# Pseudodifferential analysis for the Laplacian on noncompact symmetric spaces

Rafe Mazzeo

ABSTRACT. This is a brief and informal survey of an ongoing project, with Andras Vasy, to apply some recent advances in microlocal analysis and geometric scattering theory, to the analysis of the resolvent of the Laplacian on symmetric spaces of noncompact type. We obtain detailed results about the asymptotics of the resolvent, as well as its meromorphic continuation. The procedure outlined here involves the use of compactifications, which illuminate both the subsystem structure and the localizations required for the parametrix construction. Combining this with the more standard technique of complex scaling yields the continuation result.

## 1. Introduction

The spectral behaviour of the Laplacian on symmetric space of noncompact type is quite similar to that for Schrödinger operators with $N$-body potentials on Euclidean space. In this short note we describe some new techniques, presented in the framework of geometric scattering theory [**18**], which have been used successfully in both settings. The ideas here were first developed and applied by Andras Vasy for $N$-body Hamiltonians [**22**], [**23**], but have been adapted to the symmetric space setting in the past few years by the author and Vasy. The purpose of this note is to sketch some of the main ideas in a way intended to be accessible to those already acquainted with scattering theory. All work reported on here is part of this joint ongoing project with Vasy.

Let $M$ be a Riemannian symmetric space of noncompact type. This means that there exists a semisimple or reductive Lie group $G$ and a maximal compact subgroup $K \subset G$ such that $M = G/K$. $M$ is endowed with a metric which is invariant under the left $G$ action, and we are interested in the Laplacian of this metric. Our goal is to understand the resolvent of the Laplacian

$$R(\sigma) = (\Delta - \sigma)^{-1}, \qquad \sigma \notin \operatorname{spec}(\Delta)$$

(with the convention that $\operatorname{spec}(\Delta) \subset \mathbb{R}^+$), and more specifically, the asymptotics of its Schwartz kernel at infinity and its behaviour with respect to the spectral parameter (especially as it approaches and crosses the spectral axis).

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The most familiar example, of course, is Euclidean space $\mathbb{R}^n$ and the ordinary flat Laplacian, but also quite well known is the hyperbolic space $\mathbb{H}^n$ and its Laplacian. These cases are particularly simple because the rotational invariance of the Laplacian in each case makes it possible to reduce to a one-dimensional problem. From there, classical ODE and special function theory provides an adequate framework to obtain all of the information one seeks. The most accessible example of a 'higher rank' symmetric space is the quotient $\mathrm{SL}(n)/\mathrm{SO}(n)$, and we shall illustrate various of the constructions with this case (usually with $n = 3$). The geometry of general noncompact symmetric spaces, even in these special cases, is rather intricate. As for the corresponding analysis, the invariance of the Laplacian under the compact subgroup $K$ (thought of as acting by 'rotations' about the identity coset $o \in G/K = M$) allows us to reduce to a simpler, but still multidimensional, problem, which necessitates the use of more sophisticated tools from PDE.

We refer to **[9]**, **[10]** for all details on material related to the algebra and geometry discussed below; **[4]** is also a very good reference for the geometry of symmetric spaces. The Lie algebra $\mathfrak{g}$ of $G$ is its tangent space at the identity, and there is an involution on this vector space, the $\pm 1$ eigenspaces of which are denoted $\mathfrak{p}$ and $\mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of $K$. The exponential map identifies $\mathfrak{p}$ with the tangent space $T_oM$ at the identity coset $o$, and hence with $T_pM$ for every $p \in M$. These summands are orthogonal with respect to a natural bilinear form on $\mathfrak{g}$, called the Killing form, and in fact this form is always positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$; we shall always use the induced metric on $\mathfrak{p}$, and hence on $TM$.

The resolvent of the Laplacian associated to this metric has, for each value of the spectral parameter $\sigma$, a Schwartz kernel $R(\sigma; z, z') \in \mathcal{D}'(M \times M)$. This has a standard pseudodifferential singularity at $z = z'$ which is uniform as $z$ tends to infinity in $M$ because of the invariance relation $R(\sigma; gz, gz') = R(\sigma; z, z')$, $g \in G$. At present, our work has two main goals. First, we wish to understand the asymptotic behaviour of this distribution as $(z, z') \to \infty$ in $M \times M$ in all possible combinations of directions. This 'far-field' behaviour determines, for example, the refined mapping properties of the Laplacian on all manner of function spaces, and can be used to determine immediately the geometric structure of the Martin compactification of $M$. Second, we are also interested in the behaviour of $R$ as the spectral parameter $\sigma$ approaches and crosses $\mathrm{spec}\,(\Delta)$, and the analytic (or meromorphic) continuation of the resolvent.

Although such questions may be analyzed through direct use of Lie theory, they do not seem to have been satisfactorily addressed before. The reason is partly one of historical focus: for example, the resolvent is a central object in scattering theory, but has never attracted much attention in the symmetric space literature (but see **[5]**). On the other hand, starting from the work of Harish-Chandra, cf. **[10]**, **[6]**, one subject of intensive study has been the analytic continuation of Harish-Chandra's $\mathfrak{c}$-function and the so-called (generalized) spherical eigenfunctions, because of their role in defining the analogue of the Fourier transform in this setting. We refer also to **[8]**, **[1]** (to name just a very few sources) for other recent developments stemming from this point of view.

The material reported on here is drawn from four papers **[14]**, **[15]**, **[16]**, **[17]**, in which we have been successful in most of these basic goals: the asymptotics of the resolvent in the two simplest rank 2 cases is obtained in the first two papers (with the general case to follow) and the meromorphic continuation of the resolvent in general is obtained in the latter two. In addition, after the analytic proof of the continuation of the resolvent in [**17**], we also show how this fact follows from the continuation of the $\mathfrak{c}$-function; this was explored in somewhat more detail later in [**21**]. However, one of our main motivations is to develop methods which do not depend so heavily on the group theoretic structure and the sophisticated algebraic machinery. By developing appropriate geometric and analytic tools, one can construct the resolvent in a direct and robust manner, which allows one to progress further into the finer structure of the scattering theory on symmetric spaces, and which will also generalize to other settings. In particular, we expect to adapt our methods to the analysis of the Laplacian on more general locally symmetric spaces (both arithmetic and non-finite volume).

In the following sections, we first explain in slightly more detail some parts of the geometry and algebra of $M$, specifically the root system on the Cartan subspace $\mathfrak{a}$. We then discuss the reduction to the subspace of spherical ($K$-invariant) functions, and write an explicit expression for the radial Laplacian, $\Delta_{\mathrm{rad}}$, which is the operator induced by $\Delta$ on this subspace. Its form is manifestly that of an $N$-body Hamiltonian. Following this, we define different compactifications of $\mathfrak{a}$ and their utility in analyzing $\Delta_{\mathrm{rad}}$. These compactifications help, amongst other things, in identifying $\Delta_{\mathrm{rad}}$ in various asymptotic regions with subsystems, essentially radial Laplacians on lower rank symmetric spaces, which are assumed to be understood by induction. The main points then are a formula for the resolvent on a product space, the transplantation of these product resolvents using appropriate localizations, and the method of complex scaling. From all this we can deduce all we wish about the structure of the resolvent.

The writing style here is very informal, and statements of theorems are only discussed rather than stated precisely; the goal is to give some flavour of the setting and methods.

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## 2. The structure of $\mathfrak{a}$

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. In the example $M_n = \mathrm{SL}(n)/\mathrm{SO}(n)$, $\mathfrak{g}$ is the set of real matrices with trace zero, and $\mathfrak{p}$ and $\mathfrak{k}$ are the subspaces of symmetric and skewsymmetric matrices; a good choice for $\mathfrak{a}$ is the space of diagonal matrices with trace 0. The submanifold $A := \exp(\mathfrak{a}) \subset M$ is a flat, totally geodesic submanifold, and is maximal with these properties; its dimension is called the rank of $M$ (so $\mathrm{rank}(M_n) = n-1$). Since the exponential mapping is a diffeomorphism, we shall always identify $A$ with $\mathfrak{a}$. The geometric structure of $M = G/K$ is encoded in a collection of algebraic structures on $\mathfrak{a}$. First, $A$ is only unique up to conjugation, but even having fixed a particular choice of Cartan subspace, there is still a symmetry given by the quotient of the normalizer of $A$ in $G$ by its centralizer. This is called the Weyl group $W$, and can be realized as a finite reflection group acting on $\mathfrak{a}$. It is in this latter guise we always regard it. For $M_n$, the ambiguity is the ordering of the diagonal entries of $X \in \mathfrak{a}$, and we have $W = S_{n-1}$. The fixed point set of each element $w \in W$ is a hyperplane, called a Weyl chamber wall, and the complement of the union of all such hyperplanes in $\mathfrak{a}$ is denoted $\mathfrak{a}_{\text{reg}}$. We fix a component of this open set and denote it $\mathfrak{a}^+$; this is the positive Weyl chamber, and we set $A^+ = \exp(\mathfrak{a}^+)$.

The Cartan decomposition $G = KAK$ means that for any $g \in G$ there is an expression $g = k_1 a k_2$; the elements $k_1, k_2 \in K$ are not unique, but $a$ is unique so long as it is restricted to lie in $\overline{A^+}$. Thus $M = G/K$ can be identified with $K\overline{A^+}$. This is actually a direct product away from the boundary of $\overline{A^+}$. In other words, there is a dense 'regular part' of $M$ which is diffeomorphic to a product: $M_{\text{reg}} = K \times A^+$. This provides a convenient way to visualize $M$, as a union of orbits of the compact group $K$ over the Euclidean space $\mathfrak{a}$. At regular points, the orbit is a copy of $K$ which intersects $\mathfrak{a}$ in a finite number of points, but on the Weyl chamber walls, the orbits have smaller dimension.

Now consider the restriction of the Lie bracket

$$X, Y \longmapsto [X, Y], \qquad X \in \mathfrak{a},\ Y \in \mathfrak{g}.$$

Since $\mathfrak{a}$ is abelian, this has a simultaneous diagonalization: there exist eigenspaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ and elements $\alpha \in \mathfrak{a}^*$ such that

$$[X, y] = \alpha(Y), \qquad Y \in \mathfrak{g}_\alpha.$$

The $\mathfrak{g}_\alpha$ are called the root subspaces; the corresponding linear functionals are called roots and constitute the root system $\Lambda$, which is the other fundamental algebraic structure on $\mathfrak{a}$. The induced action of $W$ on $\mathfrak{a}^*$ permutes the elements of $\Lambda$, and the Weyl chamber walls may be identified with the the collection of zero sets $W_\alpha := \alpha^{-1}(0)$, $\alpha \in \Lambda$. We shall also consider the dual vectors $H_\alpha \in \mathfrak{a}$, determined by $B(H_\alpha, X) = \alpha(X)$, (where $B$ is the Killing form) for $X \in \mathfrak{a}$; these are normal to the $\mathcal{W}_\alpha$.

One can calculate these quantities easily for $M_n$, of course. Define the linear functional $T_i(X) = X_i$ (the $i^{\text{th}}$ diagonal entry), for $X \in \mathfrak{a}$; then $\Lambda = \{T_i - T_j\}$

## 3. The radial Laplacian

Both the Laplacian $\Delta$ and its resolvent $R(\sigma)$ commute with the action of $G$ on $M$. We shall be particularly interested in the restriction of these operators to the subspace $\mathcal{C}^\infty(M)^K$ of smooth functions which are invariant under the (left) $K$ action. Elements of this space, called spherical functions, are the symmetric space analogue of the radial functions on $\mathbb{R}^n$. In fact, just as on Euclidean space, $R(\sigma; z, o)$ is a radial distribution (which is regular away from $z = o$). Write $\Delta_{\text{rad}}$ for the restriction of $\Delta$ to the spherical subspace. It is not hard to see that the restriction of $R(\sigma)$ to the space of spherical functions completely determines the resolvent on all functions. In other words, it suffices for us to study the Schwartz kernel of the radial resolvent

$$R(\sigma)_{\text{rad}} := (\Delta_{\text{rad}} - \sigma)^{-1},$$

as a distribution on $\mathfrak{a} \times \mathfrak{a}$, because this determines the behaviour of the full resolvent.

There is an explicit formula which expresses the radial Laplacian as a perturbation of the ordinary Euclidean Laplacian on $\mathfrak{a}$:

$$\Delta_{\text{rad}} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Lambda} m_\alpha \coth \alpha H_\alpha.$$

Here $m_\alpha = \dim \mathfrak{g}_\alpha$ and each root $\alpha$ is regarded as a linear function on $\mathfrak{a}$. This formula is the starting point for our studies, for it shows that $\Delta_{\text{rad}}$ is the sum of the 'free Hamiltonian' $\Delta_{\mathfrak{a}}$ and a first-order 'interaction' potential. This perturbation is a sum of terms, each acting on one of the orthogonal lines $W_\alpha^\perp$. Alternately, one can also rewrite this operator in the positive Weyl chamber by adding and subtracting the constant coefficient operator $\sum_{\alpha\in\Lambda} m_\alpha H_\alpha$; this gives a decomposition for $\Delta_{\text{rad}}$ as a sum of a constant coefficient operator ($\Delta_{\mathfrak{a}}$ plus first order terms) and a sum of first order terms, each of which decay exponentially as the corresponding root $\alpha \to \infty$.

The coefficients of the first order terms in $\Delta_{\text{rad}}$ have relatively strong (regular singular type) singularities along the Weyl chamber walls. However, these do not present new difficulties because they arise only from the use of 'polar coordinates' near these walls. More precisely, if we were to let $\Delta_{\text{rad}}$ act on the space of *all* smooth functions on $\mathfrak{a}$, then we would have to take greater care with the definition of the domain of this operator. However, since we are only interested in the restriction of its action to the subspace $\mathcal{C}^\infty(\mathfrak{a})^W$ of $W$-invariant smooth functions, the singularities are effectively weaker. The explanation for this rests on the not altogether trivial fact that the space of smooth $W$-invariant functions on $\mathfrak{a}$ is naturally identified with the space of smooth $K$-invariant functions on $M$, i.e.

$$\mathcal{C}^\infty(M)^K \cong \mathcal{C}^\infty(\mathfrak{a})^W,$$

and on this space, $\Delta_{\text{rad}}$ is identified with $\Delta$, which has perfectly smooth coefficients.

We first illustrate these ideas with the radial Laplacian for $M = \mathbb{R}^n$ or $\mathbb{H}^n$. Then $\Delta_{\text{rad}}$ is an ODE:

$$\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \qquad \text{or} \qquad \frac{\partial^2}{\partial r^2} + \frac{(n-1)\cosh r}{\sinh r}\frac{\partial}{\partial r},$$

respectively, and is invariant under the Weyl group $W = \mathbb{Z}_2$, $r \mapsto -r$. It is easy to check now (for example, using the change of variables $s = r^2$) that this operator is effectively nonsingular on the subspace of even functions.

Next consider two slightly more complicated examples. The first is the product of hyperbolic spaces $\mathbb{H}^p \times \mathbb{H}^q$. Here $\mathfrak{a}$ is two dimensional, the Weyl group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, the positive Weyl chamber is the positive quadrant in $\mathbb{R}^2$, and the radial Laplacian is the sum of the two radial Laplacians on either factor:

$$\Delta_{\text{rad}} = \Delta_{\mathbb{H}^p,\text{rad}} + \Delta_{\mathbb{H}^q,\text{rad}}.$$

Using coordinates $(t_1, t_2)$ on $\mathbb{R}^2$, we have

$$\Delta_{\text{rad}} = \left(-\frac{\partial^2}{\partial t_1^2} + (p-2)\frac{\cosh t_1}{\sinh t_1}\frac{\partial}{\partial t_1}\right) + \left(-\frac{\partial^2}{\partial t_2^2} + (q-2)\frac{\cosh t_2}{\sinh t_2}\frac{\partial}{\partial t_2}\right).$$

Finally, to explain the situation for $M_3 = \mathrm{SL}(3)/\mathrm{SO}(3)$, we must choose good coordinates on $\mathfrak{a}$. This space may be represented by the set of 3-by-3 symmetric matrices of determinant 1 (through the mapping $\mathrm{SL}(3) \ni A \mapsto \sqrt{AA^t}$). Such a matrix has three real eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, with $\lambda_1\lambda_2\lambda_3 = 1$, and the Weyl group $W = S_3$ corresponds to permutations of these eigenvalues. The Weyl chamber walls are the sets where two eigenvalues coincide, and so a good choice for the positive Weyl chamber is the set $\mu = \lambda_1/\lambda_2 < 1$, $\nu = \lambda_2/\lambda_3 < 1$. In these coordinates, in the region where $\mu$ and $\nu$ are small, we have

$$\Delta_{\text{rad}} = \frac{1}{3}\left(-(\mu\partial_\mu)^2 - (\nu\partial_\nu)^2 + (\mu\partial_\mu)(\nu\partial_\nu) + (\mu\partial_\mu) + (\nu\partial_\nu) + E\right),$$

where $E$ is a sum of terms which are higher order when $\mu$ and $\nu$ are small, i.e. it is a sum of up to two products of the vector fields $\mu\partial_\mu$, $\nu\partial_\nu$, with at least one extra factor of $\mu$ or $\nu$. In other sectors of $\mathfrak{a}$ there are different representations for $\Delta_{\mathrm{rad}}$.

## 4. Compactifications

One of the principal tenets of geometric scattering theory is that it is extremely useful to compactify the underlying space on which the operator one is studying lives. For the goal of understanding the asymptotic behaviour of the resolvent, this is quite natural: in a compactification, the existence of asymptotics at infinity is transformed into a statement about the smoothness or polyhomogeneity of a function at a collection of boundary hypersurfaces. Compactifications are usually obtained by a combination of two processes: the first is that of 'completing' geodesic rays by adding a point to each such ray at infinity, as for example one does in radially compactifying Euclidean space (or any other Cartan-Hadamard manifold), and the second is blowing up appropriate submanifolds. Briefly, if $X$ is a manifold with corners and $Y$ is a lying in the boundary, then the blowup of $X$ along $Y$ is the disjoint union of $X \setminus Y$ and the inward-pointing spherical normal bundle of $Y$; this new space, which is denoted $[X;Y]$, is endowed with the minimal $\mathcal{C}^\infty$ structure for which all smooth functions on $X$, as well as 'polar coordinates' on $X$ around $Y$, lift to be smooth. (We note that this procedure is only well-behaved when $Y$ satisfies some special geometric hypotheses.) The other operation which plays a role here is enlarging the $\mathcal{C}^\infty$ structure. Thus, given a manifold with corners $X$, if $x$ is a boundary defining function for one particular boundary hypersurface $H$, then we can define a larger $\mathcal{C}^\infty$ structure by demanding that $-1/\log x$ is instead a boundary defining function for this hypersurface. If this is done at all boundary faces of $X$, the resulting $\mathcal{C}^\infty$ manifold with corners is denoted $X_{\log}$.

When faced with the task of selecting a particular compactification, one measure of its suitability is whether the vector fields which constitute the operator in question have lifts which lie in the Lie algebra $\mathcal{V}_b$ of smooth vector fields which are unconstrained in the interior and tangent to the boundary. For example, for the Euclidean Laplacian $\Delta_\mathfrak{a}$, the constituent vector fields are the constant coefficient ones, and the radial compactification $\widehat{\mathfrak{a}}$ is suitable in this sense. (In fact, the lifts here actually lie in a proper subalgebra of $\mathcal{V}_b$, called the algebra of scattering vector fields.) In any case, $\widehat{\mathfrak{a}}$ is a good space on which to study, for example, regularity questions for solutions of $\Delta_\mathfrak{a} u = 0$. However, given that we wish to study the Schwartz kernel of $(\Delta_\mathfrak{a} - \sigma)^{-1}$, we actually need to choose a compactification of $\mathfrak{a} \times \mathfrak{a}$. The first guess is obviously $\widehat{\mathfrak{a}} \times \widehat{\mathfrak{a}}$, but this is not large enough since it doesn't incorporate all the necessary approximate homogeneities and asymptotic behaviours of the free resolvent. The correct space is obtained by a sequence of two blowups:

$$\big[ [\widehat{\mathfrak{a}} \times \widehat{\mathfrak{a}}; \partial\widehat{\mathfrak{a}} \times \partial\widehat{\mathfrak{a}}]; \partial \mathrm{diag}_b \big],$$

where $\mathrm{diag}_b$ is the lift of the diagonal to the first blowup. On this 'scattering double space', the free resolvent kernel is actually polyhomogeneous (when $\sigma$ is away from the spectrum) at all boundary faces (but of course also has the usual polyhomogeneous singularity along the diagonal).

The constituent vector fields of $\Delta_{\mathrm{rad}}$ are not too much more complicated, but the key feature of this operator is its reduction to subsystems in different sectors of $\mathfrak{a}$, as explained more carefully in the next section, and neither the single space

$\widehat{\mathfrak{a}}$ nor the double space above is satisfactory for this. This is what necessitates Vasy's '$N$-body compactification'. We discuss this compactification, $\widetilde{\mathfrak{a}}$ from two different points of view. The first follows the thread we have already started. Starting from the collection $\{W_\alpha\}$, $\alpha \in \Lambda$, of Weyl chamber walls we define the collection of subspaces (of varying dimension) $\{S_a\}$, $a \in I$, in $\mathfrak{a}$, obtained as all possible intersections of the $W_\alpha$. The index set $I$ contains elements corresponding to the trivial (single) intersections $S_\alpha = W_\alpha$, as well as the elements 0 and $*$ with $S_0 = \{0\}$, $S_* = \mathfrak{a}$. Following standard $N$-body terminology, we write $S^a = S_a^\perp$; thus, the vector field $H_\alpha$ may be considered as acting in $S^\alpha$. Next, let $C_a = S_a \cap \partial\widehat{\mathfrak{a}}$. Then $\{C_a\}$, $a \in I$, is a lattice of great spheres of varying dimension in $\partial\widehat{\mathfrak{a}}$. The first definition of the compactification $\widetilde{\mathfrak{a}}$ is as the iterated blowup of $\widehat{\mathfrak{a}}$ along the varying submanifolds $C_a$, in order of increasing dimension. (When blowing up a collection of intersecting submanifolds, the order in which the blowups are performed can be important. In this case, the first blowup, of smallest dimensional spheres, has the effect of making disjoint all the spheres of next lowest dimension, and so the order in which these are blown up does not matter, etc.)

It is very helpful, and perhaps more geometrically appealing, to realize $\widetilde{\mathfrak{a}}$ by a different sequence of compactifications and blowups. Thus, rather than passing through the radial compactification, we first compactify $\mathfrak{a}$ as a polytope $\overline{\mathfrak{a}}$ as follows. The closure of the positive Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$ is a closed sector $\overline{\mathfrak{a}^+}$. Select a basis for $\mathfrak{a}^*$ in $\Lambda$ consisting of those roots which are positive on $\mathfrak{a}^+$ and not positive integer multiples of other elements of $\Lambda$; this is the set of positive indecomposable roots. As linear coordinates on $\mathfrak{a}$, these induce an affine mapping from $\overline{\mathfrak{a}^+}$ to $\mathbb{R}^n$, $n = \mathrm{rank}(M)$. Now replace each $\alpha$ in this basis by the function $t_\alpha = e^{-\alpha}$ and adjoin the faces $\{t_\alpha = 0\}$; this compactifies $\overline{\mathfrak{a}^+}$ as $[0,1]^n$. The Weyl group continues to act smoothly, and using it we may patch together the set of cubes which are the compactifications of the various Weyl chambers, so as to obtain the compactication $\overline{\mathfrak{a}}$ of the entire Cartan subspace. This is a smooth manifold with corners which has the structure of a closed affine polytope on which the action of the Weyl group extends. A brief calculation shows that the vector fields which comprise $\Delta_{\mathrm{rad}}$ all lift to be $b$-vector fields on this space (apart from the singularities on the $W_\alpha$). Unlike on $\widehat{\mathfrak{a}}$, these lifts don't have an additional order of vanishing, which holds because we are using the exponentials $e^{-\alpha}$, rather than the inverse of the polar distance $r$ on $\mathfrak{a}$ as the boundary defining functions.

Let $M$ be a compact manifold with corners, and define the class of $b$-differential operators, $\mathrm{Diff}_b^*(M)$, to consist of all differential operators which may be written locally as finite sums of products of $b$-vector fields. This sits inside the calculus of $b$-pseudodifferential operators $\Psi_b^*(M)$; elements of this latter space are defined so that their Schwartz kernels have the same sorts of approximate local homogeneities as elements of $\mathrm{Diff}_b^*(M)$ near $\partial M$, and one of the main properties of this calculus is that there is a parametrix construction for elliptic $b$-differential operators in it. We refer to [**19**] and also [**12**] for full details on the case where $M$ is a manifold with boundary, and to [**11**] for some aspects of this calculus in the general case.

Since $\Delta_{\mathrm{rad}}$ lifts to be an elliptic element of this calculus, it would seem that we could stop here and proceed with our analysis on this compactification. However, perhaps unexpectedly, there is problem at this point with the 'trivial' constant coefficient part $\Delta_{\mathfrak{a}}$. The parametrix construction in this calculus works directly only for those elliptic differential $b$-operators which are asymptotically of product type near the corners, which is not the case for $\Delta_{\mathfrak{a}}$. Even in the special case $M_3$, this behaviour is exhibited clearly in the explicit coordinate description of $\Delta_{\mathrm{rad}}$ presented earlier. To remedy this, we pass to a larger space, performing additional operations on $\overline{\mathfrak{a}}$, so that the lifted operator will have an asymptotic product type near the corners. The first step is to enlarge the $\mathcal{C}^\infty$ structure, replacing $\overline{\mathfrak{a}}$ by $(\overline{\mathfrak{a}})_{\log}$. Then we take the 'total boundary blowup', which is the process of blowing up all the corners of this space in order of increasing dimension. The composition of these two operations is referred to as taking the logarithmic total boundary blowup. The resulting space is denoted $\widetilde{\mathfrak{a}}$, and turns out to be naturally identified with the space defined by the other sequence of compactifications and blowups described earlier. It is a $\mathcal{C}^\infty$ manifold with corners with a smooth action by the finite Weyl group.

Let us return momentarily to the the rank two examples. For $M = \mathbb{H}^p \times \mathbb{H}^q$, $W = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts by reflecting the coordinates $(t_1, t_2) \in \mathfrak{a} = \mathbb{R}^2$, and the positive Weyl chamber is the positive quadrant. Here $\overline{\mathfrak{a}}$ is a square $[-1,1]^2$, and $\widetilde{\mathfrak{a}}$ is the octagon obtained by taking the logarithmic total boundary blowup of this square. For $M_3$, on the other hand, $W = S_3$ and the Weyl chambers are sectors of angle $\pi/3$ in $\mathbb{R}^2$. Each sector still compactifies as a square, but these are now arranged affinely into a hexagon in the compactification $\overline{\mathfrak{a}}$; logarithmically blowing up the corners of this hexagon yields the dodecagon $\widetilde{\mathfrak{a}}$.

We have gone into rather extensive detail about these compactifications because this is one of the central ideas of our methods. Although the geometry of these spaces is increasingly intricate as the rank gets larger, there is a gain: the combinatorial complexity of the collection of boundary faces of $\widetilde{\mathfrak{a}}$ very effectively replaces (or at least encodes) the analytic complexity of $\Delta_{\mathrm{rad}}$ and its solutions.

We shall not describe the double space, i.e. the sequence of compactifications and blowups of $\mathfrak{a} \times \mathfrak{a}$, needed to adequately describe all the possible asymptotic regimes of the resolvent. This is currently available in the rank two case, cf. **[14]**, **[15]**, and the corresponding details of the general higher rank case are obtained in a very similar manner; this is part of this ongoing project.

## 5. Subsystems and localization on $\widetilde{\mathfrak{a}}$

As indicated earlier, we use an inductive procedure to study the resolvent of $\Delta_{\mathrm{rad}}$ on a symmetric space of rank $n$, and assume known all required properties of the resolvents on all symmetric spaces of rank less than $n$. These lower rank resolvents appear explicitly in the parametrix construction on $\widetilde{\mathfrak{a}}$: in fact, one key feature of this space (already present on $\overline{\mathfrak{a}}$) is that the restriction of $\Delta_{\mathrm{rad}}$ to any boundary face is the radial Laplacian on some lower rank space. We now describe this correspondence.

Recall that the set $I$ indexes not only the set of all subspaces $\{S_a\}$ which are intersections of Weyl chamber walls, but also the faces of $\widetilde{\mathfrak{a}}$. For each $S_a$, we need to find a good model for $\Delta_{\mathrm{rad}}$ in some neighbourhood of infinity in this subspace. This model is a sum of two terms, each living on one of the two summands of the orthogonal decomposition

$$\mathfrak{a} = S_a \oplus S^a.$$

Note that the corresponding boundary face $F_a$ of $\widetilde{\mathfrak{a}}$ is a compactification of $S^a$.

Now define

$$\Lambda_a = \{\alpha \in \Lambda : \alpha = 0 \quad \text{on} \quad S_a\}.$$

Any $\alpha \in \Lambda_a$ extends smoothly from $\mathfrak{a}$ to $\mathfrak{a} \cup (F_a)^\circ \subset \widetilde{\mathfrak{a}}$, and vanishes on this boundary face (but is infinite on all other boundary faces). A closer examination of the definitions leads eventually to the conclusion that $S^a$ is a Cartan subspace, and $\Lambda_a$ the associated root system, for a symmetric space $M_a$ of rank strictly less than $n$. Denoting the radial Laplacian for $M_a$ by $\Delta_{a,\mathrm{rad}}$, then moreover, in a suitable neighbourhood of $S_a$,

$$\Delta_{\mathrm{rad}} = \Delta_{a,\mathrm{rad}} + L_a + E_a,$$

where $L_a$ is a constant coefficient operator on $S_a$ of the form $\Delta_{S_a} + V_a$, $\mathrm{ord}(V_a) = 1$, and $E_a$ is a decaying error term. This product operator

$$P_a = \Delta_{a,\mathrm{rad}} + L_a$$

is the subsystem Hamiltonian corresponding to $S_a$.

The form of $\Delta_{\mathrm{rad}}$ for $\mathbb{H}^p \times \mathbb{H}^q$ is already of product type, of course, and so (in terms of the expression written out near the end of §3), we see the subsystems even on $\overline{\mathfrak{a}}$. As $t_1 \to \infty$, for example, which corresponds to approaching the 'right', vertical face of $\overline{\mathfrak{a}}$, we have the approximation

$$\Delta_{\mathrm{rad}} = \left(-\frac{\partial^2}{\partial t_1^2} + (p-2)\frac{\partial}{\partial t_1}\right) + \left(-\frac{\partial^2}{\partial t_2^2} + (q-2)\frac{\cosh t_2}{\sinh t_2}\frac{\partial}{\partial t_2}\right) + E,$$

where the error term is exponentially decreasing. For this product case, it is not apparent why one would want or need to pass to $\widetilde{\mathfrak{a}}$. The real justification for this emerges when one considers the asymptotics of eigenfunctions; these have polyhomogeneous behaviour (multiplied by an oscillatory factor) on $\widetilde{\mathfrak{a}}$ but not on $\overline{\mathfrak{a}}$. In any case, the subsystem Hamiltonians on the horizontal and vertical faces of $\widetilde{\mathfrak{a}}$, those corresponding to the lifts of faces of $\overline{\mathfrak{a}}$, are the same as those above, whereas at the remaining faces the subsystem has trivial root structure and the associated $P_a$ is constant coefficient on both factors. For $M_3$, the subsystem Hamiltonian corresponding to one of the Weyl chamber walls $S_\alpha = W_\alpha$ takes the form

$$\Delta_{\mathrm{rad}} = \frac{1}{3}\left(-(\mu\partial_\mu)^2 - \left(\frac{\mu+\mu^{-1}}{\mu-\mu^{-1}} - \frac{s^2(\mu-\mu^{-1})}{s^4 - s^2(\mu+\mu^{-1})+1}\right)\mu\partial_\mu\right)$$
$$+\frac{1}{4}\left(-(s\partial_s)^2 - \frac{2(s^4-1)}{s^4 - s^2(\mu+\mu^{-1})+1}s\partial_s\right);$$

after a change of variables, the first summand here can be recognized as $\Delta_{\mathrm{rad}}$ in $\mathbb{H}^2$. In a rank three case, the restrictions could be, for example, any one of the rank two cases above, or the sum of a rank one radial Laplacian and a constant coefficient term.

The presence of cross-terms in the principal second order part near the corners of $\overline{\mathfrak{a}}$ is precisely the reason why it is necessary to pass to $\widetilde{\mathfrak{a}}$. The parametrix for the resolvent is obtained by patching together the resolvents of each of the $P_a$; the key point is that the approximation $\Delta_{\mathrm{rad}} \approx P_a$ is valid in neighbourhood of the closure of the face $F_a$ in $\widetilde{\mathfrak{a}}$, whereas this is not true in $\overline{\mathfrak{a}}$.

## 6. Product formula

In the last few sections we have described a geometric setting where $\mathfrak{a}$ can be divided into various sectors at infinity, in each of which $\Delta_{\mathrm{rad}}$ is approximated by a simpler product operator. However, if this is to be of any use, we must have some way to analyze the resolvent of such products. This problem can be stated in more generally.

Suppose that $P_1$ and $P_2$ are two self-adjoint operators, both bounded below, acting on the Hilbert spaces $H_1$ and $H_2$. Consider the operator

$$P = P_1 \otimes 1 + 1 \otimes P_2$$

acting on $H_1 \oplus H_2$. We ask whether if various properties are known to hold for the resolvents $R_j(\sigma) = (P_j - \sigma)^{-1}$, we can deduce the same properties for $R(\sigma) = (P - \sigma)^{-1}$. For specific properties involving the limiting behaviour as $\sigma$ approaches the spectrum (the limiting absorption principle), this was considered some time ago by Ben-Artzi and Devinatz [**3**], but we are interested in the finer properties of resolvent asymptotics and analytic continuation past the spectrum.

A useful tool to explore such questions, discovered in [**14**], is the identity

$$R(\sigma) = \frac{1}{2\pi i} \int_\Gamma R_1(\mu) \otimes R_2(\sigma - \mu)\, d\mu.$$

Here $R_j(\sigma)$ is defined and holomorphic in $\mathbb{C} \setminus [\sigma_0^{(j)}, \infty)$, so that as a function of $\mu$, this integrand is holomorphic in $\mathbb{C} \setminus [\sigma_0^{(1)}, \infty) \cup (-\infty, \sigma - \sigma_0^{(2)}]$. This set is connected provided $\lambda \notin [\sigma_0^{(1)} + \sigma_0^{(2)}, \infty)$, which we assume. Finally, the contour $\Gamma$ is one which diverges from the positive real axis linearly, for example has the form $c_1 + ic_2 t$, $c_2 > 0$, when $t$ is very large positive or very large negative. By standard estimates, this integral converges in the space of bounded operators $\mathcal{B}(H_1 \oplus H_2)$.

There are two immediate consequences of this formula. First, if both $R_1$ and $R_2$ have meromorphic continuations past their respective spectra, then the same is true for $R(\sigma)$. However, we note that poles in the continuations of either of the $R_j$ will produce 'Regge poles', i.e. ramification points, in the continuation of $R$, and so the continuation may only exist on a possibly quite complicated multi-sheeted covering of $\mathbb{C}$. This is proved by a straightforward contour deformation. The other consequence is more special and subtle, and is explored in detail in [**14**]: if $P_j$ is an elliptic operator on some noncompact manifold $M_j$, $H_j = L^2(M_j)$, and if we have good enough information about the asymptotics of each of the $R_j$ on $M_j \times M_j$, then a stationary phase analysis allows us to deduce similarly detailed information about the asymptotic behaviour of $R$ on $M \times M$, $M = M_1 \times M_2$.

## 7. Parametrix for the resolvent

We now collect all of the information above to produce a parametrix with compact error for the resolvent of $\Delta_{\mathrm{rad}}$ on the symmetric space $M$. We first choose two partitions of unity, $\{\chi_a\}$, $\{\tilde{\chi}_a\}$, $a \in I$, on $\tilde{\mathfrak{a}}$ such that both $\chi_a$ and $\tilde{\chi}_a$ equal one in a neighbourhood of the face $F_a$ and vanish on a neighbourhood of all faces $F_b$ where $F_b \cap F_a = \emptyset$, $\tilde{\chi}_a = 1$ on the support of $\chi_a$, and finally, such that each of these functions is $W$-invariant. For the special index $a = 0$, the functions $\chi_0$ and $\tilde{\chi}_0$ should be compactly supported in $\mathfrak{a}$. Next, for the index $a$ we use the product formula of the previous section to define the operator

$$R_a(\sigma) = (\Delta_{a,\mathrm{rad}} + L_a - \sigma)^{-1}.$$

Somewhat remarkably, the spectrum for each of these product models is the same half-line $[\sigma_0, \infty)$. Thus we can define the parametrix

$$\tilde{R}(\sigma) = \sum_{a \in I} \tilde{\chi}_a R_a(\sigma) \chi_a$$

for every $\sigma \in \mathbb{C} \setminus [\sigma_0, \infty)$. It can then be shown that

$$(\Delta_{\mathrm{rad}} - \sigma)\tilde{R}(\sigma) = I - Q,$$

where $Q$ is a compact operator. (In fact, $Q$ has smooth Schwartz kernel which vanishes at an appropriate rate at every boundary face of $\widetilde{\mathfrak{a}} \times \widetilde{\mathfrak{a}}$.)

The first main theorem, giving the asymptotics of $R(\sigma)$ on $\widetilde{\mathfrak{a}} \times \widetilde{\mathfrak{a}}$ (for $\sigma \in \mathbb{C} \setminus [\sigma_0, \infty)$), is quite complicated to state (and indeed, has been written down carefully only for special rank two cases [**14**], [**15**]), so we shall not formulate it precisely here. However, it should not be implausible, given all of the framework above, that one could deduce such asymptotics: the product formula yields the asymptotics of each of the model resolvents $R_a(\sigma)$ via stationary phase, and this in turn is parlayed into the leading part of the asymptotics for $R$ itself, etc.

## 8. Complex scaling and continuation of the resolvent

In this final section we describe the second main theorem, concerning the meromorphic continuation of the resolvent. This builds upon the parametrix construction above, and uses the technique of complex scaling, as in [**2**].

We begin by recalling this technique. Abstractly, one requires a family $U_\theta$ of operators on $L^2(\mathfrak{a})^W$, defined for $\theta$ in a contractible domain $D \subset \mathbb{C}$, such that $U_0 = \mathrm{Id}$, $U_\theta$ is unitary for $\theta \in D \cap \mathbb{R}$ and bounded on all Sobolev spaces, and with suitable holomorphy properties which is stated in the form that for $f$ in a suitable dense subclass of analytic vectors, $\theta \to U_\theta f$ is analytic with values in $L^2(\mathfrak{a})^W$. For our purposes, a good choice of the family $U_\theta$ is obtained by rotating the space $\mathfrak{a}$ in its complexification $\mathfrak{a}_{\mathbb{C}}$. More specifically, define the family of diffeomorphisms

$$\Phi_\theta(X) = e^\theta X,$$

and set

$$(U_\theta f)(a) = (\det\, D_\theta \Phi)^{\frac{1}{2}} f(e^\theta a) = J_\theta^{\frac{1}{2}} (\Phi_\theta^* f)(a), \ a \in \mathfrak{a}.$$

When $\theta \in \mathbb{R}$, each $U_\theta$ is unitary on $L^2(\mathfrak{a}, |W|^{-1}\pi_* dg)^W$. Specific properties of the analytic extension of this family can be deduced from the explicit expression

$$J_\theta(a) = (\det\, D_\theta \Phi)(a) = w^n \prod_{\alpha \in \Lambda^+} \left( \frac{\sinh(w\alpha(a))}{\sinh(\alpha(a))} \right)^{m_\alpha}, \ a \in C^+, \ w = e^\theta.$$

Now define

$$\Delta_{\mathrm{rad},\theta} = U_\theta \Delta_{\mathrm{rad}} U_\theta^{-1};$$

this is a $W$-invariant differential operator on $\mathfrak{a}$, and working with the explicit expression for $\Delta_{\mathrm{rad}}$ one sees that its coefficients depend analytically on $\theta$ in $D = \{\theta : |\mathrm{Im}\theta| < \pi/2\} \subset \mathbb{C}$. The key point in all of these definitions is that this conjugation rotates the continuous spectrum by an angle $-2\theta$, and in fact

$$\mathrm{spec}(\Delta_{\mathrm{rad},\theta}) = \sigma_0 + e^{-2i\theta}[0, \infty).$$

The corresponding resolvent

$$R_\theta(\sigma) = (\Delta_{\mathrm{rad},\theta} - \sigma)^{-1} = U_\theta R(\sigma) U_\theta^{-1}$$

is defined and holomorphic in the complement of this set and – this is the important point – can be studied by parametrix methods there.

To be concrete, suppose that $\sigma$ lies in the region $\Im\sigma < 0$, just below $\mathrm{spec}(\Delta_{\mathrm{rad}})$, and we wish to continue $R(\sigma)$ as $\sigma$ moves up to, and then beyond the spectrum. By the inductive hypothesis, the resolvents on all symmetric spaces of lower rank have a continuation; there is also a continuation of the resolvents of the constant coefficient operators $L_a$ appearing in $P_a$, and so, by the product formula, $P_a$ itself has a continuation. Very nicely, the ramification point $\sigma_0$ is the same for all of these operators. From this we see that the parametrix defined in the last section has a meromorphic continuation, and the continuation of $R(\sigma)$ itself follows from the analytic Fredholm theorem.

We note that this overall strategy was also employed by C. Gerard [**7**] and B. Simon [**20**] to prove similar continuation results for a class of $N$-body Hamiltonians.

There are some weaknesses of our analytic method. The chief one here is that we are able to get very little information about the precise region to which $R(\sigma)$ continues. The problem is that even though $\Delta_{\mathrm{rad},\theta}$ is defined for $|\Im\theta| < \pi/2$, the continuation of $R$ might be ramified at interior points in this sector. We explain this with some examples. There is an explicit formula for $R(\sigma)$ when $M = \mathbb{H}^n$, cf. [**13**], which shows that this resolvent continues to the Riemann surface for $\sqrt{\sigma}$, and the only possible poles occur along the single ray $|\arg\sigma| = 3\pi$ (corresponding to $|\Im\theta| = \pi/2$). In particular, the continuation is pole-free in $|\Im\theta| < \pi/2$. This case is used as a subsystem in the construction of the resolvent for $M_3$; however, for this rank 2 space, using only the abstract methods, we can deduce the existence of a continuation into $|\arg\sigma| < 3\pi$, but possibly with poles of finite rank. These poles do not actually occur, but must be ruled out using special global properties of the symmetric space. Next, for $M_4$ one of the subsystem Hamiltonians is $P_a = \Delta_{\mathrm{rad},M_3} + L_a$, and the resolvent of this model would extend only to a more complicated Riemann surface if the continued resolvent of the first summand had poles: by the product formula, the resolvent of $P_a$ would ramify at points in the continuation sector which are translates of the poles of the first summand by the threshold of the second. In other words, $(P_a - \sigma)^{-1}$ might continues to a highly ramified surface, and then, a fortiori, so would $R(\sigma)$ itself.

This complicated scenario could well (and probably does) occur even for spaces which are compact (metric and/or topological) perturbations of globally symmetric spaces. In order to show that the structure is simpler for symmetric spaces, one must bring in more features of the special structure. It is not clear how much is actually needed, but if one grants the the properties of the $\mathfrak{c}$ function, then it is proved in [**17**] that no extra poles or ramification points occur in the region $|\Im\theta| < \pi/2$, cf. also the subsequent paper [**21**].

## References

[1] J.-P. Anker and L. Ji. Comportement exact du noyau de la chaleur et de la fonction de Green sur les espaces symétriques non-compacts. *C. R. Acad. Sci. Paris*, Sér. I, 326:153–156, 1998.

[2] E. Balslev and J. M. Combes. Spectral properties of many body Schrödinger operators with dilation analytic potentials. *Commun. Math. Phys.*, 22:280–294, 1971.

[3] M. Ben-Artzi and A. Devinatz. Resolvent estimates for a sum of tensor products with applications to the spectral theory of differential operators. *J. d'Analyse Math.*, 43:215–250, 1983/84.

[4] P. Eberlein. *Geometry of nonpositively curved manifolds*. University of Chicago Press, 1996.

[5] Flensted-Jensen. Spherical functions on a real semisimple Lie group. A method of reduction to the complex case. *Jour. Func. Anal.*, 30: 106-146, 1978.

[6] R. Gangolli R. Gangolli and V.S. Varadarajan. Harmonic analysis of spherical functions on real reductive groups. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, Berlin (1988).

[7] C. Gérard. Distortion analyticity for $N$-particle Hamiltonians. *Helv. Phys. Acta*, 66(2):216–225, 1993.

[8] Y. Guivarc'h, L. Ji, and J. C. Taylor. *Compactifications of symmetric spaces.* Birkhäuser, Boston, 1998.
[9] S. Helgason. *Differential geometry, Lie groups and symmetric spaces.* Academic Press, 1978.
[10] S. Helgason. *Groups and geometric analysis.* Academic Press, 1984.
[11] P. Loya The structure of the resolvent of elliptic pseudodifferential operators. Jour. Funct. Anal. 184(1): 77-135, 2001.
[12] R. Mazzeo Elliptic theory of differential edge operators, I. *Comm. PDE*, 16(10): 1616-1664, 1991.
[13] R. Mazzeo and R. B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Func. Anal.*, 75:260–310, 1987.
[14] R. Mazzeo and A. Vasy. Resolvents and Martin boundaries of product spaces. *Geom. Func. Anal.*, 12:1018–1079, 2002.
[15] R. Mazzeo and A. Vasy. Analytic continuation of the resolvent of the Laplacian on SL(3)/SO(3). *Preprint*, 2002. arXiv:math.AP/0208116 To appear *Amer. Jour. Math.*
[16] R. Mazzeo and A. Vasy. Scattering theory on SL(3)/SO(3): connections with quantum 3-body scattering. *Preprint*, 2002. arXiv:math.AP/0206118.
[17] R. Mazzeo and A. Vasy. Analytic continuation of the resolvent on symmetric spaces of noncompact type. *Preprint*, 2003. arXiv:math.AP/0308043
[18] R. B. Melrose. *Geometric scattering theory.* Cambridge University Press, 1995.
[19] R. B. Melrose *The Atiyah-Patodi-Singer index theorem.* A.K. Peters, 1993.
[20] B. Simon The definition of molecular resonance curves by the method of exterior complex scaling. *Phys. Lett.*, 71A:211–214, 1979.
[21] A. Strohmaier. Analytic continuation of resolvent kernels on noncompact symmetric spaces. *Preprint*, 2003. arXiv:math.FA/0310395.
[22] A. Vasy. Propagation of singularities in three-body scattering. *Astérisque*, 262, 2000.
[23] A. Vasy. Propagation of singularities in many-body scattering. *Ann. Sci. École Norm. Sup. (4)*, 34:313–402, 2001.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305, USA
*E-mail address*: mazzeo@math.stanford.edu