Self-similar expanding solutions for the planar network flow

Rafe Mazzeo *
Stanford University

Mariel Saez †
Max Planck Institut für Gravitationsphysik

Abstract

We prove the existence of self-similar expanding solutions of the curvature flow on planar networks where the initial configuration is any number of half-lines meeting at the origin. This generalizes recent work by Schnürer and Schulze which treats the case of three half-lines. There are multiple solutions, and these are parametrized by combinatorial objects, namely Steiner trees with respect to a complete negatively curved metric on the unit ball which span k specified points on the boundary at infinity. We also provide a sharp formulation of the regularity of these solutions at $t = 0$.

1 Introduction

The detailed analysis of the curve-shortening flow for embedded closed curves in the plane was an early success in the field of geometric flows. It is not hard to extend this theory to include curves with boundary which are either fixed (a Dirichlet condition) or constrained to lie on the boundary of a convex domain, for example (a Neumann condition).

Slightly more generally, one might consider the flow by curvature for networks of curves.

Definition 1.1. *A planar network is a finite union of embedded arcs and properly embedded half-lines* $\{\gamma_i\}$ *such that for each* $i \neq j$, $\gamma_i \cap \gamma_j$ *is either empty or else consists of one (or both) boundary points of each curve. Each boundary points of every* γ_i *is either a boundary or interior vertex. These intersections are called the (interior) vertices of the network. The boundary of the network*

*Department of Mathematics, Stanford University, Stanford, CA 94305; Email: mazzeo@math.stanford.edu; Supported by the NSF under grant DMS-050579

†Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, D-14476 Golm, Germany; Email: mariel@aei.mpg.de

consists of the set of points which are endpoints of exactly one of these curves. The number of curves intersecting at each interior vertex is called the valence of that vertex. We always assume that all curves are at least $\mathcal{C}^2$

One must formulate the evolution equation for the flow near vertices weakly, and this is commonly done using Brakke's ideas [Br]. However, certain features of the short-time existence, and many other aspects of the long-time existence and convergence, for this network flow have proved elusive. The first thorough analytic attack on this problem was undertaken by Mantegazza, Novaga and Tortorelli [MNT] several years ago. Their results are primarily directed at networks which consist of only three arcs meeting at equal angles at a single interior vertex. The primary difficulties include the choice of boundary conditions at this vertex, and later (for more complicated initial networks) the possibility of creation and annihilation of such vertices during the flow. They obtain some interesting convergence results under certain hypotheses, but many difficult questions remain.

Let us say that a network is *regular* if each interior vertex is trivalent and the curves meet at equal angles (of $2\pi/3$) there. Brakke flow preserves regularity of vertices, at least locally in t. More precisely, if the edges of a network are evolving independently of one another by curvature flow in such a way that at later times t, the edges still fit together into a network and all vertices are regular, then this network is evolving by Brakke flow. Slightly more generally, if a network is evolving so that at each vertex, the sum of the unit tangents to all incoming curves vanishes, then it too is evolving by Brakke flow. In this paper we shall restrict to 'classical' solutions of Brakke flow satisfying the first condition, i.e. that all vertices are regular.

It is important to move beyond the restricted class of initial configurations considered in [MNT] and consider the flow starting at a more general network with multivalent vertices. One motivation is that such 'nonregular' networks may appear at discrete values of time when vertices collide, so it is important to understand how to flow past them. This paper takes the first step in proving short-time existence for the curvature flow on networks when the initial configuration is one of these more general networks. We prove here the existence of self-similar solutions, i.e. expanding solitons for the flow, when the initial network is a finite union of half-lines intersecting at the origin. The solution with this initial condition is far from unique, but we are able to describe the set of all solutions which are regular when $t > 0$. Finally, we also present a somewhat new perspective which leads to a sharp regularity statement at 'irregular vertices' at time 0. In a forthcoming sequel to this paper we shall apply our results here to prove short-time existence for the network flow starting from fairly general initial networks.

Our approach is inspired by a quite recent paper by Schnürer and Schulze [SS], in which they prove the existence and uniqueness of a self-similar solution when the initial condition consists of three half-lines meeting at the origin, but not necessarily in equal angles. Their solution is regular for $t > 0$ and remains a union of three properly embedded arcs meeting at a common vertex. They do not state the precise trajectory of this vertex. In contrast, the self-similar solutions here, which start from a union of at least four half-lines meeting at 0, immediately break up into a (possibly disconnected) regular network with multiple interior vertices and remain so for all later times. As we explain, the trajectories of these vertices are easy to determine. This 'explosion' of a nonregular vertex into a more complicated network provides the model for the short-time existence for the general network flow, and the multiplicity of self-similar solutions corresponds to the nonuniqueness of solutions with a given initial condition.

Imposing self-similarity is tantamount to a dimension reduction of the equation, which transforms this problem into an ODE. Schnürer and Schulze derive certain convexity properties of solutions of this ODE, which were key to their analysis. However, there is a somewhat broader and more natural geometric picture which we explain here, that solutions of this ODE are geodesics for a certain complete metric on the plane, and the curvature properties of this metric provide a concise explanation for those convexity properties. Furthermore, the identification of these curves with geodesics allows us to use variational arguments to prove the existence of the more complicated regular networks which provide the solutions to our problem.

To state our main result, let us introduce some notation. Let B be the ball, regarded as the stereographic compactification of $\mathbb{R}^2$. A union of k half-lines C_0 meeting at the origin in $\mathbb{R}^2$ determines a finite collection of points $p_1, \ldots, p_k \in \partial B$. Define the metric

$$g = e^{x^2+y^2}(dx^2 + dy^2).$$

This is complete and negatively curved, with curvature tending to 0 at infinity. We can now state our main result.

Main Theorem. *Let C_0 be a finite union of half-lines in the plane meeting at 0, and $p_1, \ldots, p_k$ the corresponding points on ∂B. The set of self-similar solutions of the curve-shortening flow with initial condition C_0 is in bijective correspondence with the set of possibly disconnected regular networks on B, each arc of which is a geodesic for g, with boundary the k prescribed points at infinity. There always exists at least one (and often very many) connected geodesic Steiner tree with these asymptotic boundary values, and indeed, there also exist geodesic Steiner trees with precisely s components corresponding to any 'increasing' partition $\{1, \ldots, k\} = I_1 \cup \ldots, I_s$, where each I_j consists of a consecutive string of integers, with each $|I_j| > 1$. Finally, these self-similar solutions lift to a smooth family of networks on the parabolic blowup of $\mathbb{R}^2 \times \mathbb{R}^+$ at $x = y = t = 0$.*

This general picture was understood qualitatively by Brakke, and hinted at in an appendix to his book [Br]. Unbeknownst to us when we were doing this work, many of the specific facts presented here were discovered in slightly different forms by Tom Ilmanen and Brian White in the mid '90's. Some discussion of this appears in [ACI] and [I], but the focus in those papers is mostly on the higher dimensional case. We hope that this independent and more elementary discussion of the one-dimensional case, along the lines of [SS], is not unwelcome. There are some interesting new points too, including the enumeration of self-similar expanding solutions in terms of (nonelementary) combinatorial data, i.e. the number of Steiner trees in $(\mathbb{R}^2, g)$ spanning the k given points at infinity, as well as the formulation of regularity. The first author wishes to thank Brian White for several very helpful conversations, and in particular for explaining certain aspects of the Brakke flow, and more importantly, 'size minimization' in the class of flat chains mod k, which provides a shortcut to the existence result of §3 which circumvents a more explicit but longer synthetic approach. Frank Morgan also provided some useful comments. The second author wishes to thank Marilyn Daily and Felix Schulze for pointing out appropriate references.

The next section describes solutions of the dimension reduced equation and the geometry of the metric g; the main existence result for regular networks with prescribed asymptotes is proved in §3; finally, we make some remarks about regularity at $t = 0$ in §4, explaining the last assertion of the main theorem.

2 Self-similar solutions of curve-shortening flow

Let γ_0 be an immersed curve in the plane. The curve-shortening flow with initial condition γ_0 is the evolution leading to the family of curves γ_t, $t \geq 0$, where

$$\frac{d}{dt}\gamma_t = \kappa(\gamma_t)\nu(\gamma_t);$$

here κ is the curvature and ν the unit normal to γ_t. The well-known theorem of Grayson asserts that if γ_0 is closed and embedded, then γ_t remains embedded and shrinks to a point at some finite time T. Moreover, for an appropriate choice of 'center' P, $(T-t)^{-1/2}(\gamma_t - P)$ converges in $\mathcal{C}^\infty$ to the circle S^1.

We are being sloppy here and conflating the curves γ_t with embeddings F_t from S^1 (for example) to $\mathbb{R}^2$ with image γ_t. It is frequently convenient to consider a modified flow equation which includes an extra tangential term; the flow leads to the same family of curves, but alters their parametrizations. We mostly work with parametrized curves below, and the equations of motion will have an extra tangential term, i.e. have the form $\dot{\gamma}_t = \kappa\nu + f\gamma'$ for some function f.

A solution $\{\gamma_t\}_{t\geq 0}$ of this equation is called self-similar, or a soliton, if γ_t is similar to γ_0 for all $t > 0$. This notion can be defined for curvature flows whenever the ambient space has a Killing field [HS]. We are particularly interested in solutions for which γ_t is simply a dilation of γ_0, $\gamma_t = \lambda(t)\gamma_0$ for some function $\lambda(t)$. An alternate way to phrase this uses the family of parabolic dilations D_λ on $\mathbb{R}^2 \times \mathbb{R}^+$, $\lambda > 0$:

$$D_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

Any solution of the curve-shortening flow determines a 'world-sheet'

$$\Gamma = \bigcup_{t\geq 0} \gamma_t \times \{t\} \subset \mathbb{R}^2 \times \mathbb{R}^+,$$

and self-similarity is equivalent to the requirement that $D_\lambda(\Gamma) = \Gamma$ for all $\lambda > 0$. (This corresponds to the expanding case; shrinking self-similar solutions have world-sheets $\Gamma \subset \mathbb{R}^2 \times \mathbb{R}^-$, so that $t = 0$ is the time of extinction, or at least, the boundary of Γ.) It is clear that such a solution, if it exists, is determined by the set $\gamma_{1/2} = \Gamma \cap \{t = 1/2\}$ (the reason for using $t = 1/2$ rather than $t = 1$ is to make various other equations neater); furthermore, γ_0 is simply the (unique) tangent cone at infinity of $\gamma_{1/2}$.

Self-similarity transforms the curve-shortening flow into a stationary equation for this curve $\gamma_{1/2}$, which is simply

$$\kappa = (x, y) \cdot \nu. \tag{1}$$

To prove this, fix a parametrization $F(u)$ of (some piece of) the curve at $t = 1/2$. Then,

$$F_t(u) = \lambda(t)F(u/\lambda(t))$$

parametrizes the corresponding part of that curve at any other time, where $\lambda(t)$ is some function to be determined with $\lambda(1/2) = 1$. Setting this expression into the equation yields

$$\frac{d}{dt}F_t(u) \cdot \nu = \left(\dot{\lambda}F(u/\lambda) - \frac{\dot{\lambda}}{\lambda}\frac{u}{\lambda}F'(u/\lambda)\right) \cdot \nu = \dot{\lambda}F(u/\lambda) \cdot \nu$$

and

$$\kappa = \frac{1}{\lambda|F'(u/\lambda)|^2}F''(u/\lambda) \cdot \nu,$$

hence

$$\frac{1}{|F'(u/\lambda)|^2}F''(u/\lambda) \cdot \nu = \lambda(t)\dot{\lambda}(t)F(u/\lambda) \cdot \nu. \tag{2}$$

Here $\dot{\lambda}$ is the derivative with respect to t. This can hold for all u and t if and only if $\lambda\dot{\lambda} = c$ is constant, so $\lambda^2 = 2ct + c'$. Since $\lambda(0) = 0$ and $\lambda(1/2) = 1$, we get $\lambda(t) = \sqrt{2t}$, and so $F_t(u) = \sqrt{2t}F(u/\sqrt{2t})$. Setting this in (2) gives (1).

In any case, we can now reformulate our problem as the

Proposition 2.1. *A self-similar solution of the curve-shortening flow with initial condition C_0, the union of a finite number of half-lines meeting at 0, is equivalent to a regular network of curves in $\mathbb{R}^2$, each of which is a solution to (1), with tangent cone at infinity equal to C_0.*

In the remainder of this section, we determine all solutions of (1).

Note first that there is a distinguished subset of solutions, namely the collection of all straight lines through the origin in $\mathbb{R}^2$. Each such line is clearly a solution, since its curvature is zero and the position vector and tangent vector are always multiples of one another, hence orthogonal to the normal at each point. This set of lines gives the full set of solutions passing through the origin, and conversely, any solution passing through 0 is a straight line. In fact, even more is true: any solution γ which has tangent vector a multiple of the position vector at any point is one of these straight lines.

The most convenient parametrization for any other solution is as a normal graph. This is because, following the last remark above, the tangent for all solutions except the ones considered above never points in the radial direction. Thus we seek a function $r(\theta)$ so that the curve is the image of the map $F(\theta) = r(\theta)R(\theta)$, where

$$R(\theta) = (\cos\theta, \sin\theta), \qquad N(\theta) = (-\sin\theta, \cos\theta).$$

We calculate

$$F' = r'R + rN, \qquad F'' = (r'' - r)R + 2r'N,$$

so in particular

$$T = \frac{1}{\sigma}F' = \frac{r'}{\sigma}R + \frac{r}{\sigma}N, \qquad \text{and} \qquad \nu = -\frac{r}{\sigma}R + \frac{r'}{\sigma}N$$

are the unit tangent and normal; here $\sigma = \sqrt{r^2 + (r')^2}$. Hence

$$\kappa\nu = \frac{1}{\sigma}\left(\frac{1}{\sigma}F'\right)' = \frac{1}{\sigma^2}F'' - \frac{\sigma'}{\sigma^3}F'.$$

Rewriting this in terms of r, R and N, we see that

$$\kappa\nu = \left(\frac{r'' - r}{\sigma^2} - \frac{(\sigma')^2}{\sigma^3}r'\right)R + \left(\frac{2r'}{\sigma^2} - \frac{\sigma'}{\sigma^2}r\right)N,$$

so finally

$$\kappa = \frac{1}{\sigma^3}\left(2(r')^2 - rr'' + r^2\right). \tag{3}$$

The right side of (1) is just

$$F \cdot \nu = -\frac{r^2}{\sigma}. \tag{4}$$

Equating these and simplifying yields, finally, the main equation

$$rr'' = r^2 + 2(r')^2 + r^2(r^2 + (r')^2). \tag{5}$$

Proposition 2.2. *Any maximally extended solution $r(\theta)$ of (5) is defined on an interval $(a, b) \subset [0, 2\pi]$ (mod 2π) with $b - a < \pi$, and satisfies:*

i) *r is convex as a function of θ;*

ii) $\lim_{\theta\searrow a} r(\theta) = \lim_{\theta\nearrow b} r(\theta) = \infty$;

iii) *$r(\theta) = r((a+b) - \theta)$, i.e. the image of r is symmetric about the ray which makes angle $(a + b)/2$ with the horizontal.*

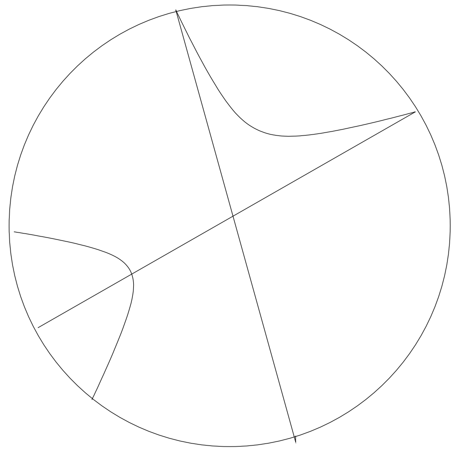

Figure 1: Geodesics for g in the stereographic compactification

Proof. Property i) is obvious directly from (5). If (a,b) is the maximal interval of existence, then by convexity, the two limits in ii) exist. By earlier remarks, neither limit can equal zero, and by maximality of (a,b), these limits cannot be finite either, which establishes ii). Using convexity and properness, we see that there is a unique point $\theta_0 \in (a,b)$ where $r'(\theta_0) = 0$. Since (5) is invariant under the reflection $\theta_0 + \eta \mapsto \theta_0 - \eta$, by uniqueness of solutions of the initial value problem we see that the solution must be invariant under this flip, and hence that $\theta_0 = (a+b)/2$.

To prove that this maximal interval of existence has length less than π, define $v = r'/r$, so that

$$v' = \frac{r''}{r} - \frac{(r')^2}{r^2}.$$

Using (5), we find that

$$v' = (1+v^2)(1+r^2) \Longrightarrow v' \geq (1+v^2).$$

Integrating gives

$$\frac{r'(\theta)}{r(\theta)} \geq \tan(\theta + C)$$

for some C, and this clearly proves the claim.

Note that if r attains its minimum at θ_0 and $r(\theta_0) = r_0$, then

$$v' \geq (1+v^2)(1+r_0^2),$$

hence

$$v(\theta) \geq \tan((1+r_0^2)\theta + C),$$

so that the maximal interval of existence of this solution is of length less than $\pi/(1+r_0^2)$. ☐

These solutions account for all remaining solutions of (1).

The most striking feature of these solutions is that they behave qualitatively exactly like the geodesics on the hyperbolic plane. In particular, we have the

Corollary 2.1. *Let B be the compactification of the stereographic projection of $\mathbb{R}^2$. Then every maximally extended solution of (1) intersects ∂B in precisely two points. Moreover, for any pair of distinct points $p, q \in \overline{B}$, there exists a unique solution γ of (1) which passes through (or terminates) at these two points.*

This behaviour is no accident, since in fact these solution curves are geodesics for a metric of negative curvature!

Proposition 2.3. *Let $g_0 = dr^2 + r^2 d\theta^2$ be the standard Euclidean metric on $\mathbb{R}^2$ and define $g = e^{r^2} g_0$. The geodesics for g are solutions to (1) and conversely.*

Proof. Since the conformal factor e^{r^2} is radial, the straight lines through the origin are geodesics for g. Suppose that $(r(u), \theta(u))$ is the polar representation of any $\mathcal{C}^2$ curve in the plane. The geodesic equations for g are

$$\begin{aligned} \ddot{r} - (r^3 + r)\dot{\theta}^2 + r\dot{r}^2 &= 0 \\ \ddot{\theta} + 2(r + r^{-1})\dot{r}\dot{\theta} &= 0 \end{aligned}$$

(where now the dot refers to derivatives with respect to the parameter u). Assuming that $\dot{\theta} \neq 0$, we consider r as a function of θ and immediately derive that

$$r'' = r + 2r^{-1}(r')^2 + r(r^2 + (r')^2),$$

which is just (5); here $r' = dr/d\theta$.

The proof is complete. □

The Gauss curvature of the metric g is equal to $K = -2e^{-r^2}$, which is everywhere negative. However, since $K \to 0$ at infinity, we cannot circumvent the proposition above about the global behaviour of geodesics by appealing to the properties of geodesics on Cartan-Hadamard manifolds.

One final remark is that any two distinct geodesics of g which converge to the same boundary point $p \in \partial B$ meet tangentially there, but the order of tangency is not quadratic. Indeed, a short estimation shows that while the tangents to these curves do coincide at p, their directions approach one another inversely proportionally to the logarithm of the distance to the origin (with respect to g).

3 Regular geodesic networks

The remaining part of the proof of our main theorem involves showing the existence of a regular network in $\mathbb{R}^2$ where each edge is a geodesic arc or ray for the metric g, and whose tangent cone at infinity is C_0, the union of k half-lines meeting at 0. Alternately, this network spans k specified points $p_1, \ldots, p_k \in \partial B$, where B is the compactification of the stereographic projection of $\mathbb{R}^2$. For convenience, we assume that these points are labelled in consecutive order around the circle (mod k).

When $k = 3$, the existence and uniqueness of this regular network may be accomplished directly by degree theory, cf. [SS]. Certain special cases are also quite easy to handle: for example, if k is even, then a particular solution is the disjoint collection of geodesics $\gamma_1, \ldots, \gamma_{k/2}$, where γ_j connects p_{2j-1} to p_{2j}, $j = 1, \ldots, k/2$. Such a network would correspond to a 'complete dissolution' of this vertex into smooth curves. For the general case, consider any increasing partition $\{1, \ldots, k\} = I_1 \cup \ldots \cup I_s$ where each I_j is a collection of consecutive integers with $|I_j| > 1$. The remainder of this section is devoted to proving the existence of at least one regular geodesic network with precisely s components which spans these k points. Note in particular that this includes the case $s = 1$, where the resulting network is connected. The number of solutions corresponding to any given partition is, by definition, just the number of Steiner trees (with respect to g) where the boundary of the ℓ^{th} component is equal to $\{p_i : i \in I_\ell\}$.

It is possible to construct connected regular geodesic networks with arbitrary prescribed boundary values using synthetic geometry. However, another – particularly efficient – way to obtain existence uses geometric measure theory. The precise formulation from this point of view, as well as relevant literature, was suggested to us by Brian White, to whom we are very grateful.

We shall work in the class $\mathcal{F}_1(\mathbb{R}^2; \mathbb{Z}_k)$ of flat chains of dimension 1 in $\mathbb{R}^2$ with coefficients in the group $\mathbb{Z}_k$, endowed with the norm $|g| = 1$ for all $g \in \mathbb{Z}_k$. Let us explain what this means. First, the space of flat chains of dimension 1 is the completion of the space of polygonal curves with respect to the flat norm. A flat chain with coefficients in $\mathbb{Z}_k$ is an ordinary flat chain such that its (integer) multiplicity function is reduced mod k. The norm on $\mathbb{Z}_k$ appears in the definition of the *size* (rather than mass) of a flat chain $c = \sum g_\tau \tau$, where each $g_\tau \in \mathbb{Z}_k$, which is defined by

$$\mathbb{S}(c) = \sum_\tau |g_\tau| \mathbb{M}(\tau) = \sum_\tau \mathbb{M}(\tau).$$

Here $\mathbb{M}(\tau)$ denotes mass with respect to the metric g, which for a $\mathcal{C}^1$ arc corresponds to the usual length in that metric. We refer to [W] and also [M2] for these facts and for more complete references.

For each j, let ℓ_j denote the ray from 0 to the point p_j at infinity, and for each radius R, set $p_j^R = \ell_j \cap \partial B_R$.

Lemma 3.1. *Fix the partition $\{1, \ldots, k\} = I_1 \cup \ldots I_s$ as above. Then there exists a 1-dimensional flat chain T^R with with coefficients in $\mathbb{Z}_k$ such that $\partial T^R = \{p_1^R, \ldots, p_k^R\}$ and which is size-minimizing, i.e.*

$$\mathbb{S}(T^R) = \inf \{\mathbb{S}(S) : S \in \mathcal{F}_1(B_R; \mathbb{Z}_k),\ \textit{supp}(S) \textit{ connected},\ \partial S = \{p_1, \ldots, p_k\}\}.$$

The support of this minimizer T^R is a regular geodesic network with precisely s components: each edge is a geodesic arc or ray for the metric g and there are a *finite number of interior vertices, each of which is trivalent, where these edges meet in equal angles.*

Proof. This is all standard, but we review the argument briefly. (There is one twist at the end, about the valence of boundary vertices.) For the existence we use the compactness theorem for flat chains, as in [W], cf. also [M1]. Let $\{T_j\}$ be a sequence of elements in $\mathcal{F}_1(B_R;\mathbb{Z}_k)$ where the support of each T_j has precisely s components, i.e. $T_j = T_j^{(1)} \cup \ldots T_j^{(s)}$ where $\partial T_j^{(\ell)} = \{p_j : j \in I_\ell\}$, and such that $\mathbb{S}(T_j)$ converges to the minimum possible value in this class of competitors. Note that we can separate the supports of the individual $T_j^{(\ell)}$ by geodesics. For simplicity in the remainder of this argument, assume that $s = 1$ so that the support of T_j is connected; the modifications for the general case are straightforward. Clearly $\mathbb{S}(T_j) \leq C$ for some $C > 0$, and in addition $\mathbb{S}(\partial T_j) = k$, for all j. The compactness theorem implies that there is a convergent subsequence, relabeled again as T_j, with limit T^R. This has the same boundary, and lower-semicontinuity of $\mathbb{S}$ implies that

$$\mathbb{S}(T^R) \leq \lim_{j\to\infty} \mathbb{S}(T_j),$$

so that T^R is indeed a size minimizer.

For the regularity of T^R, we use [AA], but see also [M1], [M2] and [T]. The regularity theorem in [AA] implies that the singular set has Hausdorff dimension 0, and in fact consists of a finite number of points. By the first variation formula, the regular set consists of a finite number of geodesic arcs or rays. At each interior vertex p, following [M1], consider the following variation. Let v, w be unit vectors tangent to two adjacent edges ℓ_v and ℓ_w which meet at p. Let u be another unit vector in the positive cone determined by v and w, and consider the new network which replaces these two edges by a triod with one very short edge of length ϵ along the geodesic starting at p in the direction u and the other two edges the geodesics from the other end of that short geodesic to the other ends of ℓ_v and ℓ_w. The multiplicity of the short geodesic should equal the sum of the multiplicities of ℓ_v and ℓ_w, while the two new longer geodesics should have the same multiplicities as those, respectively. We compute that

$$0 \leq \left. \frac{d}{d\epsilon}\mathbb{S}(T^R_\epsilon)\right|_{\epsilon=0} = 1 - (v+w)\cdot u;$$

the inequality holds because T^R is minimizing. In particular, setting $u = \frac{v+w}{|v+w|}$, we conclude that $1 \leq |v+w|$, or equivalently $v \cdot w \leq -\frac{1}{2}$. Hence the angle between v and w is at least $2\pi/3$. Therefore, at most three edges can meet at p, and if there are three incoming vertices, then these must meet at $2\pi/3$. If only two edges meet at an interior vertex p, then the geodesic connecting the other two endpoints of these edges would be shorter.

It can happen in certain geometries that precisely two vertices meet at a boundary point, though of course by the preceding argument, the angle between them must be at least $2\pi/3$. Since ∂B_R is convex, neither of these edges will lie along this boundary. We claim that this is impossible once R is sufficiently large. Indeed, the convex hull of the points $p_1^R, \ldots, p_k^R$ in B_R is a polygon with geodesic sides, and with angle at each vertex tending to 0 as $R \to \infty$, cf. the final remark of §2. Hence these boundary vertices must be univalent as soon as R is sufficiently large so that the opening angle of this convex hull is less than $2\pi/3$.

To conclude, we must show that T^R is connected. Since convergence in flat norm implies convergence as currents (see [S]),

$$\int_{T_j} \phi \, dT_j \to \int_{T^R} \phi \, dT^R \tag{6}$$

for every compactly supported ϕ. Suppose that the support of T^R is disconnected. Then there is a curve γ in B_R such that $B_R \backslash \gamma$ has two components, each intersecting the support of T^R nontrivially. Denote by $\mathcal{U}_\epsilon$ the ϵ-neighbourhood around γ. For sufficiently small ϵ, $\mathcal{U}_\epsilon \cap \operatorname{supp}(T^R) = \emptyset$. Consider a nonnegative $\phi \in \mathcal{C}_0^\infty$ with support in $\mathcal{U}_\epsilon$ which equals 1 in $\mathcal{U}_{\epsilon/2}$. Since T_j is connected

$$\int_{T_j} \phi \, dT_j \geq \frac{\epsilon}{4}$$

for all j, but on the other hand

$$\int_{T^R} \phi \, dT^R = 0,$$

which contradicts (6). This finishes the proof. □

To obtain a network which spans the points $p_1, \ldots, p_k \in \partial B$, we take the limit of T^R as $R \to \infty$.

Proposition 3.1. *Let R_j be a sequence of radii tending to infinity, and let T_j be one of the connected size-minizing flat chains with coefficients in $\mathbb{Z}_k$ with $\partial T_j = \{p_1^{R_j}, \ldots, p_k^{R_j}\}$ obtained in the previous lemma. As $j \to \infty$, some subsequence of the T_j converges (in the flat topology) to a locally size-minimizing connected flat chain with $\partial T = \{p_1, \ldots, p_k\}$.*

Proof. Let $\mathcal{P}$ denote the ideal k-gon which is the convex hull of the points $p_1, \ldots, p_k$, and $\mathcal{P}_j$ the convex hull of $p_1^{R_j}, \ldots, p_k^{R_j}$. As already used in the last proof, the support of each T_j lies in $\mathcal{P}_j \subset \mathcal{P}$, and $\mathcal{P}_j \nearrow \mathcal{P}$.

Next, let us observe that the total number of interior vertices in the support of each T_j remains fixed. Indeed, if ℓ denotes the number of interior vertices and e the number of edges, then

$$\begin{aligned} 3\ell + k \quad &= 2e \\ \ell + k \quad &= e + 1. \end{aligned}$$

The first equation uses that each interior vertex is trivalent and each boundary vertex connects to only one edge; the second equation asserts that the Euler characteristic of a tree is equal to 1. Subtracting the second equation from the first gives $2\ell = e - 1$, and hence, after some manipulation

$$\ell = k - 2.$$

We now claim that no interior vertex can converge to any one of the boundary vertices, and hence disappear in the limit. Indeed, if an interior vertex q lies in the cusp of this convex hull corresponding to some p_i, then at most one of the three edges which meet at q is directed 'outwards', toward p_i and the others must be pointed inward. However, since the tree remains inside the convex hull, the extensions of either of the other edges must hit the boundary of the convex hull in some distance which we can estimate from the position of q. Since this is impossible, there must be two new interior vertices. We can repeat this argument a finite number of times. If q is sufficiently deep into this corner, the tree would have more than $k-2$ additional vertices in just this neighbourhood, which is impossible.

This shows that all $k-2$ interior vertices remain within some fixed ball B_R, and hence we can take a limit of the T_j and obtain a nontrivial limit T. Clearly $\partial T = \{p_1, \ldots, p_k\}$, as required, and by the same argument as above, the support of T is connected. Finally, it is a standard fact that the limit of a convergent sequence of mass-minimizing currents is again mass-minimizing. This transfers immediately to the setting of flat chains with coefficients in $\mathbb{Z}_k$, with the given norm on this cyclic group. □

It is worth remarking that this result (and argument) is close in spirit to the construction in [An] of complete mass-minimizing submanifolds in hyperbolic space which have a prescribed asymptotic boundary at infinity.

4 Behaviour of the flow at $t = 0$

We conclude this paper with some brief remarks about a geometric way to formulate the precise regularity of these self-similar solutions to the network flow. This will be expanded on considerably in our subsequent paper on general short-term existence results, where it plays a more crucial role, but is worth sketching here since it provides a conceptually appealing way to think of this flow.

Even when $k = 3$ but the initial configuration is not regular (i.e. the half-lines do not meet at equal angles), there seems to be a sudden jump in the configuration as soon as t becomes positive. When $k > 3$, this jump is even more pronounced since new vertices and edges are created instantaneously. As we now explain, there is a way of viewing all of this, however, which makes this behaviour continuous. Introduce the parabolic blowup of $\mathbb{R}^2 \times \mathbb{R}^+$ at $\{x = y = t = 0\}$. This is a manifold X with corners up to codimension two which is obtained by taking the union of $\{(x, y, t) : t \geq 0, (x, y, t) \neq (0, 0, 0)\}$ and a new hypersurface F, which is the 'parabolic spherical normal bundle' of the origin with respect to the family of dilations D_λ introduced in §2, but more colloquially also the front face of this new space. In other words, every point of F corresponds to an orbit of this dilation group. If we were using ordinary dilations $((x, y, t) \to (\lambda x, \lambda y, \lambda t))$ then this would be the more familiar normal blowup, which can be described easily in terms of polar coordinates around the origin: indeed, the front face in a normal blowup is the one obtained by setting the radial variable equal to 0. The picture is still qualitatively the same here since the front face is diffeomorphic to a half-sphere. The two codimension one boundaries of X are the front face F, described above, and the compactification of the original boundary minus the origin, which we denote T. Note that T is naturally the complement of a ball in $\mathbb{R}^2$, while F is a half-sphere.

In the previous sections, we were using an identification of the slice $\{t = 1\}$ with an open hemisphere via ordinary stereographic projection. There is a similar identification of this slice with the interior of the face F defined by the dilations D_λ. This identification is *not* conformal.

The world-sheet Γ of any self-similar solution of the network flow is, away from the origin in $\mathbb{R}^2 \times \mathbb{R}^+$, a union of a finite number of surfaces with boundary, mutually intersecting in triples along their boundaries; these intersection curves are also orbits of the family of dilations. The entire surface Γ is a union of dilation orbits $\{(\lambda x_0, \lambda y_0, \lambda^2) : \lambda > 0\}$. By definition of the parabolic blowup, the closure of $\Gamma \setminus (0, 0, 0)$ in X intersects each of the two boundary faces F and T in a collection of curves. The intersection $\Gamma \cap T$ consists of the original collection of half-lines (now lifted to X so that they no longer intersect), while $\Gamma \cap F$ is a regular network on the hemisphere. The intersections of these curves at the corner $F \cap T$ yield the k boundary points $p_1, \ldots, p_k$. We illustrate this when $k = 4$.

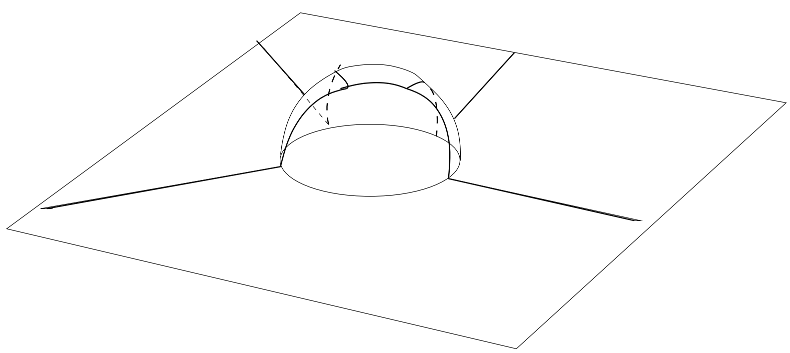

Figure 2: The blowup and induced network at $t = 0$

The important observation, which is essentially tautological, is that all of these component surfaces in Γ, as closed surfaces with boundary, are smooth submanifolds (up to the boundary!) in X.

Proposition 4.1. *The closure of* $\Gamma \setminus \{(0, 0, 0)\}$ *in* X *is a union of a finite number of smooth closed surfaces with boundary and corners in* X*; these intersect along a finite number of smooth curves, each of which is a dilation orbit.*

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