \exp_p(sn)$, for some $p \in N$, and $n$ the unit normal to $N$ at $p$, with initial conditions $J_w(0) = w \in T_pN$, $J_w'(0) = \nabla_w n \in T_pN$. (The initial derivative for $J_w$ is the one ensuring that $J_w$ corresponds to a family of geodesics emanating orthogonally from a hypersurface.) Then

$$(13) \quad k(s)(w, w) = g(J_w(s), J_w(s)).$$

Because $g$ tends to an isotropic metric exponentially along $\gamma$, the field $J_w(s)$ tends strongly to a Jacobi field $J^0_w(s)$ for a hyperbolic metric, along with all its derivatives (cf. the theorems on asymptotic integration in [H]). In particular, $|J^0_w(s) - J_w(s)| = o(|J_w(s)|)$ as $s \to \infty$. Thus, differentiating (13) with respect to $s$, and recalling that $\nabla_{\omega\omega}g = 0$, we find that

$$k^{(n)}(s) - 2^nk(s) = o(k), \quad n = 1, 2, \ldots$$

which is actually far stronger than (4), and also proves that $|\nabla_s II| \leq C$.

Finally, to bound the whole covariant derivative $\nabla II$ it suffices to bound each $\nabla_{v(s)} II$ where $v(s)$ is a parallel vector field along $\gamma(s)$ and orthogonal to it. For this we need to consider a family of Jacobi fields $J_{w(t)}(s)$, where $w(t)$ is a parallel field along a curve through $p$ in $N$ with $w(0) = w$ and initial tangent vector $v(0)$, and prove that $X(s) = \nabla_{\omega\omega} J_{w(t)}(s) |_{t=0}$ grows at the same rate as $J_w(s)$. But this vector field satisfies an inhomogeneous Jacobi equation with exponentially decreasing inhomogeneous term, so a simple perturbation argument provides the correct growth rate for $X(s)$ and $X'(s)$, and by taking derivatives in (13) we arrive at the necessary bound.

We may now apply Theorem (7) to obtain a Carleman estimate for the principal part of the Laplacian for this metric. Using the remarks made earlier in this section, we finally get a Carleman estimate for a general second-order elliptic $\mathcal{V}_0$ operator. This last step requires that, using the coordinate $s$ above and $z$ along $N$, the vector fields $x \partial_x$ and $x \partial_y$ span the same space over bounded $C^\infty$ functions as $\partial_s$ and $e^{-s}\partial_z$, also, $x \to 0$ as $s \to \infty$. Finally, applying Corollary (11), we have proved
(14) **Theorem.** Let $P \in \text{Diff}^2_0(M)$ be elliptic; suppose that $u \in C^\infty(M)$ satisfies $Pu = 0$ and $u$ vanishes to infinite order along an open set of $\partial M$. Then $u \equiv 0$.

**Remark.** It should be noted that Theorem (14) contains a uniqueness result for second order elliptic operators of the usual sort. Indeed, if $P \in \text{Diff}^2(M)$ is elliptic and $\rho$ is a defining function for $\partial M$, then $\rho^2 P \in \text{Diff}^2_0(M)$ is elliptic in the $\mathcal{V}_0$ sense. Hence, since $Pu = 0$ implies $\rho^2 Pu = 0$, any such function $u$ could not vanish to infinite order on an open set of $\partial M$ without vanishing identically. While this is well known, the same reasoning shows that (14) cannot be improved to a strong uniqueness theorem, which would force any solution of $Pu = 0$, $P \in \text{Diff}^2_0(M)$ elliptic, which vanishes to infinite order at a single point to vanish identically. This follows from the fact that there are harmonic functions on a smoothly bounded domain which vanish to infinite order at a boundary point, e.g. $e^{-1/\sqrt{2}}$ on the right half space in $\mathbb{R}^2$.

**IV. Embedded eigenvalues.** In this last section we turn to the study of eigenvalues embedded in the continuous spectrum for the Hodge-Laplacian acting on $L^2 \Omega^k(M)$, $L^2$ differential forms of degree $k$ on $M$. The underlying metric $g$ is assumed to be conformally compact, that is, of $\mathcal{V}_0$ type as discussed in the last section, on the interior of the compact manifold with boundary $M$. Examples of such metrics are those geometrically finite quotients of hyperbolic space $\mathbb{H}^n/\Gamma$ where $\Gamma$ has no parabolic or elliptic elements. The following arguments are local though, so we obtain the same results for any infinite volume geometrically finite hyperbolic quotient.

Recall from the first section that the spectrum of $\Delta$ acting on $L^2 \Omega^k(\mathbb{H}^n)$ is purely absolutely continuous, except for an eigenvalue of infinite multiplicity at zero when $k = n/2$. It consists of the two half-lines

\[ S_1(k) = [(n - 2k - 1)^{1/4}, \infty) \]

\[ S_2(k) = [(n - 2k + 1)^{1/4}, \infty) \]

each taken with uniform infinite multiplicity. Each of these occurs as the spectrum of $\Delta$ acting on the invariantly defined subspaces of $L^2 \Omega^k$ given as $dL^2 \Omega^{k-1}$ and $\delta L^2 \Omega^{k+1}$, respectively. Thus it is quite natural that...
$S_2(k + 1) = S_1(k)$. For a geometrically finite quotient $M = \mathbf{H}^n/\Gamma$, the essential spectrum is exactly as above [Ma-P]. When $k = 0$ all essential spectrum is known to be absolutely continuous. The analogous statement when $k > 0$ (excepting the one eigenvalue in the middle degree), does not appear in the literature but undoubtedly follows from the techniques of [Ma-Me].

Let us denote the limiting curvature function $-(\partial \rho/\partial v)^2$ for $g = \rho^{-2}h$ by $-a^2$, and set $a_0 = \inf\{a(y), y \in \partial M\}$, $a_1 = \sup\{a(y), y \in \partial M\}$. From now on we shall suppose for notational convenience that $k \leq n/2$. The results for $k > n/2$ are obvious from duality. It is shown in [Ma 2] that the essential spectrum for $\Delta$ on $L^2\Omega^k(M, g)$ consists of the half-line $[a_0^2(n - 2k - 1)^2/4, \infty)$, along with $\{0\}$ when $k = n/2$. We may now formulate

(16) Theorem. Let $g = \rho^{-2}h$ be a conformally compact metric on $M$. Then there are no $L^2$ eigenforms for $\Delta$ of degree $k \leq n/2$ in the half-line $(a_0^2(n - 2k + 1)^2/4, \infty)$. The lower endpoint of this interval may be included if $a$ equals $a_0$ on an open set of $\partial M$. There may be eigenvalues in the interval $[a_0^2(n - 2k - 1)^2/4, a_0^2(n - 2k + 1)^2/4]$, but an eigenform corresponding to an eigenvalue in the open interval (or closed interval given the stronger assumption on $a$) must be closed, hence arise as the differential of an eigenform of degree $k - 1$ with eigenvalue lying below the essential spectrum for $\Delta$ on $L^2\Omega^{k-1}(M, g)$. The only exception occurs when $k = (n - 1)/2$ and $\lambda = 0$, in which case there is an eigenspace isomorphic to $H^k(M, \partial M)$.

Remark. The last statement concerning the space of $L^2$ harmonic forms when $k = (n - 1)/2$ is analogous to the main result of [Ma 2], but this particular case is joint work with Ralph Phillips [Ma-P] and will not be proved here. As noted in the introduction, Theorem (16) is still valid when $M = \mathbf{H}^n/\Gamma$ is any infinite volume geometrically finite quotient of hyperbolic space. The dimension of the space of harmonic forms when $k = (n - 1)/2$ is still topologically determined, but the formulation is a bit more complicated.

Proof. The details of the construction of the parametrices for $\Delta - \lambda$ used below will not be given. They are treated in [Ma 2]. We shall also assume that $a$ is constant, so that $a_0 = a_1$, and we even renormalize so that $a = 1$. The more general statements in the theorem are proved as below; one need only choose appropriate neighborhoods near $\partial M$ in which to argue.
The basic fact is that when $g = \rho^{-2} h$, $\Delta - \lambda$ is an elliptic element of $\text{Diff}_\partial(M; \wedge^k, \wedge^k)$, hence may be written

$$\Delta - \lambda = \sum_{j + |\alpha| \leq 2} a_{j,\alpha}(x, y) (x \partial_x)^j (x \partial_y)^\alpha$$

for certain linear transformations $a_{j,\alpha}$ on $\wedge^k_{(x,y)} M$. In [Ma 2] it is shown that

$$\Delta - \lambda = -x^2 \partial_x^2 + (n - 2k - 2) x \partial_x$$

$$+ (A - \lambda) + \sum_{1 \leq |\alpha| \leq 2} b_\alpha(y) (x \partial_y)^\alpha + xQ.$$

Here $Q$ is some element of $\text{Diff}_\partial(M; \wedge^k, \wedge^k)$, and $A$ is the constant matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & n - 2k \end{pmatrix}$$

according to the identification valid near $\partial M : \wedge^k M = \wedge^k \partial M \oplus \wedge^{k-1} \partial M$, $\omega = \alpha + dx \wedge \beta$, using the coordinates $x, y$.

Let $L$ denote the Fuchsian ordinary differential operator occupying the first three terms of (18). It has indicial roots

$$\frac{(n - 2k - 1)/2 \pm ((n - 2k - 1)^2/4 - \lambda)^{1/2}}{(n - 2k - 1)/2 \pm ((n - 2k + 1)^2/4 - \lambda)^{1/2}}$$

with each pair of roots corresponding to one of the summands of the splitting of $\wedge^k M$. When $\lambda \geq (n - 2k + 1)^2/4$ (and $k \leq n/2$), all roots in (19) are complex and have real part $(n - 2k - 1)/2$. Thus no solution of $Lu = 0$ lies in $L^2(x^{2k-n}dx)$ near $x = 0$ then. We point this out because the natural measure for coefficients of $k$-forms with respect to the basis generated by the $dx, dy'$ is essentially $x^{2k-n}dxdy$.

We now describe a local parametrix for $\Delta - \lambda$ which is constructed in detail in [Me]. Its Schwartz kernel is a distribution on $M \times M$. Let $(x, y)$ and $(\bar{x}, \bar{y})$ be two sets of identical coordinates on each copy of $M$ near the boundary. A parametrix should be smooth except along the
diagonal \( \{x = \tilde{x}, y = \tilde{y}\} \), where it must be conormal of the appropriate order, see \([H\tilde{o} 2]\), Chapter XVIII. The problem arises as \( x, \tilde{x} \to 0 \), for \( \Delta - \lambda \) is not uniformly elliptic then.

This nonuniformity is countered by introducing a singular coordinate system in \( M \times M \) around the boundary of the diagonal \( \{x = \tilde{x} = 0, y = \tilde{y}\} \). Actually, the correct viewpoint is that we are introducing a new manifold \( M \times_0 M \) and a map \( b \) to \( M \times M \) which is a diffeomorphism except over the above-mentioned submanifold, and such that polar coordinates around this submanifold lift under \( b \) to be smooth on \( M \times_0 M \). If these polar coordinates are given by

\[
\begin{align*}
  r &= (x^2 + |y - \tilde{y}|^2 + \tilde{x}^2)^{1/2} \in \mathbb{R}^+ \\
  \Theta &= (x, y - \tilde{y}, \tilde{x})/r \in \mathbb{S}^n_+ = \{\Theta \in \mathbb{S}^n : \Theta_0, \Theta_n \geq 0\},
\end{align*}
\]

then \( M \times_0 M \) has a new codimension one boundary component defined by \( \{r = 0\} \) and coordinatized by \( \Theta, \tilde{y} \). It is a quarter-sphere bundle over \( \partial M \). In these coordinates \( \Delta - \lambda \) lifts to an operator which is transversely elliptic to the lifted diagonal, even down to \( r = 0 \). Thus an initial parametrix \( E_0 \) is easy to construct; it is the microlocal inverse to this lifted operator depending parametrically on the complementary coordinates. \( E_0 \) may be chosen to be supported in a small neighborhood of the lifted diagonal, or equivalently, in a small conic neighborhood around the diagonal in the original coordinates.

It solves

\[
(\Delta - \lambda)E_0 = I - R_0
\]

where the error term \( R_0 \) is \( C^\infty \) in polar coordinates and supported in the same neighborhood. Since \( R_0 \) is not smooth on \( M \times M \), (20) can never suffice to prove Fredholm properties.

Equation (20) does have the following use though. If \( \omega \) is an eigenform, then applying the transpose of (20) to \( \omega \) shows that \( \omega = Q_0\omega \), where \( Q_0 = R_0^\ast \). The vector fields \( x\partial_x, x\partial_y \) lift in polar coordinates to nonsingular smooth vector fields on \( M \times_0 M \) which are tangent to all boundary components including the ‘front face’ \( \{r = 0\} \). Thus \( x\partial_xQ_0, x\partial_yQ_0 \) are operators with Schwartz kernels possessing the same degree of regularity as \( Q_0 \) itself. It is straightforward to show that any one of these operators, which are \( C^\infty \) on \( M \times_0 M \) and have supports intersecting
the boundary only in the interior of \( \{ r = 0 \} \), are bounded on any of the weighted Sobolev spaces we consider below. We have proved, assuming \( \omega \in L^2 \Omega^k(M, g) \) is an eigenform, that any coefficient \( \omega_i \) with respect to the usual basis for \( k \)-forms must satisfy

\[
(x \partial_x)^j (x \partial_y)^\alpha \omega_i \in L^2(\mathbb{R}^n_+, x^{2k-n}dx dy)
\]

\( \forall j, \alpha \).

Here and in similar statements below, we mean that the relevant function is defined and in \( L^2 \) only in a neighborhood of the origin.

Let us now combine this information with (17). From the above we find that

\[
L \omega_i = - (\sum_{|\alpha| = 2} b_\alpha (x \partial_y)^\alpha \omega + xQ \omega)_i
\]

\[
= x^s L^2(\mathbb{R}^n_+, x^{-1}dx; L^2(\mathbb{R}^{n-1}, dy))
\]

where we have set

\[
s = (n - 2k - 1)/2.
\]

But reinterpreting the right side of this equation actually shows that

\[
L \omega_i \in x^{s+1} L^2(\mathbb{R}^n_+, x^{-1}dx; H^{-1}(\mathbb{R}^{n-1}, dy)).
\]

It is a consequence of (22) that each component \( \omega_i \) itself must lie in \( x^{s+1} L^2(\mathbb{R}^n_+, x^{-1}dx; H^{-1}(\mathbb{R}^{n-1}, dy)) \). A convenient way to see this is using the Mellin transform, cf. [Me]. Let us replace the typical \( \omega_i \) by the function \( u \). Then the Mellin transform of \( u \)

\[
u_M(\xi, y) = \int_0^\infty x^{s} u(x, y) dx/x
\]

is nothing more than the Fourier transform in logarithmic coordinates. We need to assume that \( u \) is cut-off so as to vanish for \( x \geq 1 \), say. If \( u \in x^s L^2(\mathbb{R}^n_+, x^{-1}dx; L^2(dy)) \), it follows from usual properties of the Fourier
transform that \( u_M \) is defined and holomorphic in the complex half-plane \( \{ \text{Im} \ \xi < s \} \) with values in \( L^2(dy) \) and satisfies

\[
\sup_{r < s} \int_{\text{Im} \ \xi = r} |u_M(\xi, y)|^2 \, d\xi < \infty
\]

locally uniformly in \( y \). Conversely, let \( MH^i(t) \) denote the space of functions \( u_M(\xi, y) \) holomorphic in \( \{ \text{Im} \ \xi < t \} \) with values \( H^i(dy) \) and satisfying (24) with \( s \) replaced by \( t \). Then \( u_M \) is the Mellin transform of a function \( u \in x^i L^2(x^{-1} dx; H^i(dy)) \). Again, one needs to work locally in \( y \). The last property of the Mellin transform we need is that it converts differentiation by \( x \partial_x \) into multiplication by \( -i\xi \).

Now, take the Mellin transform of (22). We find that

\[ L(-i\xi)u_M \in MH^{-1}(s + 1) \]

where \( L(-i\xi) \) is the indicial polynomial of \( L \) evaluated at \( -i\xi \). But \( L(-i\xi) \) vanishes only when \( -i\xi \) assumes one of the values (19), or equivalently, at four points on the line \( \{ \text{Im} \ \xi = s \} \). Thus setting \( L(-i\xi)u_M = \phi \), then \( u_M = \phi/L(-i\xi) \) might fail to lie in \( MH^{-1}(s + 1) \) only because of possible poles on the critical line \( \{ \text{Im} \ \xi = s \} \). Clearly we may write \( u_M = v_M + w_M \) where \( v_M \in MH^{-1}(s + 1) \) and \( w_M \) is meromorphic in \( C \) with poles only at the roots (19). The corresponding function \( w \) may be taken to be a sum of terms \( x^i\omega(y) \), \( i \) being one of the roots (19). Hence \( w \) cannot lie in \( L^2(x^{2k-n} dy) \), so that \( u_M = v_M \) and \( u_M \in MH^{-1}(s + 1) \). We have proved that \( u \in x^{s+1} L^2(x^{-1} dx; H^{-1}(dy)) \).

This process may be continued inductively, so as to conclude that

\[
(25) \quad u \in x^{s+j} L^2(x^{-1} dx, H^{-j}(dy)), \quad j = 1, 2, \ldots
\]

This is far weaker than the proposed rapid decrease. Order of vanishing is gained only at the expense of lost tangential regularity. This loss is recovered by using a somewhat finer parametrix for \( \Delta - \lambda \).

This new parametrix \( E_1 \), is obtained by adding a correction term \( E_0^j \) to \( E_0 \). Recall that the error term \( R_0 \) above does not vanish at the
front face \( \{ r = 0 \} \). \( E'_0 \) is chosen mainly to ensure that the new error term does. Namely, we seek to solve

\[
(\Delta - \lambda)E'_0 \big|_{r=0} = R_0 \big|_{r=0}.
\]

Since \( \Delta - \lambda \) is formed from the vector fields \( x\partial_x, x\partial_y \), which lift to \( M \times_0 M \) to be tangent to \( r = 0 \), (26) is a well-posed equation. Furthermore it is discussed in [Ma 1] that solving this equation amounts to inverting \( \Delta - \lambda \) on \( H^s \) with compactly supported right hand side. A solution is readily found, but its support spreads to all boundary components of the front face. \( E'_0 \) is chosen to have this solution as its boundary value at \( r = 0 \), and we finally may correct \( E_0 \) using the ordinary differential operator \( L \) so that the error term \( R_1 \) in

\[
(\Delta - \lambda)(E_0 + E'_0) = I - R_1
\]

not only vanishes to first order at \( r = 0 \), but also to infinite order on the boundary face \( x = 0 \). (This last vanishing is obtained by using \( L \) to solve away the terms in the asymptotic expansion of the intermediate error term at \( x = 0 \). Here we use that all roots of \( L(\xi) \) have real parts strictly less than any term in this expansion.)

Let us now apply the transpose of (27) to \( \omega \). We find that \( \omega = Q_1 \omega \), where \( Q_1 \) is the transpose of \( R_1 \), vanishes to first order at \( r = 0 \) and to infinite order at \( \bar{x} = 0 \). Note that \( Q_1 \omega \) is defined because of (25). Now we apply \( \partial_x \) to this equation. The Schwartz kernel of \( \partial_x Q_1 \) is still smooth at \( r = 0 \), although now nonvanishing there, and has an asymptotic expansion at \( x = 0 \) no worse than the one for \( Q_1 \). One may find in [Ma 1] a proof of the fact that such a kernel induces a bounded transformation on \( x' L^2(x^{-1}dx; H^{-j}(dy)) \) for sufficiently negative \( r \) and for any \( j \). The main point is that such a kernel acts as a pseudo-differential operator of order zero in \( y \) uniformly as \( x \to 0 \).

Next, using (25), this boundedness implies that any coefficient \( \omega_i \) satisfied

\[
\partial_x \omega_i = (\partial_x Q_1 \omega)_i \in x' L^2(x^{-1}dx, H^{-i}(dy)),
\]

so that

\[
\omega_i \in x' L^2(x^{-1}dx; H^{1-i}(dy))
\]
for some fixed $r \ll 0$. Iterating this now proves that

$$\omega_i \in x^j L^2(x^{-1}dx; H^j(dy))$$

for any $j \in \mathbb{Z}$. Finally, use (28) in our initial procedure to conclude that

$$\omega_i \in x^{i+1}L^2(x^{-1}dx, H^i(dy)) \quad i, j = 0, 1, 2, \ldots$$

Of course, (29) is equivalent to the statement that $\omega$ is $C^\infty$ on $M$ and vanishes to infinite order on $\partial M$.

Combining this with Corollary (11) proves that there are no $L^2$ solutions of $(\Delta - \lambda)\omega = 0$ when $\lambda \geq (n - 2k + 1)^2/4$. In order to prove the remaining statements, observe that if $\omega$ is an eigenform then so is $d\omega$. But if $\Delta\omega = \lambda\omega$ and $(n - 2k - 1)^2/4 \leq \lambda < (n - 2k + 1)^2/4$, then the eigenvalue $\lambda$ is in the forbidden range for the $(k + 1)$-form $d\omega$ except when $k = (n - 1)/2$. The previous argument implies that $d\omega = 0$, so by the equation, $\omega = d(\lambda^{-1}\delta\omega)$. The $(k - 1)$-form $\lambda^{-1}\delta\omega$ is also an eigenform with eigenvalue $\lambda$ now outside the essential spectrum for $(k - 1)$-forms.

The same conclusion is more difficult to reach when $k = (n - 1)/2$. It will now be more convenient to work in the dual equivalent case $k = (n + 1)/2$. We seek to prove that if $\omega$ is an $L^2$ eigenform with eigenvalue $\lambda$ satisfying $0 \leq \lambda < 1$, then $\omega$ is coclosed, so that by the equation $\omega = \delta(\lambda^{-1}d\omega)$, where the $(k + 1)$-eigenform $\lambda^{-1}d\omega$ has eigenvalue outside the essential spectrum for $(k + 1)$-forms. Equivalently, we must show that if $\omega$ is an eigenform of degree $k$ which is closed, then $\omega = 0$. This will show that the part of $\omega = d(\lambda^{-1}\delta\omega) + \delta(\lambda^{-1}d\omega)$ in the range of $d$, which is also an eigenform, must vanish.

For this we must consider the equations more closely. Let us set $\omega = \alpha + dx \wedge \beta$, where $\alpha$ is a $k$-form and $\beta$ a $(k - 1)$-form on $\partial M$ depending parametrically on $x$. That $d\omega = 0$ means

$$d_\alpha = 0$$
$$\alpha' = d_\beta.$$

Next, if we freeze the coefficients of the expression (17) for $(\Delta - \lambda)$ at the point $(0, y_0) \in \partial M$, then the remaining “constant coefficient” operator in $x\partial_x, x\partial_y$, which might be regarded as acting on the inward half
of the tangent space to $M$ at $(0, y_0)$, may be identified by a linear transformation with the Laplacian on $H^n$ in the upper half-space model. When $k = (n + 1)/2$, this constant curvature operator acts as

$$
(\Delta_H - \lambda)(\alpha + dx \wedge \beta)
= (x^2\Delta_x \alpha - x^2\alpha'' - 3x\alpha' - \lambda \alpha + 2xd_x \beta)
+ dx\wedge(x^2\Delta_x \beta - x^2\beta'' - 3x\beta' - (\lambda + 1)\beta + 2x\delta_x \alpha).
$$

Hence $(\Delta - \lambda)\omega$ equals the right side of (31) plus an error $xQ\omega$ for some $Q \in \text{Diff}^\infty(M; \wedge^k, \wedge^n)$.

By the naturality of $d_x$ on $\partial M$ we may substitute (30) into (31) so as to conclude that

$$
-x^2\alpha'' - x\alpha' - \lambda \alpha = -xQ'\omega - x^2\Delta_x \alpha
-x^2\beta'' - 3x\beta' - (\lambda + 1)\beta = -xQ''\omega - x^2\Delta_x \beta - 2x\delta_x \alpha,
$$

where $Q'\omega$ and $Q''\omega$ are the tangential and normal components of $Q\omega$. The Fuchsian ordinary differential operators in (32) have indicial roots $\pm i\sqrt{\lambda}$, and $-1 \pm i\sqrt{\lambda}$, respectively. But when $k = (n + 1)/2$, the functions $x^{z+2\sqrt{\lambda}}$ are well within $L^2(x^{2k-\eta}dx dy) = L^2(xdx dy)$. Hence when we apply our usual iteration procedure with the Mellin transform to (32) we cannot rule out poles first for $\alpha_m$ and then $\beta_m$ at $\xi = \pm \sqrt{\lambda}$. In fact, upon iteration the error term in (32) produces poles for $\alpha_m$ and $\beta_m$ at the whole discrete set $\pm \sqrt{\lambda} + i\mathbf{Z}^+, \mathbf{Z}^+ = \{0, 1, 2, \ldots \}$. The functions $\alpha_m$ and $\beta_m$ are meromorphic in $\mathbb{C}$ with poles at these locations.

Finally we use (30) again. The Mellin transform of $\alpha' = d_x \beta$ is

$$
-i\xi \alpha_m(\xi + i, y) = d_x \beta_m(\xi, y).
$$

But this implies that $\alpha_m$ only has poles at $\pm \sqrt{\lambda} + i(\mathbf{Z}^+ + 1)$. Since the poles of $\beta_m$ arise from the poles of $\alpha_m$ in the interaction procedure, this shows that $\beta_m$ can only have poles on this smaller set. Now use (33) to conclude that the poles of $\alpha_m$ are contained in $\pm \sqrt{\lambda} + i(\mathbf{Z}^+ + 2)$, and so on. This implies as before that $\alpha$ and $\beta$ are $C^\infty$ and vanish to infinite order at $x = 0$. By Corollary (11), $\omega \equiv 0$. This concludes the proof.
REFERENCES


