

# Poincaré-Einstein metrics and the Schouten tensor

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## Abstract

We examine here the space of conformally compact metrics  $g$  on the interior of a compact manifold with boundary which have the property that the  $k^{\text{th}}$  elementary symmetric function of the Schouten tensor  $A_g$  is constant. When  $k = 1$  this is equivalent to the familiar Yamabe problem, and the corresponding metrics are complete with constant negative scalar curvature. We show for every  $k$  that the deformation theory for this problem is unobstructed, so in particular the set of conformal classes containing a solution of any one of these equations is open in the space of all conformal classes. We then observe that the common intersection of these solution spaces coincides with the space of conformally compact Einstein metrics, and hence this space is a finite intersection of closed analytic submanifolds.

Let  $\overline{M}^{n+1}$  be a smooth compact manifold with boundary. A metric  $g$  defined on its interior is said to be *conformally compact* if there is a nonnegative defining function  $\rho$  for  $\partial\overline{M}$  (i.e.  $\rho = 0$  only on  $\partial\overline{M}$  and  $\nabla\rho \neq 0$  there) such that  $\overline{g} = \rho^2 g$  is a nondegenerate metric on  $\overline{M}$ . The precise regularity of  $\rho$  and  $\overline{g}$  is somewhat peripheral and shall be discussed later. Such a metric is automatically complete. Metrics which are conformally compact and also Einstein are of great current interest in (some parts of) the physics community, since they serve as the basis of the AdS/CFT correspondence [23], and they are also quite interesting as geometric objects. Since they are natural generalizations of the hyperbolic metric on the ball  $B^{n+1}$ , as well as the complete constant negative Gauss curvature metrics on hyperbolic Riemann surfaces – which exist in particular on the interiors of arbitrary smooth surfaces with boundary – and which are often called Poincaré metrics [19], we say that a metric which is both conformally compact and Einstein is *Poincaré-Einstein* (or P-E for short). Until recently, beyond a handful of examples, the only general existence result concerning the existence of P-E metrics was the local perturbation theory of Graham and Lee [10], which gives an infinite dimensional family of such metrics in a neighbourhood of the

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hyperbolic metric on the ball, parametrized by conformal classes on the boundary sphere near to the standard one. Recently many new existence results have been obtained, including further perturbation results by Biquard [6] and Lee [12], and Anderson has announced some important global existence results in dimension four [3]. Many interesting geometric and topological properties of these metrics have also been found [9], [22], [1], [2]; this last paper also surveys a number of intriguing examples of P-E metrics.

A common thread through the analytic approaches to the construction of these metrics is the possible existence of an  $L^2$  obstruction, or more simply a finite dimensional cokernel of the (suitably gauged) linearization of the Einstein equations around a solution. For any P-E metric where this obstruction is trivial, the implicit function theorem readily implies that the moduli space  $\mathcal{E}$  of P-E metrics is (locally) a Banach manifold, parametrized by conformal classes of metrics on  $\partial\overline{M}$ . (Actually, the smoothness of  $\mathcal{E}$  is true in generality [3], but this geometric parametrization breaks down.) Unfortunately, the only known geometric criteria ensuring the vanishing of this obstruction are strong global ones [12].

The purpose of this note is to introduce some new ideas into this picture which may help elucidate the structure of this moduli space. We consider a related family of conformally compact metrics which satisfy a (finite) set of *scalar* nonlinear equations, including and generalizing the familiar Yamabe equation, which we introduce below. These are sometimes called the  $\sigma_k$ -Yamabe equations,  $k = 1, \dots, n+1$ . The hyperbolic metric on the ball, or indeed an arbitrary P-E metric on any manifold with boundary satisfies each of these equations, and conversely, in this particular (conformally compact) setting, metrics which satisfy every one of these scalar problems are also P-E. The punchline is that the deformation theory for the  $\sigma_k$ -Yamabe equations is always unobstructed! This fact seems to have been unappreciated even in the setting of compact manifolds without boundary (except for the case  $k = 1$ ). The full implications of this statement in the conformally compact case for the moduli space of P-E metrics is not completely evident at this point, but this relationship seems quite likely to be of some value. Furthermore, the deformation theory for these  $\sigma_k$ -Yamabe metrics is new, and also of some interest.

To define these equations, recall the Schouten tensor  $A_g$ , defined for any metric  $g$  on a manifold of dimension  $n + 1$  by the formula

$$A_g = \frac{1}{n-1} \left( \text{Ric} - \frac{R}{2n}g \right); \quad (1)$$

here  $\text{Ric} = \text{Ric}_g$  and  $R = R_g$  are the Ricci tensor and scalar curvature function for  $g$ . This tensor occupies a prominent position in conformal geometry because it transforms quite nicely under conformal changes of metric. In fact, if  $\tilde{g} = e^{2u}g$ , then

$$A_{\tilde{g}} = A_g - \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g. \quad (2)$$

Notice also that  $g$  is Einstein if and only if  $A_g$  is diagonal:

$$\text{Ric}_g = \frac{R}{n+1}g \iff A_g = \frac{R}{2n(n+1)}g. \quad (3)$$

The metric  $g$  is a  $\sigma_k$ -Yamabe metric if the  $k^{\text{th}}$  elementary symmetric function of the eigenvalues of  $A_g$  are constant,  $\sigma_k(A_g) = C$ . This is usually posed as a problem in conformal geometry: starting with an arbitrary metric  $g$ , the  $\sigma_k$ -Yamabe problem is to find a new metric  $\tilde{g} = e^{2u}g$  in the conformal class of  $g$  such that  $\sigma_k(A_{\tilde{g}}) = C$ . Notice that  $\sigma_1(A_g) = R/2n$ , and so  $\tilde{g}$  is a  $\sigma_1$ -Yamabe metric if and only if its scalar curvature is constant. When  $M$  is compact (without boundary), this equation is semilinear (for the function  $v$  defined by  $v^{4/(n-1)} = e^{2u}$ ) and the existence theory is complete and by now well known [13]. However, when  $k > 1$  these equations are fully nonlinear and the existence theory is much less well understood. Recent significant progress has been made by Chang-Gursky-Yang [7] when  $k = 2$ , and also by Viaclovsky [21], but much remains to be understood. In particular, in contrast with the ordinary Yamabe problem, for  $k > 1$  the  $\sigma_k$ -Yamabe problem seems to be somewhat more tractable for positively curved metrics: a crucial a priori  $C^2$  estimate is missing in the negative case [21].

We now write out the  $\sigma_k$ -Yamabe equations (within a conformal class) more explicitly. Fixing  $g$  and using (2), we see that

$$\tilde{g} = e^{2u}g \quad \text{satisfies} \quad \sigma_k(A_{\tilde{g}}) = (-1)^k C \quad (4)$$

provided

$$\mathcal{F}_k(u, C) = \sigma_k(\nabla^2 u - du \otimes du + \frac{1}{2}|\nabla u|^2 g - A_g) - C e^{2ku} = 0. \quad (5)$$

The symmetric function of the eigenvalues of  $A_{\tilde{g}}$  here is computed with respect to  $g$  rather than  $\tilde{g}$ , which accounts for the exponential factor; the sign on the final term comes from taking  $\sigma_k$  of  $-A_{\tilde{g}}$ . We have (momentarily) suppressed the dependence of the functional  $\mathcal{F}_k$  on  $g$ , although this will be important later. In fact we shall also regard  $\mathcal{F}_k$  as a functional on the space of metrics as follows. Let  $\mathfrak{S}$  be a (local) slice of the set of conformal classes of  $M$ ; in other words, we assume that  $\mathfrak{S}$  contains one metric from each conformal class in a small neighbourhood in the space of all conformal classes. It is customary when defining these slices also to mod out by the diffeomorphism group, but we do not do this here as there is no particular need. Then we may also consider  $(g, u) \rightarrow \mathcal{F}_k(g, u, C)$ , for  $g \in \mathcal{S}$  and  $u$  a (smooth) function on  $M$ . For any constant  $\beta_k$ , define

$$\Sigma_k(\beta_k) = \{(g, u) : \mathcal{F}_k(g, u, \beta_k) = 0\}. \quad (6)$$

Clearly this set does not depend on the choice of slice  $\mathfrak{S}$ , and so defines a subset within the space of all metrics on  $M$ .

As already indicated, the main result here involves the perturbation theory for solutions of  $\mathcal{F}_k$ , or equivalently, the structure of the sets  $\Sigma_k(\beta_k)$ , in the case

where  $M^{n+1}$  is a manifold with boundary and all metrics are conformally compact. We use the same notation, namely  $\Sigma_k(\beta_k)$ , to denote the set of conformally compact  $\sigma_k$ -Yamabe metrics on  $M$ . Since any conformally compact metric has asymptotically negative (in fact, isotropic) sectional curvatures, we see that of necessity  $\beta_k > 0$ . In particular, the particular constants  $\beta_k$  corresponding to the hyperbolic metric  $g_0$  on  $B^{n+1}$  are

$$\beta_k^0 = 2^{-k} \binom{n+1}{k}. \quad (7)$$

**Theorem 1** *Fix  $\beta_k > 0$ . If  $g \in \Sigma_k(\beta_k)$  (of a regularity to be specified later), then there is a neighbourhood  $\mathcal{U}$  of  $g$  in the space of all conformally compact metrics on  $M$  such that  $\mathcal{U} \cap \Sigma_k(\beta_k)$  is an analytic Banach submanifold of  $\mathcal{U}$ , with respect to an appropriate Banach topology.*

This theorem gives a rich class of conformally compact  $\sigma_k$ -Yamabe metrics on the manifold  $M$ , granting the existence of at least one such metric. In particular, it states that the deformation theory for this problem is always unobstructed whenever  $\beta_k > 0$ . As noted above, the analogue of this theorem holds also when  $M$  is compact without boundary, and the proof is similar but even more straightforward. For the record, we state this result too:

**Theorem 2** *Fix  $\beta_k > 0$ . Let  $g$  be a metric on the compact manifold  $M$  and  $[g]$  its conformal class. Suppose that  $\sigma_k(A_g) = (-1)^k \beta_k$ . Then there is a neighbourhood  $\mathcal{U}$  of  $[g]$  in the space of conformal classes on  $M$  such that every conformal class  $[g']$  sufficiently near to  $[g]$  contains a unique metric  $g'_u = e^{2u} g'$  with  $\sigma_k(A_{g'_u}) = (-1)^k \beta_k$  which is near to  $g$ ; the set of these solution metrics fills out an (open piece of an) analytic Banach submanifold, with respect to an appropriate Banach topology.*

Let us return to conformally compact metrics, and connect Theorem 1 with the first theme discussed in the introduction. We recall that a Poincaré-Einstein metric is also a  $\sigma_k$ -Yamabe metric for every  $k = 1, \dots, n+1$ . The constant  $\beta_k$  must equal the constant  $\beta_k^0$  for hyperbolic space, so in particular the moduli space  $\mathcal{E}$  of P-E metrics is included in the intersection of the  $\Sigma_k(\beta_k^0)$  over all  $k$ . However, more is true:

**Theorem 3** *Within the class of conformally compact metrics on  $M$ , there is an equality between the sets of Poincaré-Einstein metrics and the metrics which are in  $\Sigma_k(\beta_k^0)$  for every  $k = 1, \dots, n+1$ . In other words, with  $\beta^0 = (\beta_1^0, \dots, \beta_{n+1}^0)$ ,*

$$\Sigma(\beta^0) \equiv \bigcap_{k=1}^{n+1} \Sigma_k(\beta_k^0) = \mathcal{E}.$$

*Hence  $\mathcal{E}$  is a finite intersection of locally closed Banach submanifolds, and in particular is always closed in the space of conformally compact metrics on  $M$ .*

Notice that if  $\beta = (\beta_1, \dots, \beta_{n+1})$  is any other  $(n+1)$ -tuple of numbers with  $\beta_k > 0$ , and if  $M$  is compact without boundary, then

$$\Sigma(\beta) = \bigcap_{k=1}^{n+1} \Sigma_k(\beta_k)$$

is typically empty. As we explain later, if  $g$  is any metric in  $\Sigma(\beta)$ , then its Ricci tensor has constant eigenvalues. In particular, metrics with  $\nabla \text{Ric} = 0$  are in  $\Sigma(\beta)$  for some  $\beta$ . However, the reverse inclusion may not be true and in any case is not well understood; only a few partial results are known, e.g. in the Kähler case [4].

The plan for the rest of this paper is as follows. §1 reviews the structure of the functionals  $\mathcal{F}_k$  and their linearizations  $\mathcal{L}_k$ , and this is followed in §2 by a discussion of the function spaces and of the mapping properties of the  $\mathcal{L}_k$  on these spaces. The deformation theory for the  $\sigma_k$ -Yamabe equations and the proof of Theorems 1 and 2 is the topic of §3. The (very brief) proof of Theorem 3 and further discussion of some geometric aspects of the moduli space  $\mathcal{E}$  is contained in §4. Finally, §5 contains a list of some interesting open questions raised by the results here.

We wish to thank Paul Yang for providing initial inspiration for these results during a fortuitous conversation at MSRI, during which he pointed out the dearth of examples of  $\sigma_2$ -Yamabe metrics in the negative case, and suggested some possible lines of enquiry. Matt Gursky's encouragement was also quite helpful.

## 1 The functionals $\mathcal{F}_k$

Let us fix a conformally compact metric  $g_0$ , which we may as well take to be smooth, i.e.  $g_0 = \rho^{-2}\bar{g}_0$ , where both  $\rho$  and  $\bar{g}_0$  are  $\mathcal{C}^\infty$  on  $\bar{M}$ . Fix also a constant  $\beta_k > 0$ . Recall that the metric  $g = e^{2u}g_0$  is in  $\Sigma_k(\beta_k)$ , and so has  $\sigma_k(A_g) = (-1)^k \beta_k$ , provided

$$\mathcal{F}(g_0, u, \beta_k) = \sigma_k(\nabla^2 u - du \otimes du + \frac{1}{2}|\nabla u|^2 g_0 - A_{g_0}) - \beta_k e^{2ku} = 0.$$

In this section we recall some facts about the ellipticity of this operator and the structure of its linearization. These facts are taken from [21], and we refer there for all proofs and further discussion.

To approach the issue of ellipticity, first consider the  $k^{\text{th}}$  elementary symmetric function  $\sigma_k$  as a function on vectors  $\lambda = (\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ . Let  $\Gamma_k^+$  denote the component of the open set  $\{\lambda : \sigma_k(\lambda) > 0\}$  containing the positive orthant  $\{\lambda : \lambda_j > 0 \forall j\}$ . We note that these are all convex cones with vertices at the origin and

$$\{\lambda : \lambda_j > 0 \forall j\} = \Gamma_{n+1}^+ \subset \Gamma_n^+ \subset \dots \subset \Gamma_1^+ = \{\lambda : \sigma_1(\lambda) > 0\}.$$

Also, let  $\Gamma_k^- = -\Gamma_k^+$ . A real symmetric matrix  $A$  is said to lie in  $\Gamma_k^\pm$  if its eigenvalues lie in the corresponding set.

**Proposition 1** *If  $A_g \in \Gamma_k^-$ , then  $u \rightarrow \mathcal{F}_k(g, u, \beta_k)$  is elliptic at any solution of  $\mathcal{F}_k(g, u, \beta_k) = 0$ .*

The proof of this, in [21], relies on the computation of the linearization of  $\mathcal{F}_k$  in the direction of the conformal factor  $u$ . The neatest formulation of this requires a definition from linear algebra. For any real symmetric matrix  $B$ , and any  $q = 0, \dots, n+1$ , define the  $q^{\text{th}}$  Newton transform of  $B$  as the new (real, symmetric) matrix

$$T_q(B) = \sigma_q(B)I - \sigma_{q-1}(B)B + \dots + (-1)^q B^q.$$

Of course,  $T_{n+1}(B) = 0$ . Now suppose that  $B = B(\epsilon)$  depends smoothly on a parameter  $\epsilon$ , and write  $B'(0) = \dot{B}$ . It is proved in [20] that

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \sigma_k(B(\epsilon)) = \langle T_{k-1}(B), \dot{B} \rangle_g.$$

We apply this to the Schouten tensors associated to the family of metrics  $g(\epsilon) = e^{2\epsilon\phi}g$  where  $g \in \Sigma_k(\beta_k)$ . We have

$$B(\epsilon) = -A_g + \epsilon \nabla^2 \phi + \epsilon^2 \left( \frac{1}{2} |\nabla \phi|^2 - d\phi \otimes d\phi \right)$$

so that  $B = -A_g$  and  $\dot{B} = \nabla^2 \phi$ . Hence

$$D\mathcal{F}_k|_{g,0}((0, \phi) \equiv \mathcal{L}_k \phi = \langle T_{k-1}(-A_g), \nabla^2 \phi \rangle - 2k\beta_k \phi. \quad (8)$$

The proof of Proposition 1 in general (i.e. when  $g$  is not necessarily a solution itself and when the linearization is computed at some solution  $u \neq 0$ ) relies on the following ingredients: the convexity of  $\Gamma_k^+$ , the *concavity* of the function  $\sigma_k^{1/k}$  in  $\Gamma_k^+$  and finally the fact that  $T_{k-1}(B)$  is positive definite when  $B \in \Gamma_k^+$ .

Let us compute  $\mathcal{L}_k$  more explicitly when  $g$  is hyperbolic, or in fact, when  $g$  is an arbitrary Poincaré-Einstein metric. We shall always normalize the metric so the Einstein condition is  $\text{Ric} = -ng$ . Therefore, by (3), if  $g$  is P-E then  $A_g = -\frac{1}{2}g$ , and so

$$T_{k-1}(-A_g) = 2^{1-k} T_{k-1}(g) = 2^{1-k} \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{k-1-j} \equiv c_{k,n}.$$

It is well-known, and easy to prove by induction, that

$$c_{k,n} = 2^{1-k} \binom{n}{k};$$

in particular, this constant is always positive. Hence we obtain the useful formula

$$\mathcal{L}_k \phi = c_{k,n} \Delta \phi - 2k\beta_k \phi, \quad (9)$$

which holds when  $g$  is Poincaré-Einstein.

If  $g \in \Sigma_k(\beta_k)$  is a more general solution (i.e. not necessarily P-E), then  $\mathcal{L}_k$  is more complicated. However, certain properties remain valid.

**Proposition 2** *Suppose  $g \in \Sigma_k(\beta_k)$ ,  $\beta_k > 0$ , and let  $\mathcal{L}_k$  denote the linearization of  $\mathcal{F}_k(g, u, \beta_k)$  at  $u = 0$ . Then*

$$\mathcal{L}_k \phi = c_{k,n} \Delta_g \phi - 2k\beta_k \phi + \rho^3 E \phi, \quad (10)$$

where  $E$  is a second order operator with bounded coefficients on  $\overline{M}$  (smooth if  $\rho$  and  $\overline{g} = \rho^2 g$  are smooth), and without constant term. Furthermore, if  $\mathcal{L}_k \phi = 0$  and  $\|\phi\|_{L^\infty} < \infty$ , then  $\phi \equiv 0$ .

The final statement follows directly from the asymptotic maximum principle. A direct calculation yields the form of  $\mathcal{L}_k$ .

We note that (10) may also be obtained from general principles involving the theory of uniformly degenerate operators [15], [16]. Since some of the main results of this theory will be invoked later anyway, we digress briefly to explain this setup. Choose coordinates  $(x, y_1, \dots, y_n)$ ,  $x \geq 0$  near a point of the boundary of  $\overline{M}$ . A second order operator  $L$  is said to be uniformly degenerate if it may be expressed in the form

$$L = \sum_{j+|\alpha| \leq 2} a_{j,\alpha}(x, y) (x \partial_x)^j (x \partial_y)^\alpha. \quad (11)$$

The coefficients may be scalar or matrix-valued, and although we usually assume they are smooth, it is easy to extend most of the main conclusions of this theory when they are polyhomogeneous, or of some finite regularity. Operators of this type arise naturally in geometry, and in particular all of the natural geometric operators associated to a conformally compact metric are uniformly degenerate. Note that the error term  $\rho^3 E$  in (10) is actually of the form  $\rho E'$  where  $E'$  is some second order uniformly degenerate operator without constant term.

The ‘uniformly degenerate symbol’ of this operator is elliptic provided

$$\sigma(L)(x, y; \xi, \eta) = \sum_{j+|\alpha|=2} a_{j,\alpha}(x, y) \xi^j \eta^\alpha \neq 0 \quad \text{when } (\xi, \eta) \neq 0.$$

(For systems, we require  $\sigma(L)$  to be invertible as a matrix when  $(\xi, \eta) \neq 0$ .) We also define the associated *normal operator*

$$N(L) = \sum_{j+|\alpha| \leq 2} a_{j,\alpha}(0, y) (s \partial_s)^j (s \partial_u)^\alpha.$$

The boundary variable  $y$  enters only as a parameter, while the ‘active’ variables  $(s, u)$  in this expression may be regarded as formal, but in fact are naturally identified with linear coordinates on the inward pointing half-tangent space  $T_{(0,y)}^+ M$ . In particular

**Proposition 3** *If  $g$  is a smooth conformally compact metric, then its Laplace-Beltrami operator  $\Delta_g$  is an elliptic uniformly degenerate operator with normal operator*

$$N(\Delta_g) = \Delta_{\mathbb{H}^{n+1}} = (s \partial_s)^2 + s^2 \Delta_u - ns \partial_s. \quad (12)$$

Furthermore, if  $g \in \Sigma_k(\beta_k)$  for some  $\beta_k > 0$ , then the linearization  $\mathcal{L}_k$  of  $\mathcal{F}_k(g, u, \beta_k)$  at  $u = 0$  is also elliptic and uniformly degenerate, with normal operator

$$N(\mathcal{L}_k) = \mathcal{L}_k^0 \equiv c_{k,n}((s\partial_s)^2 + s^2\Delta_u - ns\partial_s) - 2k\beta_k. \quad (13)$$

As we explain in the next section, the operator  $\mathcal{L}_k$  is Fredholm on various natural function spaces. This specializes a criterion which is applicable to other more general uniformly degenerate operators  $L$ , namely that  $L$  is Fredholm if and only if two separate ellipticity conditions hold: first, the symbol  $\sigma(L)$  should be invertible, and in addition, the normal operator  $N(L)$  must be invertible on certain weighted  $L^2$  spaces.

## 2 Function spaces and mapping properties

Let  $\mathcal{L}_k$  be the linearization considered in the last section. We shall now describe some of its mapping properties. As indicated above, these properties also hold for more general elliptic, uniformly degenerate operators  $L$ .

We first review one particular scale of function spaces which is convenient in the present setting, and then state the mapping properties on them enjoyed by  $\mathcal{L}_k$ . The material here is taken from [15], to which we refer for further discussion and proofs.

Fix a reference (smooth) conformally compact metric  $g_0 = \rho^{-2}\bar{g}_0$ ; also, choose a smooth boundary coordinate chart  $(x, y)$  as in the previous section, and recall the basic vector fields  $x\partial_x$  and  $x\partial_{y_j}$ ,  $j = 1, \dots, n$ . Since  $x$  is a smooth nonvanishing multiple of  $\rho$  near  $\partial\bar{M}$ , these vector fields are all of uniformly bounded lengths with respect to  $g_0$ , and are also uniformly independent as  $x \searrow 0$ . There are two equivalent ways to define the Hölder space  $\Lambda_0^{0,\alpha}(M)$ ,  $\ell \in \mathbb{N}$ ,  $0 < \alpha < 1$ . In either case, it suffices to work in a boundary coordinate chart. The first is to set

$$\Lambda_0^{0,\alpha}(M) = \left\{ u : \sup \frac{|u(x, y) - u(x', y')|(x + x')^\alpha}{|x - x'|^\alpha + |y - y'|^\alpha} \right\},$$

where the supremum is taken first over all points  $z = (x, y)$ ,  $z' = (x', y')$ ,  $z \neq z'$ , which lie in some coordinate cube  $B$  centered at a point  $z_0 = (x_0, y_0)$  of sidelength  $\frac{1}{2}x_0$ , and then over all such cubes. The other is to let  $B$  denote a ball of unit radius with respect to the metric  $g_0$  centered at  $z_0$ , and to replace the denominator in this definition by  $\text{dist}_{g_0}(z, z')^\alpha$ , for  $z, z' \in B$ , and then take the same sequence of suprema.

This latter definition is more geometric, while the former definition clearly implies the scale invariance of these spaces, namely that if  $u(z)$  is defined (and, say, compactly supported) in one of these coordinate charts and if we define  $u_\epsilon(z) = u(z/\epsilon)$ , then the associated norms of  $u$  and  $u_\epsilon$  are the same.

From here we can define a few other closely related spaces which will be useful:

- For  $\ell \in \mathbb{N}$ , let

$$\Lambda_0^{\ell, \alpha}(M) = \left\{ u : (x\partial_x)^j (x\partial_y)^\beta u \in \Lambda_0^{0, \alpha}(M) \ \forall j + |\beta| \leq \ell \right\}.$$

- For  $0 \leq \ell' \leq \ell$ ,  $\ell, \ell' \in \mathbb{N}$ , let

$$\Lambda_0^{\ell, \alpha, \ell'}(M) = \left\{ u : \partial_y^{\beta'} u \in \Lambda_0^{\ell - |\beta'|, \alpha}(M) \text{ for } |\beta'| \leq \ell' \right\}.$$

- Finally, for  $\gamma \in \mathbb{R}$ , and  $0 \leq \ell' \leq \ell$ ,  $\ell, \ell' \in \mathbb{N}$ , let

$$\rho^\gamma \Lambda_0^{\ell, \alpha, \ell'}(M) = \left\{ u : u = \rho^\gamma \tilde{u}, \quad \text{where } \tilde{u} \in \Lambda_0^{\ell, \alpha, \ell'}(M) \right\}.$$

Thus the first of these are just the natural higher order Hölder spaces associated to the geometry of  $g_0$ , or equivalently, to differentiations with respect to the vector fields  $x\partial_x$  and  $x\partial_y$ . The second of these allows up to  $\ell'$  of the derivatives to be taken with respect to the ‘nongeometric’ vector fields  $\partial_y$ , which are exponentially large with respect to  $g_0$  as  $x \rightarrow 0$ , but which are needed if one desires any sort of boundary regularity. The final spaces are just the usual weighted analogues. The corresponding norms are  $\|\cdot\|_{\ell, \alpha}$ ,  $\|\cdot\|_{\ell, \alpha, \ell'}$ , and  $\|\cdot\|_{\ell, \alpha, \ell', \gamma}$ , respectively.

We could equally easily have defined  $L^2$ - and  $L^p$ -based Sobolev spaces, corresponding to differentiations with respect to the vector fields  $x\partial_x$  and  $x\partial_y$ , as well as the corresponding partially tangentially regular and weighted versions. The mapping properties we state below all have direct analogues for these spaces. However, as usual the Hölder spaces are perhaps the simplest to deal with for nonlinear PDE.

As a further note, still fixing the reference metric  $g_0$  we can use these definitions to define appropriate finite regularity spaces of tensor fields on  $M$ . In particular, we set, for  $\gamma > 0$ ,

$$\mathfrak{M}_\gamma^{\ell, \alpha, \ell'}(M) = \{g = g_0 + h : g > 0 \text{ and } h \in \rho^{-2+\gamma} \Lambda_0^{\ell, \alpha, \ell'}(M; S^2(M))\}.$$

Now let us turn to the mapping properties of  $\mathcal{L}_k$ . First of all, it follows immediately from the definitions that

$$\mathcal{L}_k : \rho^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}(M) \longrightarrow \rho^\gamma \Lambda_0^{\ell, \alpha, \ell'}(M) \tag{14}$$

is a bounded mapping for any  $\gamma \in \mathbb{R}$  and  $0 \leq \ell' \leq \ell$ . However, this map is not well-behaved for many values of the weight parameter  $\gamma$ . There are two ways this may occur. First if  $\gamma$  is sufficiently large positive, then it is not hard to see that (14) has an infinite dimensional cokernel, while dually, if  $\gamma$  is sufficiently large negative, then (14) has an infinite dimensional nullspace. Although we do not use it here, less trivial is the fact that in either of these two cases the mapping is semi-Fredholm (i.e. has closed range and either the kernel or cokernel are finite-dimensional).

However, for certain values of  $\gamma$  the range of this mapping may not be closed. This is determined by a consideration of the indicial roots of  $\mathcal{L}_k$ . We say that  $\gamma$  is an indicial root of  $\mathcal{L}_k$  if  $\mathcal{L}_k(\rho^\gamma) = O(\rho^\gamma + 1)$  (note that because of the uniform degeneracy of  $\mathcal{L}_k$ ,  $\mathcal{L}_k(\rho^\gamma) = O(\rho^\gamma)$  is always true). Thus  $\gamma$  is an indicial root only if some special cancellation occurs. Using Proposition 3, it is clear that the indicial roots of  $\mathcal{L}_k$  agree with those of  $\mathcal{L}_k^0$ , and then (13) shows that  $\gamma$  is an indicial root if and only if

$$c_{k,n}(\gamma^2 - n\gamma) - 2k\beta_k = 0,$$

or in other words

$$\gamma = \gamma_\pm = \frac{c_{k,n}n \pm \sqrt{(c_{k,n}n)^2 + 8k\beta_k c_{k,n}}}{2c_{k,n}} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{2k}{c_{k,n}}}.$$

In particular

$$\gamma_- < 0 < \gamma_+.$$

The relevance of these indicial roots to the mapping properties of (14) is that when  $\gamma$  is equal to one of these two values, then (14) does not have closed range. At heart, this stems from the fact that the equation

$$\mathcal{L}_k^0 u = s^{\gamma_\pm}$$

has solution  $u = cs^{\gamma_\pm}(\log s)$  for some constant  $c$ , i.e. the inhomogeneous term is in the appropriate weighted Hölder space but the solution  $u$  just misses being in this space.

Despite these cautions, we have the following basic result:

**Mapping properties:** *If  $\gamma_- < \gamma < \gamma_+$ , then the mapping (14) is Fredholm of index zero.*

The main result of [15] is a considerably more general theorem of this sort for more general elliptic uniformly degenerate differential operators. There are two special features of  $\mathcal{L}_k$  which enter into the precise form of the statement here. First, there is a nontrivial interval  $(\gamma_-, \gamma_+)$  between the two indicial roots  $\gamma_\pm$ , allowing for the possibility of a ‘Fredholm range’. Second, the Fredholm index is zero for  $\gamma$  in this interval ultimately because  $\mathcal{L}_k$  is self-adjoint on  $L^2(dV_g)$ .

To show that  $\mathcal{L}_k$  is actually invertible when  $\gamma$  is in this Fredholm range, we note that at least when  $\gamma \geq 0$  we could use the maximum principle to assert that the nullspace is trivial, and then use the vanishing of the index to conclude that the mapping is actually an isomorphism. In fact, this same reasoning may be used whenever  $\gamma > \gamma_-$  on account of another basic from [15]:

**Regularity of solutions:** *If  $\gamma_- < \gamma < \gamma_+$  and  $\phi \in x^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}$  is a solution of  $\mathcal{L}_k \phi = f$  where  $f$  vanishes to all orders at  $\partial\overline{M}$ , then as  $x \rightarrow 0$ ,*

$$\phi(x, y) \sim \sum_{j=0}^{\infty} \phi_j(y) x^{\gamma+j}, \quad \text{with} \quad \phi_j(y) \in \mathcal{C}^\infty(\partial\overline{M}),$$

i.e.  $\phi \in \rho^{\gamma+} \mathcal{C}^\infty(\overline{M})$ .

In particular, if  $\phi$  is in the nullspace of  $\mathcal{L}_k$  when  $\gamma > \gamma_-$  then we may apply this (with  $f = 0$ ) to obtain that  $\phi = O(\rho^{\gamma+})$  and so the maximum principle may be used as above to conclude that  $\phi = 0$ .

### 3 Perturbation theory in $\Sigma_k$

Before preceding to the main deformation result, we review the the structure of the space of (conformally compact) metrics on  $M$ .

There is a multiplicative action by scalar functions on the set of conformally compact metrics:

$$\rho^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}(M) \times \mathfrak{M}_\gamma^{\ell+2, \alpha, \ell'}(M) \longrightarrow \mathfrak{M}_\gamma^{\ell+2, \alpha, \ell'}(M),$$

$$(u, g) \longrightarrow e^{2u} g.$$

The tangent space to this action at  $(0, g)$  consists of the set

$$\left\{ h \in \rho^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}(M; S^2(T^*M)) : \text{tr}_g(h) = 0 \right\}.$$

It is customary to model the set of conformal classes  $\mathfrak{C}$  in a neighbourhood of  $g$  by the set of tensor fields  $h$  which are traceless, and also divergence free. The latter condition corresponds to fixing a gauge transverse to the orbit of the action of the diffeomorphism group; however, it is not really necessary for our purposes to mod out by this action, and so we dispense with this gauge.

Now define the slice

$$\mathfrak{S}_\epsilon = \left\{ h \in \rho^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}(M; S^2(T^*M)) : \text{tr}_g(h) = 0, \|h\|_{\ell+2, \alpha, \ell', \gamma} < \epsilon \right\}, \quad (15)$$

and identify a neighbourhood of 0 in this space with the space of conformal classes of conformally compact metrics near  $g$  (but not modulo diffeomorphisms!).

**Theorem 4** *Fix  $\beta_k > 0$  and suppose that  $g \in \Sigma_k(\beta_k)$ . Then there is a neighbourhood  $\mathcal{U}$  of  $g$  in  $\mathfrak{M}_\gamma^{\ell+2, \alpha, \ell'}$  such that  $\mathcal{U} \cap \Sigma_k(\beta_k)$  is a Banach manifold modelled on the slice  $\mathfrak{S}_\epsilon$  defined in (15).*

Having set things up carefully, the proof is almost immediate. Consider the mapping

$$\mathcal{H} : \mathfrak{S}_\epsilon \times \rho^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}(M) \longrightarrow \mathfrak{M}_\gamma^{\ell, \alpha, \ell'} \quad (16)$$

defined by

$$\mathcal{H}(h, u) = \mathcal{F}_k(g + h, u, \beta_k).$$

In a neighbourhood of  $g$ , the set  $\Sigma_k(\beta_k)$  is identified with the zero set of  $\mathcal{H}$ . In particular,  $(h, u) = (0, 0) \in \mathcal{H}^{-1}(0)$ .

To find all other nearby solutions, we shall apply the implicit function theorem, very much in the spirit of the closely related papers [18] and [10]. Thus we must check two things:

- The mapping  $\mathcal{H}$  in (16) is a smooth mapping of open sets of Banach spaces, and
- The linearization  $D\mathcal{H}|_{0,0}$  is surjective between the appropriate tangent spaces.

The first of these is straightforward from the definitions and (2), provided we choose the weight parameter  $\gamma > 0$ . As for the other, recall that the restriction of this Frechét derivative to tangent vectors of the form  $(0, \phi)$  corresponds to the operator  $\mathcal{L}_k$ . We have already checked that this is surjective on the tangent spaces provided we choose the weight parameter  $\gamma \in (\gamma_-, \gamma_+)$ . Since  $\gamma_+ > 0$ , these two restrictions on  $\gamma$  are not inconsistent, and so we now fix any  $\gamma \in (0, \gamma_+)$ . With this choice, we obtain the existence of a smooth map

$$\Phi : \mathfrak{S}_\epsilon \longrightarrow \rho^\gamma \Lambda_0^{\ell+2, \alpha, \ell'}(M)$$

with  $\Phi(0) = 0$ ,  $D\Phi|_0 = 0$  and such that

$$\mathcal{H}(h, \Phi(h)) \equiv 0.$$

Furthermore, all solutions of  $\mathcal{H}(h, u)$  in a sufficiently small neighbourhood of  $(0, 0)$  are of this form. This concludes the proof.  $\square$

We omit the proof of Theorem 2 because it is nearly identical; indeed, the only difference is that standard elliptic theory replaces the Fredholm theory for uniformly degenerate operators we have quoted.

We have shown that  $\Sigma_k(\beta_k)$  is a Banach submanifold in a neighbourhood of  $g$ , and furthermore that it may be regarded as a graph over the space of conformal classes, or at least those conformal classes near to  $g$ . For  $k = 1$ , every conformal class on  $\bar{M}$  contains a unique representative lying in  $\Sigma_k(\beta_k)$ , and thus  $\Sigma_1(\beta_1)$  is a graph globally over the space  $\mathfrak{C}$  of all conformal classes. It is not known whether this remains true when  $k > 1$ , and thus we define

$$\mathfrak{C}_k = \{c \in \mathfrak{C} : c \text{ contains at least one } g \in \Sigma_k(\beta_k)\}. \quad (17)$$

(Note we have suppressed the dependence of  $\beta_k$  in this notation because, if there is a solution  $g \in c$  for the equation with one constant  $\beta_k > 0$ , then, simply by scaling, we can produce a solution for every other positive constant.) Note that  $\mathfrak{C}_1 = \mathfrak{C}$ , and Theorem 4 shows that  $\mathfrak{C}_k$  is open in  $\mathfrak{C}$  for every  $k$ .

## 4 The moduli space $\mathcal{E}$ of Poincaré-Einstein metrics

We begin by recalling some previously known results about the moduli space  $\mathcal{E}$  of Poincaré-Einstein metrics on the manifold with boundary  $M$ , and then proceed to the (nearly obvious) relationship of this space with the moduli spaces  $\Sigma_k(\beta_k)$ .

It is natural to formulate the existence theory of P-E metrics as an asymptotic boundary problem. Recall that any conformally compact manifold  $g$  determines a conformal class  $[\rho^2 g|_{\partial\bar{M}}]$  on the boundary of  $\partial\bar{M}$ . We then pose the following

**Problem:** *Given a conformal class  $[h_0]$  on  $\partial\bar{M}$ , is it possible to find a Poincaré-Einstein metric  $g$  on  $M$ , such that the conformal class determined by  $g$  on  $\partial\bar{M}$  is the specified class  $[h_0]$ ?*

There are many particular solutions in the physics literature, cf. [14], and this problem was considered on a formal level by Fefferman and Graham [8]. The first general existence result was obtained by Graham and Lee [10], and consisted of a perturbation analysis about the standard hyperbolic metric on the ball  $B^{n+1}$ . They showed that this problem is solvable for conformal classes sufficiently near the standard one. There are now several other related perturbation results [6], [12], and Anderson has announced a more global existence theory in four dimensions [3].

There are many interesting questions, including the description of asymptotic regularity of these metrics in terms of the regularity of the conformal class  $[h_0]$ , problems involving uniqueness, etc. Anderson [2] settles (appropriate versions of) these particular issues in the four dimensional case while [11] contains a different approach to regularity and uniqueness valid in all dimensions.

The goal in this section is simply to provide a slightly different perspective onto this class of metrics. Fix the constants  $\beta_k = \beta_k^0$  corresponding to the standard hyperbolic metric, and henceforth write  $\Sigma_k(\beta_k^0)$  simply as  $\Sigma_k$ . In any case, for any  $\ell$ -tuple  $J = \{j_1, \dots, j_\ell\} \subset \{1, \dots, n+1\}$ , consider the intersection

$$\Sigma_J = \Sigma_{j_1} \cap \dots \cap \Sigma_{j_\ell},$$

and

$$\Sigma = \Sigma_1 \cap \dots \cap \Sigma_{n+1}.$$

**Proposition 4** *The moduli space of P-E metrics on  $M$  agrees with this intersection of the submanifolds  $\Sigma_k$ :*

$$\mathcal{E} = \Sigma. \tag{18}$$

The fact that  $\mathcal{E} \subset \Sigma$  is obvious. On the other hand, if  $g \in \Sigma$ , then we first notice that all the eigenvalues of  $A_g$  (or equivalently, of  $\text{Ric}_g$ ) are constant; this is simply because they are the roots of the characteristic polynomial, the coefficients of which are precisely these higher ( $\sigma_k$ ) traces of  $A_g$ , which are

by assumption constant. To see that these constant eigenvalues  $\lambda_j$  agree with one another, and are equal to the value on hyperbolic space, we can either proceed geometrically and note that near infinity (i.e. as  $\rho \rightarrow 0$ ), the metric  $g$  is asymptotically hyperbolic so we may evaluate the  $\lambda_j$  in this limit, or else algebraically, by observing that by the particular choice of constants  $\beta_k^0$ , the characteristic polynomial of  $A_g$  must be simply that of hyperbolic space, i.e.  $(\lambda + 1/2)^{n+1}$ . By either approach, we conclude that  $g$  is Einstein, and this gives the opposite inclusion.

There are various ways to interpret this equality of moduli spaces.

- Proposition 4 shows that the somewhat less tractable space  $\mathcal{E}$  is a finite intersection of submanifolds  $\Sigma_j$ , each of which is an *analytic* submanifold, but more importantly, each of which has an unobstructed deformation theory. This amounts to some sort of figurative ‘factorization’ of the Einstein equations into  $n + 1$  scalar (albeit fully nonlinear) equations.
- Each  $\Sigma_k$  is a (perhaps multiply sheeted) graph over the open set of conformal classes  $\mathfrak{C}_k$ . But the intersection  $\Sigma$  lies over a much thinner, infinite codimension set  $\mathfrak{C}_{\mathcal{E}} \subset \mathfrak{C}$ . Elements of  $\mathfrak{C}_{\mathcal{E}}$  are precisely the conformal classes which contain P-E metrics! Another way to think of the basic asymptotic boundary problem for P-E metrics is that the set  $\mathfrak{C}_{\mathcal{E}}(\partial\overline{M})$  of conformal classes on  $\partial\overline{M}$  for which this problem is solvable coincides with the restrictions to  $\partial\overline{M}$  of classes in  $\mathfrak{C}_{\mathcal{E}}$ . This near-tautology presents an interesting (though probably intractable) way to think about the boundary problem: one might first seek to extend the conformal class  $[h_0]$  from the boundary to a ‘good’ one (i.e. one in  $\mathfrak{C}_{\mathcal{E}}$ ) in the interior, and then consider the Einstein equation as a conformal problem within this class.

## 5 Open questions and further directions

We conclude this note by raising a few other problems and questions related to the results and methods here.

- a) Because of the difficulty in obtaining a  $\mathcal{C}^2$  estimate for the  $\sigma_k$ -Yamabe problem when  $k > 1$ , it is worth wondering whether it might be worthwhile to pose a weaker version of this problem, at least for conformally compact metrics on manifolds with boundary: namely, given a conformal class  $[h_0]$  on  $\partial\overline{M}$ , is it possible to extend this conformal class to at least some conformal class  $[\overline{g}]$  on the interior such that the  $\sigma_k$ -Yamabe problem is solvable in  $[\overline{g}]$ ? Probably there are infinitely many such extensions, as is the case when  $k = 1$ , but the added flexibility in this formulation may be of some use.
- b) It seems central to understand whether  $\mathfrak{C}_k = \mathfrak{C}$ , or in other words, whether every conformal class on  $M$  contains a conformally compact  $\sigma_k$ -Yamabe metric. Related to this is the observation that we do not know whether

each of the submanifolds  $\Sigma_k$  itself is closed; this depends ultimately on whether some version of this  $C^2$  estimate holds. It seems interesting, though, to ask which of the finite intersections  $\Sigma_J = \Sigma_{j_1} \cap \dots \cap \Sigma_{j_\ell}$  are closed. Notice that if  $1 \in J$ , then this is certainly true because the  $C^2$  estimate for the conformal factor is routine for the scalar curvature equation.

- c) The regularity of the metrics  $g \in \Sigma_k(\beta_k)$  is an interesting question. When  $k = 1$  this is resolved in [16], cf. also [19]: if  $g$  is a smooth conformally compact metric, then the conformal factor  $u$  corresponding to the unique solution  $\tilde{g} = e^{2u}g \in [g]$  has a polyhomogeneous expansion. Presumably a similar result holds for all  $k$ . Note that unless  $\gamma_+ \in \mathbb{N}$ , this expansion will involve nonintegral powers of  $\rho$ ; this should not be viewed negatively, since functions with expansions of this form may be manipulated just as easily as smooth functions.
- d) The  $\sigma_k$ -Yamabe problem considered here extends naturally to the more general setting of the singular  $\sigma_k$ -Yamabe problem: given a smooth metric  $g_0$  on a compact manifold  $M$  and a closed subset  $\Lambda \subset M$ , when is it possible to find a conformally related metric  $g = e^{2u}g_0$  which is both complete on  $M \setminus \Lambda$  and a  $\sigma_k$ -Yamabe metric? When  $k = 1$  it is known that the dimension of  $\Lambda$  is intimately related to the sign of the imposed scalar curvature of the solution, and very good existence results are known when  $\Lambda$  is a submanifold [5], [17]. What is the correct statement, and to what extent is this true when  $k > 1$ ? There are a number of interesting analytic problems of this nature, and we shall return to this soon.
- e) In general (not just in the conformally compact setting),  $\mathcal{E}$  sits inside the finite intersection  $\cap \Sigma_k$ . Does it appear here as a finite codimensional analytic set, and if so, is this related to some sort of Kuranishi reduction for the perturbation theory for  $\mathcal{E}$ ?
- f) Finally, it appears that very little is known about metrics with Ricci tensor having constant eigenvalues, but cf. [4]. The (presumably) smaller class of metrics with parallel Ricci tensor is more tractable, but it does not seem to be known if the eigenvalues can be constant without the Ricci tensor being parallel. Examples would be very welcome. Also, in any setting (compact or conformally compact or ...) it seems to be a very basic problem in Riemannian geometry to ask what are the possible  $(n + 1)$ -tuples  $(\beta_1, \dots, \beta_{n+1})$  for which  $\Sigma_1(\beta_1) \cap \dots \cap \Sigma_{n+1}(\beta_{n+1})$  is nonempty?

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