

On the topology of vacuum spacetimes

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Abstract

We prove that there are no restrictions on the spatial topology of asymptotically flat solutions of the vacuum Einstein equations in $(n + 1)$ -dimensions. We do this by gluing a solution of the vacuum constraint equations on an arbitrary compact manifold Σ^n to an asymptotically Euclidean solution of the constraints on \mathbb{R}^n . For any Σ^n which does not admit a metric of positive scalar curvature, this provides for the existence of asymptotically flat vacuum spacetimes with no maximal slices. Our main theorem is a special case of a more general gluing construction for nondegenerate solutions of the vacuum constraint equations which have some restrictions on the mean curvature, but for which the mean curvature is not necessarily constant. This generalizes the construction [16], which is restricted to constant mean curvature data.

1 Introduction

A basic question in general relativity is whether there are any restrictions on the topology of the spacetime manifold M^{n+1} of a physically reasonable solution of Einstein's equations. If we restrict attention to globally hyperbolic solutions, so that $M^{n+1} = \Sigma^n \times \mathbb{R}$, and appeal to well-known results on the local well-posedness of the Einstein equations [6], then this question reduces to whether there are any restrictions on the topology of manifolds Σ^n which carry physically realistic solutions of the Einstein constraint equations. The initial data on Σ is a pair of symmetric 2-tensors (γ, Π) , where γ is a Riemannian metric and Π represents the second fundamental form in a Lorentzian development. The vacuum constraint equations are the compatibility conditions on these initial data sets arising from the putative embedding in a Ricci-flat Lorentz manifold. They take the form

$$\operatorname{div} \Pi - \nabla \operatorname{tr} \Pi = 0 \tag{1}$$

$$R - |\Pi|^2 + (\operatorname{tr} \Pi)^2 = 0 \tag{2}$$

All geometric quantities, norms and operators here are computed with respect to γ , and in particular R is the scalar curvature of this metric.

We write $\tau = \operatorname{tr} \Pi$ and call this the mean curvature function of the initial data set. Of particular interest and simplicity are the data sets with τ constant, and these are called constant mean curvature, or CMC. For Σ compact, there are always CMC data sets on Σ which satisfy the constraints :

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Proposition 1 *If Σ^n is compact, then it admits solutions of the vacuum constraint equations (1) and (2) with constant mean curvature $\tau \neq 0$.*

In fact, any compact Σ admits a metric γ of constant scalar curvature $R = -n(n-1)$. Setting $\Pi = \gamma$, then it is straightforward to check that (γ, γ) solves both (1) and (2) with $\tau = n$. By rescaling we obtain solutions with τ an arbitrary positive constant.

Related to this result is the work of Witt [24]. When Σ is compact, any smooth function which is negative somewhere is the scalar curvature function for some metric on Σ [18], and he uses this to produce non-vacuum ‘dust’ solutions to the constraints on Σ with arbitrarily prescribed non-negative energy density.

Witt also addresses the question of whether there are topological restrictions for asymptotically Euclidean solutions of the constraints. His results in this setting only hold for nonvacuum solutions, however. More specifically, he describes a procedure for gluing a solution of the constraints on an arbitrary manifold Σ with nonvanishing energy density to a set of time symmetric initial data for a Schwarzschild solution. This relies crucially on the nonvanishing of the energy density; hence his construction shows that there are asymptotically Euclidean solutions of the *non-vacuum* constraints on $\Sigma \setminus \{p\}$ for any compact manifold Σ , but it says nothing about solutions of the vacuum constraints. In fact, in his construction, the mass of the exterior Schwarzschild solution depends on the energy density of the interior, with the Schwarzschild mass equal to zero if the energy density vanishes. So if Witt’s construction could be extended to the vacuum (non-flat) case, it would produce nonflat solutions of the vacuum constraints which would be exactly Euclidean outside a compact set. This would violate the positive mass theorem [23]. The recent work of Corvino [11] shows that one can glue an exact exterior Schwarzschild metric (with non-zero mass) to compact subsets of fairly general time symmetric, asymptotically Euclidean initial data sets (with vanishing energy density). We emphasize, though, that Corvino’s construction begins with a pre-existing asymptotically Euclidean solution of the vacuum constraint equations. Hence, Corvino’s work does not bear on the issue of topological restrictions for asymptotically Euclidean solutions.

Are there any restrictions on the topology of asymptotically Euclidean vacuum initial data sets? We shall prove that this is not the case.

Theorem 1 *Let Σ be any closed n -dimensional manifold, and $p \in \Sigma$ arbitrary. Then $\Sigma \setminus \{p\}$ admits an asymptotically Euclidean initial data set satisfying the vacuum constraint equations.*

We are *not* claiming that this solution is CMC. In fact, a CMC asymptotically Euclidean initial data set necessarily has $\tau = 0$, so that (2) becomes

$$R = |\Pi|^2 \geq 0.$$

In addition, an asymptotically Euclidean metric on $\Sigma \setminus \{p\}$ with non-negative scalar curvature is conformally equivalent to (the restriction of) a metric on Σ with positive scalar curvature. This limits the possibilities for the topology of Σ dramatically, cf. [22], [12] and [13]. For example, when $n = 3$, this implies that Σ is the connected sum of manifolds with finite fundamental group (i.e. the quotient of a homotopy sphere with positive scalar curvature by a finite group of isometries) and a finite number of copies of $S^2 \times S^1$. The paper [20] surveys some of what is known in higher dimensions.

Suppose we apply this construction when Σ is a compact manifold which admits no metric of positive scalar curvature. We show in §6 that, subject to an extra hypothesis which appears in the statement of Theorem 4, the maximal spacetime development of this data, which is asymptotically flat, admits no maximal slices. The existence of spacetimes with this property was heretofore unknown.

Theorem 1 is proved by an analytical gluing method closely related to our earlier work [16]. More specifically, we produce solutions on $\Sigma \setminus \{p\}$ by joining together a CMC solution on Σ with a non-CMC solution on \mathbb{R}^n . The main result of [16] is that arbitrary nondegenerate (in a sense we explain below) solutions of the constraints on manifolds which are CMC and either compact, asymptotically Euclidean or asymptotically hyperbolic may be glued together. In particular, the method of [16] shows that if Σ is any compact 3-manifold, then $\Sigma \setminus \{p\}$ admits an asymptotically hyperbolic solution of the vacuum constraint equations. The gluing construction in the present paper closely follows that earlier work, but the new feature here is the use of non-CMC solutions on the asymptotically Euclidean summand. As we explain in the next section, this complicates the analysis slightly because the linearizations of (1), (2) uncouple when τ is constant, but not otherwise.

In the next section we review the conformal method for solving the vacuum constraints. In §3 we use the implicit function theorem to find appropriate non-CMC asymptotically Euclidean solutions on \mathbb{R}^n . The gluing is done in two steps: a family of approximate solutions is produced on $\Sigma \# \mathbb{R}^n$, and then these are perturbed using a contraction mapping argument to exact solutions. These steps are reviewed in §4 and §5, respectively. In §6 we discuss the existence of asymptotically flat vacuum spacetimes with no maximal slices.

A crucial idea in this analysis is the notion of nondegeneracy of a solution, which concerns the surjectivity of the linearized operator. We explain this concept in the next section.

Theorem 1 is a special case of a more general gluing theorem for non-CMC initial data sets.

Theorem 2 *Let $(\Sigma_j, \gamma_j, \Pi_j)$, $j = 1, 2$, be two initial data sets which solve the vacuum constraint equations; these may be either compact, asymptotically Euclidean or asymptotically hyperbolic. Suppose that both solutions are nondegenerate with respect to the appropriate function spaces (which contain functions weighted at infinity if either factor is noncompact). If the mean curvature functions τ_j are both equal to the same constant τ_0 in a neighborhood of the points $p_j \in \Sigma_j$, $j = 1, 2$, then the manifold $\Sigma_1 \# \Sigma_2$ obtained by forming a connected sum based at these two points again carries a one parameter family of solutions of the vacuum constraint equations. Moreover, for large values of the parameter, these solutions are small perturbations of the original initial data sets (γ_j, Π_j) outside of small balls around the points p_j .*

This more general result is proved in almost exactly the same way as the more specialized Theorem 1, and so we provide details only in the special case. Note that in Theorem 2 one can also take the points $p_j \in \Sigma$, $j = 1, 2$ as lying in the same connected manifold. In this case, rather than forming the connected sum, the theorem produces a family of solutions on the manifold obtained from Σ by adding a handle diffeomorphic to $S^{n-1} \times \mathbb{R}$.

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2 The conformal method

A very useful tool for the construction and enumeration of solutions to the vacuum constraint equations is the conformal method of Lichnerowicz, Choquet-Bruhat and York, and many of the basic existence results for these equations, e.g. [14], [17], [5], [9], [10], [7], [8], [1], rely on it. This method is most successful when dealing with CMC data because in this case the equations decouple. It can also handle non-CMC data, as shown in some of these references, and as our work shows here, although results to date suggest that one must place restrictions on the size of the gradient of the mean curvature.

Solutions are constructed via the conformal method as follows. One begins by fixing a background metric γ (representing a given conformal structure), and a symmetric $(0, 2)$ tensor Π which decomposes into trace-free and pure trace parts as $\mu + \frac{\tau}{n}\gamma$. The mean curvature function τ is specified through the second term on the right. (Note that one often makes the additional demand that μ is transverse-traceless, i.e. also divergence-free. This is useful in parametrizing the set of solutions to the vacuum constraint equations but is misleading for our current purposes.) We then modify this data by a conformal factor and a ‘gauging’ term by setting

$$\tilde{\gamma} = \phi^{\frac{4}{n-2}}\gamma, \quad \tilde{\Pi} = \phi^{-2}(\mu + \mathcal{D}W) + \frac{\tau}{n}\phi^{\frac{4}{n-2}}\gamma, \quad (3)$$

where ϕ and W are a positive function and a vector field, respectively. Note that the mean curvature is preserved, $\tau = \text{tr}_{\tilde{\gamma}}(\tilde{\Pi})$. The operator \mathcal{D} , which maps vector fields to trace-free symmetric $(0, 2)$ tensors, is the conformal Killing operator, and is given in local coordinates by the formula

$$\mathcal{D}X = \frac{1}{2}\mathcal{L}_X\gamma - \frac{1}{n}(\text{div}X)\gamma, \quad (\mathcal{D}X)_{jk} = \frac{1}{2}(X_{j;k} + X_{k;j}) - \frac{1}{n}\text{div}(X)\gamma_{jk}.$$

We have $\mathcal{D}X = 0$ if and only if X is a conformal Killing field. The formal adjoint of \mathcal{D} on trace-free tensors is $\mathcal{D}^* = -\text{div}$ and the operator $L \equiv \mathcal{D}^* \circ \mathcal{D}$ is formally self-adjoint, nonnegative and elliptic.

The modified data (3) satisfies the vacuum Einstein constraint equations (1) and (2) if and only if

$$\Delta_\gamma\phi - \frac{n-2}{4(n-1)}R_\gamma\phi + \frac{n-2}{4(n-1)}|\mu + \mathcal{D}W|^2\phi^{\frac{-3n+2}{n-2}} - \frac{n-2}{4n}\tau^2\phi^{\frac{n+2}{n-2}} = 0 \quad (4)$$

$$LW - (\text{div}\mu - \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\nabla\tau) = 0 \quad (5)$$

The first of these is usually called the Lichnerowicz equation. We write this coupled system as $\mathcal{N}(\phi, W, \tau) = 0$. The mean curvature τ is emphasized in this notation; however, the dependence of \mathcal{N} on γ and μ is suppressed. The linearization \mathcal{L} of \mathcal{N} in the directions (ϕ, W) (but not τ) is central to our construction.

Definition 1 A solution to the constraint equations $\mathcal{N}(\phi, W, \tau) = 0$ is nondegenerate with respect to Banach spaces \mathbf{X} and \mathbf{Y} provided $\mathcal{L} : \mathbf{X} \rightarrow \mathbf{Y}$ is an isomorphism.

Remark 1 It might seem more natural to require only that \mathcal{L} be surjective. However, in the main cases of interest, when Σ is compact, asymptotically Euclidean or asymptotically hyperbolic, these are equivalent (provided we use spaces of functions which decay at infinity).

Nondegeneracy conditions like this one are crucial to any gluing construction. The main result of [16] is that any two nondegenerate solutions of the vacuum constraint equations with the same constant mean curvature τ can be glued. For compact CMC solutions, nondegeneracy is equivalent to $\Pi \neq 0$ together with the absence of conformal Killing fields. On the other hand, asymptotically Euclidean or asymptotically hyperbolic CMC solutions are always nondegenerate (cf. §7 of [16]). While the paper [16] only treats the case $n = 3$, the generalization to higher dimensions is not difficult; this is discussed in [15], which also considers the extension to various types of non-vacuum solutions.

Suppose (ϕ, W, τ) solves (4) and (5) with background data (γ, Π) ; then we can choose the resulting solution $(\tilde{\gamma}, \tilde{\Pi})$ of the constraints (1) and (2), defined by (3), as the new background data. For the moment, all objects with tildes are associated to this new data. Let us determine the solution of $\tilde{\mathcal{N}}(\cdot, \cdot) = 0$ associated to this new data. Obviously the conformal factor $\tilde{\phi} = 1$, but we need to find the new vector field. If we assume that our solution $(\tilde{\gamma}, \tilde{\Pi})$ is nondegenerate, then we may *uniquely* solve $\tilde{L}\tilde{W} = (\operatorname{div} \mu - \tilde{\nabla}\tau)$ (since $\tilde{\phi} = 1$) for \tilde{W} in the appropriate summand of \mathbf{X} . Thus $\tilde{\mathcal{N}}(1, \tilde{W}) = 0$. We drop the tildes henceforth.

3 Non-CMC asymptotically Euclidean initial data on \mathbb{R}^n

A metric γ on \mathbb{R}^n is said to be asymptotically Euclidean, or AE, if γ decays to the Euclidean metric at some rate. More precisely, we assume that there is a $\nu > 0$ so that in Euclidean coordinates z , $|\gamma_{ij}(z) - \delta_{ij}| \leq C|z|^{-\nu}$, along with appropriate decay of the derivatives, as $z \rightarrow \infty$. To formulate this precisely, we make the

Definition 2 The family of weighted Hölder spaces $\mathcal{C}_{-\nu}^{k,\alpha}(\mathbb{R}^n)$ is defined as follows:

$$\mathcal{C}_0^{0,\alpha}(\mathbb{R}^n) = \left\{ u \in \mathcal{C}_{\text{loc}}^{0,\alpha}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : \sup_{R \geq 1} \sup_{\substack{z \neq z' \\ R \leq |z|, |z'| \leq 2R}} \frac{|u(z) - u(z')| R^\alpha}{|z - z'|^\alpha} < \infty \right\}$$

$$\mathcal{C}_0^{k,\alpha}(\mathbb{R}^n) = \{ u : (1 + |z|^2)^{|\beta|/2} \partial_z^\beta u \in \mathcal{C}_0^{0,\alpha}(\mathbb{R}^n), \text{ for } |\beta| \leq k \},$$

$$\mathcal{C}_{-\nu}^{k,\alpha}(\mathbb{R}^n) = \{ u = (1 + |z|^2)^{-\nu/2} v : v \in \mathcal{C}_0^{k,\alpha}(\mathbb{R}^n) \}.$$

Thus $u \in \mathcal{C}_{-\nu}^{k,\alpha}(\mathbb{R}^n)$ provided it is in the ordinary Hölder space $\mathcal{C}^{k,\alpha}$ on any compact set and also satisfies $|\partial_z^\beta u| \leq C_\beta |z|^{-\nu-|\beta|}$ as $|z| \rightarrow \infty$, for $|\beta| \leq k$, with an appropriate decay condition for the α Hölder seminorm on the k^{th} derivatives of u .

We define

$$\mathcal{M}_{-\nu}^{k,\alpha} = \{\text{metrics } \gamma \text{ on } \mathbb{R}^n : \gamma_{ij} - \delta_{ij} \in \mathcal{C}_{-\nu}^{k,\alpha}\}.$$

Note that if $\gamma \in \mathcal{M}_{-\nu}^{k+2,\alpha}$, then its scalar curvature function R_γ is in $\mathcal{C}_{-\nu-2}^{k,\alpha}$. Thus (4) and (5) suggest that we should assume that

$$W \in \mathcal{C}_{-\nu/2}^{k+1,\alpha}(\mathbb{R}^n, T\mathbb{R}^n), \quad \text{and} \quad \tau \in \mathcal{C}_{-\nu/2-1}^{k,\alpha}(\mathbb{R}^n).$$

We now assume that $\mu \equiv 0$ and look for solutions of the two equations (4) and (5) with (ϕ, W) close to $(1, 0)$ in appropriate weighted Hölder spaces. Set

$$\mathbf{X} = \left\{ (\phi, W, \tau) : \phi \in \mathcal{C}_{-\nu}^{k+2,\alpha}(\mathbb{R}^n), W \in \mathcal{C}_{-\nu/2}^{k+1,\alpha}(\mathbb{R}^n, T\mathbb{R}^n), \tau \in \mathcal{C}_{-\nu/2-1}^{k,\alpha}(\mathbb{R}^n) \right\}$$

and

$$\mathbf{Y} = \mathcal{C}_{-\nu-2}^{k,\alpha}(\mathbb{R}^n) \times \mathcal{C}_{-\nu/2-2}^{k-1,\alpha}(\mathbb{R}^n, T\mathbb{R}^n).$$

Then it is obvious that

$$\mathbf{X} \ni (\phi, W, \tau) \longrightarrow \mathcal{N}(\phi, W, \tau) \in \mathbf{Y}$$

is a \mathcal{C}^1 mapping in some neighbourhood \mathcal{U} of $(1, 0, 0) \in \mathbf{X}$. Let \mathcal{L} denote the linearization $D_{12}\mathcal{N}$ of \mathcal{N} in the (ϕ, W) directions, evaluated at the solution $(1, 0, 0)$. Then

$$\mathcal{L}(\psi, Z) = (\Delta\psi, LZ),$$

and clearly

$$\mathcal{L} : \mathcal{C}_{-\nu}^{k+2,\alpha}(\mathbb{R}^n) \times \mathcal{C}_{-\nu/2}^{k+1,\alpha}(\mathbb{R}^n, T\mathbb{R}^n) \longrightarrow \mathbf{Y}. \quad (6)$$

The fact that this linearization decouples reflects that we are linearizing about a CMC solution.

Theorem 3 *Let \mathcal{N} denote the vacuum constraint operator as above evaluated at the flat metric on \mathbb{R}^n with vanishing extrinsic curvature. Fix $0 < \nu < n - 2$. Then for some $\delta > 0$, there exists a \mathcal{C}^1 mapping*

$$\mathcal{F} : \{\tau : \|\tau\|_{k,\alpha,-\nu/2-1} < \delta\} \longrightarrow \mathcal{C}_{-\nu}^{k+2,\alpha}(\mathbb{R}^n) \times \mathcal{C}_{-\nu/2}^{k+1,\alpha}(\mathbb{R}^n, T\mathbb{R}^n)$$

such that $\mathcal{N}(\mathcal{F}(\tau), \tau) \equiv 0$ (and all solutions near $(1, 0, 0)$ are of this form). In particular, we may solve the vacuum constraint equations for any specified mean curvature function τ with sufficiently small norm in $\mathcal{C}_{-\nu/2-1}^{k,\alpha}(\mathbb{R}^n)$.

Proof: This follows immediately from the implicit function theorem once we recognize that for ν in this range, \mathcal{L} is an isomorphism, cf. [9], [7] for the analogous statement on weighted Sobolev spaces. \square

We shall apply this theorem by choosing a function $\tau \in \mathcal{C}_{-\nu/2-1}^{k,\alpha}(\mathbb{R}^n)$ with sufficiently small norm, and which is equal to a nonzero constant on a ball around the origin. Writing $\mathcal{F}(\tau) = (\phi, W)$, then (ϕ, W, τ) is a non-CMC solution of the vacuum constraints.

Proposition 2 *The linearization $\mathcal{L} = D_{12}\mathcal{N}$ of the operator \mathcal{N} in the first two slots, evaluated at this special solution (ϕ, W, τ) , is an isomorphism in (6) when δ is sufficiently small. Hence (ϕ, W, τ) is a nondegenerate solution.*

This follows simply because the invertibility is an open condition, hence is stable under perturbations of small norm.

Finally, notice that we can scale this solution by dilations on \mathbb{R}^n so that τ is equal to any desired constant on a (possibly smaller) ball around the origin. We assume that this has been done so that $\tau = n$ on $B_{2R}(0)$ for some $R > 0$. These dilated solutions are still nondegenerate. We summarize this as

Proposition 3 *There exist nondegenerate, asymptotically Euclidean, non-CMC solutions of the vacuum constraint equations (γ, Π) on \mathbb{R}^n which satisfy $\tau \equiv n$ on $B_{2R}(0) \subset \mathbb{R}^n$ for some $R > 0$. These solutions are given as $(1, W, \tau)$ (or equivalently as $(1, \mu, \tau)$ with $\mu = DW$) relative to the metric $\phi^{4/(n-2)}\delta$ with $(\phi, W, \tau) \in \mathbf{X}$.*

4 The approximate solutions

We now sketch the construction of the family of approximate solutions. This proceeds exactly as in [16] and so we refer to §2 of that paper for details.

Our two summands are the manifolds Σ and \mathbb{R}^n . These each have solutions of the vacuum constraints, which we write as $(\gamma_j, \phi_j, \mu_j, \tau_j)$, $j = 1, 2$ (with $j = 1$ corresponding to Σ and $j = 2$ to \mathbb{R}^n). The metric γ_1 is provided by Proposition 1 and corresponds to the solution with $\phi_1 \equiv 1$, $\mu_1 \equiv 0$, and $\tau_1 \equiv n$. Note that if (Σ, γ_1) has non-trivial conformal Killing fields, hence is degenerate, small perturbations of the conformal class of γ_1 generically have none. We then appeal to [14] to find the corresponding solutions of the constraints, which by construction are now nondegenerate. The solution $(\gamma_2, \phi_2, \mu_2, \tau_2)$ on \mathbb{R}^n is provided by Proposition 3 with $\phi_2 \equiv 1$, $\tau_2 \equiv n$ in $B_{2R}(0) \subset \mathbb{R}^n$. Note that we are including the functions $\phi_j \equiv 1$, for $j = 1, 2$, to emphasize that, relative to the other pieces of data, these are solutions to the Lichnerowicz equation (4) on each summand.

We begin by removing a small ball of radius R around the point $p \in \Sigma$ and also around the origin in \mathbb{R}^n . (These points will be written p_1 and p_2 .) The remaining manifolds both have boundaries diffeomorphic to S^{n-1} . The usual connected sum construction proceeds by identifying these copies of S^{n-1} , and we denote the resulting manifold $\widehat{\Sigma} = \Sigma \# \mathbb{R}^n$. The mean curvature functions τ_1 and τ_2 have an obvious smooth extension, which we denote by τ , to all of $\widehat{\Sigma}$.

We now construct a one-parameter family of metrics γ_T and symmetric 2-tensors Π_T which serve as background data (in the conformal method) for a family of approximate solutions. More specifically, we construct on $\widehat{\Sigma}$ a family of metrics γ_T , functions ψ_T (which are equal to 1 on $(\Sigma \setminus B_R(p)) \cup (\mathbb{R}^n \setminus B_R(0))$), and trace-free $(0, 2)$ tensors μ_T (which are equal to 0 on $\Sigma \setminus B_R(p)$ and $\mu_2 =$ on $\mathbb{R}^n \setminus B_R(0)$, respectively), such that

$$\mathcal{N}(\psi_T, 0, \tau) = E_T,$$

where the error term E_T is exponentially small in T as $T \rightarrow \infty$ (see [16] for a geometric description of the parameter T). Note that here the constraint operator, $\mathcal{N}(\cdot, \cdot, \tau)$, is

computed with respect to γ_T and μ_T . This part of the construction is completely localized on the ‘neck region’ bridging the two summands. These approximate solutions are perturbed in the next section to a family of exact solutions of the constraints, and it is only at this last step that we introduce a global correction term, which is exponentially small in T .

To construct these approximate solutions we first choose conformal factors ψ_j on each of the summands which are identically one outside the balls $B_{3R/2}(p_j)$ and which are equal to $(\text{dist}(\cdot, p_j))^{(n-2)/2}$ in $B_R(p_j)$. We then define

$$(\gamma_j)_c = \psi_j^{-\frac{4}{n-2}} \gamma_j.$$

These are complete metrics with asymptotically cylindrical ends in place of the punctured balls $B_R(p_j)$. There is a decomposition of the Π_j into trace-free and pure-trace parts,

$$\Pi_1 = \frac{\tau_1}{n} \gamma_1, \quad \Pi_2 = \mu_2 + \frac{\tau_2}{n} \gamma_2,$$

(Notice that μ_2 is not transverse-traceless: it has non-vanishing divergence since $\nabla \tau_2 \neq 0$.) Let $(\mu_2)_c = \psi_2^2 \mu_2$ and set $(\mu_1)_c = \mu_1 = 0$. The discussion from §2 shows that $((\gamma_j)_c, \psi_j, (\mu_j)_c, \tau_j)$ are solutions (with complete metrics) of the constraint equations on $\Sigma \setminus \{p\}$ and $\mathbb{R}^n \setminus \{0\}$, respectively.

If $r_j = \text{dist}_{\gamma_j}(\cdot, p_j)$, then $t_j = -\log r_j$ is a natural linear coordinate on each of these cylindrical ends. Let T be a large parameter and $A = -\log R$. Truncate the cylindrical ends of each of these manifolds by omitting the regions where $t_j > A + T$. A smooth manifold diffeomorphic to $\widehat{\Sigma}$ is obtained by identifying the two finite cylindrical segments $\{(t_j, \theta) : A \leq t_j \leq A + T\}$ via the map $(t_1, \theta) \rightarrow (T - t_1, -\theta)$. We call this new manifold Σ_T , and denote the long cylindrical tube it contains by C_T . It is convenient to use

$$s = t_1 - A - T/2 = -t_2 + A + T/2.$$

as a linear coordinate on C_T . This parametrizes C_T via the chart $(s, \theta) \in [-T/2, T/2] \times S^{n-1}$.

We next define the family of metrics γ_T and trace-free tensors μ_T on Σ_T . Choose cutoff functions χ_1, χ_2 so that $\chi_1 = 1$ on all of Σ and vanishes in the ball $B_{e^{-T/2}R}(p_1)$ and similarly for χ_2 , and moreover such that when these functions are transferred to Σ_T , then $\chi_2 = 1 - \chi_1$ and the supports of $d\chi_j$ are contained in the region $Q = [-1, 1]_s \times S_\theta^{n-1} \subset C_T$. Now define

$$\gamma_T = \chi_1(\gamma_1)_c + \chi_2(\gamma_2)_c, \quad \mu_T = \chi_2(\mu_2)_c.$$

Notice that $\gamma_T = (\gamma_j)_c$ and $\mu_T = (\mu_j)_c$, where $j = 1$ when $s \leq -1$ and on the rest of Σ and $j = 2$ when $s \geq 1$ and on the rest of \mathbb{R}^3 . The key point to emphasize here is that $(\mu_2)_c$ is very close to zero in the region where we have cut it off to be exactly zero, so this introduces only a small error. This is described in more detail in §2 of [16].

The conformal factors ψ_j on either summand must be joined together somewhat differently, by cutting them off at the far ends of the cylinder (relative to their domain of definition), as follows. We choose nonnegative cutoff functions $\tilde{\chi}_1$ on $\Sigma \setminus \{p\}$ and $\tilde{\chi}_2$ on $\mathbb{R}^n \setminus \{0\}$ which are identically one for $t_j \leq A + T - 1$ and which vanish when $t_j \geq A + T$. Now set

$$\psi_T = \tilde{\chi}_1(\psi_1)_c + \tilde{\chi}_2(\psi_2)_c.$$

This is defined on Σ_T , is identically 1 away from C_T , and equals $\psi_1 + \psi_2$ on most of the cylinder except at the ends.

The estimates for the error term $E_T = \mathcal{N}(\psi_T, 0, \tau)$ now follow readily from the estimates in §3.4, §4 and §6 of [16]. In particular, writing $E_T = (E_T^1, E_T^2) \in \mathbf{Y}$ (corresponding to (4) and (5), respectively) we have that

$$\|E_T^1\|_{k,\alpha,-\nu-2} + \|E_T^2\|_{k-1,\alpha,-\nu/2-2} \leq C e^{-T/4}, \quad (7)$$

and furthermore, the components E_T^1 and E_T^2 of E_T are supported on all of C_T and $Q \subset C_T$, respectively. In establishing this estimate it is important to note that, since $\nabla\tau \equiv 0$ on C_T , the vector constraint equation (5) does not introduce any new error terms beyond those previously encountered in [16].

5 The perturbation result

We now sketch the argument to perturb the approximate solution

$$\tilde{\gamma}_T = \psi_T^{\frac{4}{n-2}} \gamma_T, \quad \tilde{\mu}_T = \psi_T^{-2} \mu_T + \frac{\tau}{n} \psi_T^{\frac{4}{n-2}} \gamma_T$$

on Σ_T , to an exact, asymptotically Euclidean solution $(\bar{\gamma}_T, \bar{\mu}_T)$ of the vacuum constraints, corresponding to a solution (ϕ_T, Z_T) to $\mathcal{N}(\phi_T, Z_T, \tau) = 0$ on Σ_T (where $\mathcal{N}(\cdot, \cdot, \tau)$ is computed relative to γ_T and μ_T), when T is sufficiently large. Here $\phi_T = \psi_T + \phi$ and the pair (ϕ, Z_T) are small in the appropriate function space.

Definition 3 *Let w_T be an everywhere positive smooth function on Σ_T which equals $e^{-T/4} \cosh(s/2)$ on C_T and which equals 1 outside both balls $B_{2R}(p_j)$. For any $\delta \in \mathbb{R}$, and any $\phi \in \mathcal{C}_{\nu'}^{k,\alpha}(\Sigma_T)$, set*

$$\|\phi\|_{k,\alpha,\nu',\delta} = \|w_T^{-\delta} \phi\|_{k,\alpha,\nu'};$$

the corresponding space is denoted $\mathcal{C}_{\nu',\delta}^{k,\alpha}(\Sigma_T)$.

Thus elements of this function space not only have restricted growth or decay at infinity, but their norms are also measured in C_T with an extra weighting factor. We let

$$\mathbf{X}_{T,\delta} = \left\{ (\phi, Z) : \phi \in \mathcal{C}_{-\nu,\delta}^{k+2,\alpha}(\Sigma_T), Z \in \mathcal{C}_{-\nu/2}^{k+1,\alpha}(\Sigma_T; T\Sigma_T) \right\}$$

and

$$\mathbf{Y}_{T,\delta} = \mathcal{C}_{-\nu-2,\delta}^{k,\alpha}(\Sigma_T) \times \mathcal{C}_{-\nu/2-2}^{k-1,\alpha}(\Sigma_T; T\Sigma_T).$$

Notice that we are only including the weight δ on the first component of this space, but do not measure the vector field Z with a weighted norm in the neck. We use the obvious product norm on $\mathbf{X}_{T,\delta}$, but the less obvious one

$$\|(f, Y)\|_{\mathbf{Y}_{T,\delta}} = \|f\|_{k,\alpha,-\nu-2,\delta} + T^{-3} \|Y\|_{k-1,\alpha,-\nu/2-2}$$

on $\mathbf{Y}_{T,\delta}$.

We also assume that the weight ν is always in the range $(0, n - 2)$, so that the conclusion of Proposition 2 is valid.

The mapping

$$\mathbf{X}_{T,\delta} \ni (\phi, Z) \longrightarrow \tilde{\mathcal{N}}(\phi, Z) \equiv \mathcal{N}(\psi_T + \phi, Z, \tau) \in \mathbf{Y}_{T,\delta}$$

is \mathcal{C}^1 in some small neighbourhood \mathcal{U} of the origin in $\mathbf{X}_{T,\delta}$ for each T . The only subtlety here is that even when it has small norm, the function ϕ may be rather large in a pointwise sense in C_T . However, $\nabla\tau = 0$ in C_T and so this does not affect (5) there. Furthermore, if we write the linearization of $\tilde{\mathcal{N}}$ at $(\phi, Z) = (0, 0)$ as \mathcal{L}_T , then

$$\mathcal{L}_T : \mathbf{X}_{T,\delta} \longrightarrow \mathbf{Y}_{T,\delta}$$

is bounded as well.

We state the two fundamental results, which are essentially obtained by combining Propositions 7 and 8 from §5 and Corollary 1 from §3.3 in [16].

Proposition 4 *Fix any $\delta \in \mathbb{R}$. For T sufficiently large, the mapping*

$$\mathcal{L}_T : \mathbf{X}_{T,\delta} \longrightarrow \mathbf{Y}_{T,\delta}$$

is an isomorphism.

Let \mathcal{G}_T denote the inverse of \mathcal{L}_T provided by this proposition. Thus

$$\mathcal{G}_T : \mathbf{Y}_{T,\delta} \longrightarrow \mathbf{X}_{T,\delta} \tag{8}$$

and $\mathcal{L}_T \mathcal{G}_T = \mathcal{G}_T \mathcal{L}_T = I$. Of course \mathcal{G}_T also depends on δ , but we suppress this in the notation.

Proposition 5 *If $0 < \delta < 1$, then the norm of \mathcal{G}_T is uniformly bounded as $T \rightarrow \infty$.*

Although we refer to [16] for the proofs, let us make a few comments. Proposition 4 reflects the fact that the solutions we are joining are nondegenerate on their respective summands. One way to prove this result is to patch together the (pseudodifferential) inverses on each piece with the inverse on C_T (which is constructed rather explicitly in [16]). The resulting parametrix G_T satisfies $\mathcal{L}_T G_T = I - R_T$, where R_T has very small norm. As for Proposition 5, the reason we have added the T^{-3} factor in the second component in the definition of $\mathbf{Y}_{T,\delta}$ is because the inverse of the vector Laplacian $L = \mathcal{D}^* \mathcal{D}$ localized to the neck region C_T has norm bounded by T^3 (cf. §3.3 of [16]). This proposition is most handily proved by assuming this uniform bound fails and arguing to a contradiction.

From here, the rest of the proof of Theorem 1 is straightforward. The system we wish to solve is written as

$$\mathcal{N}(\psi_T + \phi, Z, \tau) = 0.$$

We write this as

$$\mathcal{L}(\phi, Z) = F_T(\phi, Z)$$

where F_T depends on all of the approximate data and consists of the error term E_T together with a nonlinear operator which is quadratically small. Using Proposition 4 this may in turn be written as

$$(\phi, Z) = \mathcal{G}_T(F_T(\phi, Z))$$

The existence of a fixed point for this map then follows from an application of the contraction mapping principle using Proposition 5 and the error estimate (7). This is explained carefully in §6 of [16]. This completes the proof of Theorem 1.

We emphasize that the reason no substantial changes need to be made to any of these arguments is that their most difficult aspects involve the explicit analysis of the linearizations of the Lichnerowicz operator (4) and the vector Laplacian (5) in C_T and the estimates of the approximate solutions in this same region. Because we are always assuming that $\nabla\tau = 0$ there, these parts of the arguments carry through completely unchanged. The remaining more global parts are quite general and it can be readily verified that they do not ‘notice’ the fact that these operators are now coupled.

Very few modifications are required to prove the more general Theorem 2. In fact, we need only note that now each of the initial solutions may have a transverse-traceless part, but these are easily incorporated into all the arguments.

6 Asymptotically flat vacuum spacetimes with no maximal slices

It has been commonly believed that every physically reasonable asymptotically flat solution of the Einstein equations should admit a foliation by asymptotically Euclidean, maximal ($\tau = 0$) slices. Such foliations are unique when they exist, and they are useful because they have desirable “singularity avoidance” properties. Roughly 20 years ago, Brill showed [4] that there exist asymptotically flat spacetimes which are *dust* solutions of the Einstein equations and which admit no maximal spacelike hypersurfaces. These are constructed by explicitly gluing a Friedman-Robertson-Walker spacetime to a Schwarzschild spacetime. The presence of dust is crucial to this argument.

Our analysis here shows that there exist asymptotically flat *vacuum* spacetimes which admit no maximal Cauchy slices (satisfying certain decay conditions). Indeed, if (M, g) is the maximal development of any initial data set constructed as in Theorem 1, with Σ admitting no metric of positive scalar curvature, then (M, g) admits no such maximal slice. As mentioned earlier, such manifolds Σ are quite abundant; for example, any closed hyperbolic 3-manifold has this property. The precise result is as follows.

Theorem 4 *Suppose that Σ is a closed n -manifold which admits no metric of positive scalar curvature. For any $p \in \Sigma$, let (γ, Π) be an asymptotically Euclidean solution of the vacuum constraint equations on $\Sigma \setminus \{p\}$ provided by Theorem 1, and let (M, g) be the maximal development of this data. Then there exists no maximal ($\tau = 0$) asymptotically Euclidean Cauchy surface $\widehat{\Sigma}$ in M for which the induced metric $\hat{\gamma} \in \mathcal{M}_{-\nu}^{k, \alpha}$ for some $\nu > \frac{n-2}{2}$, and for which the scalar curvature $R(\hat{\gamma}) \in L^1(\widehat{\Sigma})$.*

Proof: We have already observed that any maximal slice $(\widehat{\Sigma}, \hat{\gamma})$ has nonnegative scalar curvature. Schoen and Yau have proved, cf. [19] or [21], that an asymptotically Euclidean manifold with nonnegative scalar curvature may be perturbed to an asymptotically Euclidean manifold which is scalar flat and in addition conformally flat near infinity. Let γ' denote this new metric. Then there exists a compact set $K \subset \widehat{\Sigma}$ which is diffeomorphic to an exterior region in \mathbb{R}^n such that in the associated Euclidean coordinates, $\gamma' = u(x)^{\frac{4}{n-2}} \delta$

with $u(x) \rightarrow 1$ as $|x| \rightarrow \infty$ (δ is the Euclidean metric). Since both δ and γ' are scalar flat, $u(x)$ is harmonic and thus has an expansion

$$u(x) = 1 + E|x|^{2-n} + \mathcal{O}(|x|)^{1-n}, \quad |x| \rightarrow \infty;$$

here E is the total energy of γ' , see [21]. The Kelvin transform $\bar{u}(x) = |x|^{2-n}u(|x|^{-2}x)$ of u is harmonic on some punctured ball $B(0, \rho) \setminus \{0\}$. Hence $(\bar{\Sigma}, \bar{\gamma}')$ may be conformally compactified as follows. Choose a positive smooth function G on $\bar{\Sigma}$ so that $G(x)^{-1} = \bar{u}(x)$ when $|x| > 2R$ and $G \equiv 1$ on K . Then $\bar{\gamma} = G^{-\frac{4}{n-2}}\bar{\gamma}'$ extends smoothly to $\Sigma \cong \bar{\Sigma} \cup \{\infty\}$. Regard G as a function on $\Sigma \setminus \{p\}$. The facts that it is positive, diverges near p like $\text{dist}(\cdot, p)^{2-n}$, and is in the nullspace of the conformal Laplacian away from p (since γ' is scalar flat) shows that it is a constant multiple of Green's function for the conformal Laplacian (for $\bar{\gamma}$) on Σ . The positivity of Green's function is equivalent to the positivity of the first eigenvalue of the conformal Laplacian for $\bar{\gamma}$, which in turn implies that this metric is conformally equivalent to a metric on Σ with positive scalar curvature. This is a contradiction and so the proof is finished. \square

Remark 2 *This equivalence between asymptotically Euclidean metrics of nonnegative scalar curvature and metrics of positive scalar curvature on 'stereographic compactifications' is well-known, although the proof does not seem to be readily available in the literature. Note that by choosing mean curvature functions τ on \mathbb{R}^n with sufficiently fast decay one can use Theorem 1 to produce initial data sets which satisfy the asymptotic conditions required in Theorem 4. It is not at all clear, however, and may be quite subtle to prove, that any other asymptotically Euclidean Cauchy surface in the resulting maximal development must also satisfy these same decay conditions.*

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