ELLIPTIC THEORY OF DIFFERENTIAL EDGE OPERATORS I

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Abstract. Examples of edge operators include Laplacians on asymptotically flat and asymptotically hyperbolic manifolds. Edge operators also arise in boundary problems around higher codimension boundaries. This paper is concerned with the analysis of general elliptic edge operators with constant indicial roots. We determine when such an operator has closed range, and also prove that any element of the nullspace of such an operator has a distributional asymptotic expansion. Conditions are given to guarantee that the coefficients of this expansion are smooth. In Part I of this paper we only study the case when the operator is semi-Fredholm. Part II will examine edge operators with infinite dimensional kernel and cokernel, as well as develop the theory of Poisson edge operators.

§1. Introduction

The theory of linear elliptic operators has reached a rather refined state, yet many aspects still remain relatively unexplored. Of these few have attracted as much attention in recent years as elliptic theory on noncompact or singular spaces. Much effort in geometry and other fields has been directed at such spaces also because just as on compact manifolds elliptic operators can play a very useful role. Unfortunately too many phenomena occur in these contexts for there ever to be a coherent general theory encompassing any but the most simple results. However certain types of singular or noncompact manifolds arise quite frequently, and in this limited context a general theory is sometimes possible. This paper develops a method for analyzing elliptic operators on a certain class of noncompact or singular manifolds which seem quite common. The general class of spaces is abstracted from and includes manifolds with isolated conic or wedge type singularities, as well as asymptotically flat and asymptotically hyperbolic manifolds. We shall devote some time in this introduction to describing some of the main examples of edge operators, as the rest of the paper is somewhat general and analytic. We have attempted to give a broad treatment of the analysis of edge operators, encompassing all 'natural' examples, but have also tried to limit generality in an attempt to make this paper more readable.

We first state rather sketchily the general definition of the class of edge operators. These are degenerate operators on compact manifolds with boundaries. Although this

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may seem at odds with our interest in operators on noncompact manifolds or singular spaces, the examples below should make it clear that many types of singular or noncompact spaces can be reduced to manifolds with boundary by a process of compactification and desingularization. In this process many natural operators are transformed to operators with rather simple degeneracies. Suppose X is a compact manifold with boundary, and also that its boundary ∂X is the total space of a fibration

$$\begin{array}{ccc} F^q & \longrightarrow & \partial X \\ & & \downarrow \pi \\ & & Y^k \end{array} \tag{1.1}$$

with compact fibre F^q and base Y^k. Now consider the space of smooth vector fields $\mathcal{V}_e$ which are unrestricted in the interior and which lie tangent at the boundary to the leaves of the fibration. By taking those differential operators generated by $\mathcal{V}_e$ and $\mathcal{C}^\infty(X)$ we obtain the full class of edge operators. In terms of local coordinates (x, y, z), where x is a defining function for ∂X, y restricts to coordinates on ∂X lifted from Y and z restricts to coordinates on each F, any such operator takes the form

$$L = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x, y, z)(x\partial_x)^j(x\partial_y)^\alpha \partial_z^\beta. \tag{1.2}$$

L is said to be elliptic if it is an elliptic combination of the vector fields $x\partial_x, x\partial_{y^\alpha}, \partial_{z^\beta}$. The two extreme cases occur when either y or z is absent, and then we call an edge operator totally characteristic or uniformly degenerate, respectively.

We now discuss some examples. Consider first a Riemannian manifold with isolated conic singularities. This means that near each singular point the metric takes the form $g = dx^2 + g_{0\alpha} dx\, dz^\alpha + g_{\alpha\beta} dz^\alpha\, dz^\beta$, where (x, z) are polar coordinates around the singularity. The associated Laplacian may be written $\Delta = x^{-2}L$ where L is an elliptic edge (in fact totally characteristic) operator. Closely related is the Laplacian on Euclidean space in polar coordinates around infinity. By this we mean the Laplacian written in coordinates (x, z) where $r = x^{-1}$ is the usual polar distance and z is the spherical variable. It looks like $\Delta = x^2 L$, again with L totally characteristic elliptic. These examples are of edge type only up to a multiplicative factor, which can be traced back to the metric. If g is a metric with isolated conic singularity as above, then define the family of weighted metrics $g_s = x^{2s-2}g$. These have Laplacians $\Delta_s = x^{-2s}L$ with L of edge type. The original conic example corresponds to $s = 1$ while the Euclidean metric in polar coordinates around infinity has weight $s = -1$. Note that metrics with weight $s \leq 0$ are complete, and metrics with the borderline weight $s = 0$ are 'cylindrical'. Usually this weight causes no difficulty in the analysis. Parametrices are first constructed for the edge operator L, and the weight factor can then be multiplied in. However this can cause difficulties, for example if an eigenvalue parameter is added. Then $\Delta + \lambda = x^{-2s}(L + x^{2s}\lambda)$ with L of edge type, so unless $2s$ is a nonnegative integer or $\lambda = 0$ this falls outside of the edge class. In particular, the resolvent of the Laplacian on Euclidean space lies in a more degenerate space of operators. On the other hand if $s = 1$ for example, then we can take the product of a compact manifold Y with a manifold with isolated conic singularities, and the corresponding Laplacian is of the form $x^{-2}L$ with L of 'full' edge type, i.e. with all variables present. More generally such operators occur as Laplacians for metrics with 'wedge' type singularities along whole submanifolds. In particular, the Laplacian in polar coordinates around any smooth closed submanifold of arbitrary codimension is of this type.

Another class of examples originates from hyperbolic space. Using upper half-space coordinates (x, y) with $x = 0$ on the boundary, the scalar Laplacian on $\mathbf{H}^n$ equals

$$\Delta_{\mathbf{H}^n} = x^2 \partial_x^2 + (2-n)x\partial_x + x^2 \Delta_{\mathbf{R}^{n-1}}, \tag{1.3}$$

which is uniformly degenerate. Of the same form are the Laplacians (scalar or on the various geometric bundles) of any 'conformally compact' metric $g = x^{-2}h$, where h is a smooth nondegenerate metric on the compact manifold with boundary X and as before x is a defining function for the boundary [M1]. We can also take the product of these with a compact manifold F to obtain spaces with Laplacians of full edge type. (Observe the different rôles of F and Y here and earlier.) In particular, the eigenvalue problem for asymptotically hyperbolic manifolds is of edge type [M-M1]. We can also introduce weighted analogues of conformally compact metrics $g_s = x^{2s-2}h$. When $s = 1$ this is simply a smooth metric on a manifold with boundary, and its Laplacian is an ordinary uniformly elliptic operator. Similarly if P is uniformly elliptic of order m on a compact manifold with boundary, then $L = x^m P$ is uniformly degenerate, hence of edge type. Analyzing P by making it degenerate might seem misguided, but actually makes full use of the natural homogeneities of the problem, which are sometimes overlooked. It also indicates the scope of any complete analysis of the elliptic theory for edge operators since it must encompass the classical theory of elliptic boundary problems.

Our basic goals are to understand the mapping and regularity properties of elliptic edge operators. However unlike standard elliptic theory we rarely expect these operators to be Fredholm and to obtain smoothness up to the boundary for solutions. Everything is formulated somewhat differently. The most natural formulation of mapping properties involve the use of a scale of weighted Sobolev spaces. The basic theorem is that if an elliptic edge operator satisfies certain hypotheses, the most important of which are (2.22), the constancy of indicial roots, and (5.14), the unique continuation property for the model operator, then as a mapping between these spaces it has closed range for all but a discrete set of values of the weight parameter. (This weight parameter has, of course, nothing to do with the weight of a metric in the sense described above.) Closed range is generally all that can be hoped for: the operator is semi-Fredholm with either infinite dimensional kernel or cokernel if the weight parameter is sufficiently large negative or positive, and only sometimes Fredholm and then only for a bounded set of weights. Sometimes it is never Fredholm, cf. [M-S]. In some sense the most difficult case is when the operator has infinite dimensional kernel and cokernel on the same space. We formulate a similar result for edge operators as mappings between suitable weighted Hölder and L^p spaces. The precise statements are given in Theorems (4.20) and (6.2).

The regularity properties we prove concern the existence of asymptotic expansions at the boundary for solutions of a homogeneous equation; this replaces smoothness up to the boundary. Here too results are not as one might wish: unless the base space Y in (1.1) is trivial, there exist elements of the nullspace of an elliptic edge operator (in a weighted space with sufficiently negative weight) which have asymptotic expansions at the boundary only in a distributional sense. The coefficients of these expansions are not smooth, and in fact become progressively worse high up in the expansion. Simple examples of this phenomenon will be presented. However, in certain cases, for example when the solution has sufficient decay at the boundary, these coefficients are smooth and the expansion is 'polyhomogeneous'.

These results are proved using pseudodifferential operators which incorporate the degeneracies of the edge operators. This class of operators is sufficiently broad to contain parametrices, and in fact the generalized inverses and 'harmonic projectors', for elliptic edge operators, when these exist. In particular, the structure of these inverses and projectors is quite explicit and simple, and this leads to very refined understanding of the original edge operator. The structure theory of the (infinite rank) orthogonal projectors onto the nullspace should be regarded as a real analogue of the theory of the Bergman projector in the theory of several complex variables.

Laplacians of metrics with nonzero weights are not considered in detail here. As noted above, the first step of their analysis involves constructing a parametrix for the underlying edge operator. However, the extra multiplicative factor can radically affect various mapping properties, cf. [Me2]. We also treat scalar operators for the most part. Generalizations of the results here to edge operators acting between bundles are quite straightforward.

There are definite limitations to this theory and to the types of operators it can be used to study. In particular, there are much weaker notions of asymptotically flat [B], [Mc1] or asymptotically hyperbolic [A] manifolds than those described above. Standard perturbation arguments in combination with the results here imply such things as Fredholm properties for the Laplacians on these spaces. However, if more refined properties are required, such as information about the scattering matrix for long range potentials, these methods are at present inadequate. Also, we have limited our focus here to operators with constant indicial roots. Although this is justified by the fact that most naturally arising operators seem to have this property, a more complete understanding would of course be welcome. The mapping properties probably go through with the obvious modifications in this more general setting, but the regularity properties are quite different. Schulze and Rempel [R-S], [S] have developed a parallel theory of pseudodifferential operators and in particular have undertaken the analysis of operators with variable indicial roots.

Many people have worked on special or general classes of operators related to the ones considered here; we mention only those whose work is most directly related. An early indication of the value of working on weighted Sobolev spaces is contained in the work of Nirenberg and Walker [N-W], and later McOwen [Mc1], [Mc2] concerning the Laplacian and its perturbations on asymptotically flat manifolds, see also [B] for a thorough account of this topic. As is well known, this theory has been used extensively in applications, for example in general relativity [B], [A-F] and Yang-Mills theory [Tb]. Ma and McOwen [Ma-Mc] recently treated certain operators in the more general edge class. Other papers include those of Brüning and Seeley [B-S1], [B-S2], and that of Andersson [A]. The work of Rempel and Schulze [R-S], [S] is the most similar to that here, and in certain respects goes even further. However, it has certain limitations; all operators are reduced to Fredholm ones by adding boundary conditions, or rather trace and Poisson operators. Therefore direct analysis of operators with infinite rank kernel and cokernel are not immediately accessible by their methods. In general outline and many specifics the present methods are due to Richard Melrose, who was led to consider the structures underlying this theory in his study of propagation of singularities for boundary problems. The original account of "totally characteristic" or b-pseudodifferential operators is contained in [Me1]. He later went on to develop the elliptic theory with Mendoza in [Me-Me]. Following this, a study of the present class of operators was initiated in Spring 1983 by Melrose and Mendoza; unfortunately that work was never completed, although its aims were very close to those here. The purpose of the present work is to complete their undertaking. The need for a general treatment became clear to this author in attempts to apply these methods to various problems in geometry, cf. [M1], [M-M1], [M2], [M3], [M-P], [M-S], [M4]. During the intervening years, many new ideas have refined and simplified these techniques, again mainly due to Melrose, c.f. [Me2]. They have also been used in a variety of problems involving degenerating families of elliptic operators [M-M2], [E-M], [McD], and in real and complex hyperbolic geometry [M-M1], [M-P], [E-M-M], [E-M], [L-M].

Due to its length, this paper has been divided into two parts. This first part treats elliptic edge operators only on weighted spaces with respect to which they are 'semi-Fredholm'. This is more elementary for various reasons, the main one being that semi-Fredholm operators possess considerable stability properties. In addition, many natural edge operators are semi-Fredholm for every admissible weight [M-S], although not always [B-M]. The present work is arranged as follows. §2 contains a thorough discussion of the class of edge operators, and several of the attendant geometric constructions used to study them. In an appendix §2A we collect the main facts concerning the theory of polyhomogeneous conormal distributions we shall later use. In §3 the spaces of edge pseudodifferential operators are introduced, and the main points of their calculus, e.g.composition and mapping properties, are then proved. §4 contains the parametrix construction for totally characteristic (conic) operators; this is a simpler case of the later construction and also the later construction uses these results. In §5 the model 'normal operator' of an elliptic edge operator is analyzed, and this is used to complete the parametrix construction in §6 in the special case when the the model operator is either injective or surjective. §7 contains a complete discussion of the regularity theory associated to the edge class.

The second part of this paper, to appear later, treats edge operators which are not necessarily semi-Fredholm. This requires a substantial detour into and development of the theory of Calderón projectors and Poisson operators in this context.

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§2. Edge Structures

In this section we introduce the geometric and analytic constructions central to the analysis of edge operators. Recall from §1 that an edge structure $\mathcal{V}_e$ on the smooth compact manifold X is uniquely associated to a fibration of the boundary $\pi : \partial X \to Y$ with fibre F, as the space of all smooth vector fields on X tangent to the fibres of π at the boundary. It is closed under the ordinary bracket on vector fields, hence defines a Lie algebra. $\mathcal{V}_e$ is also a finitely generated $\mathcal{C}^\infty(X)$-module. It suffices to check this locally, and we use local coordinates (x, y, z) where x vanishes simply on the boundary, $(y_1, \dots, y_k)$ are coordinates on Y lifted to ∂X and then extended inwards, and $(z_1, \dots, z_q)$ restrict to coordinates along each fibre at ∂X. Then all elements of the set

$$\{x\partial_x, x\partial_{y_1}, \cdots, x\partial_{y^k}, \partial_{z_1}, \cdots, \partial_{z^q}\} \tag{2.1}$$

belong to $\mathcal{V}_e$, are independent, and generate it locally over $\mathcal{C}^\infty(X)$.

The maximality and independence of the set (2.1) means that there is a bundle ${}^eTX$ naturally associated to $\mathcal{V}_e$ such that $\mathcal{V}_e = \mathcal{C}^\infty(X, {}^eTX)$ and which is equipped with a map to the ordinary tangent bundle, $\iota_e : {}^eTX \to TX$, induced by the proper inclusion $\mathcal{V}_e \hookrightarrow \mathcal{C}^\infty(X, TX)$. ι_e is an isomorphism over $\mathring{X}$ because $\mathcal{V}_e$ consists of all smooth vector fields there, but is neither injective nor surjective at the boundary. ${}^eTX$ is defined by simply demanding that the vector fields (2.1) be a spanning set of sections. Namely, any $V \in \mathcal{V}_e$ may be uniquely expressed as $ax\partial_x + b^i x\partial_{y^i} + c^j\partial_{z^j}$, and then the coefficients a, b^i, c^j evaluated at $p \in X$ are linear coordinates in the fibre ${}^eT_pX$. Another more intrinsic definition is given in (2.13). The dual to ${}^eTX$, denoted ${}^eT^*X$, is spanned locally by the 1-forms

$$\left\{\frac{dx}{x}, \frac{dy^1}{x}, \cdots, \frac{dy^k}{x}, dz^1, \cdots, dz^q\right\} \tag{2.2}$$

which are singular as forms in the usual sense but smooth as sections of ${}^eT^*X$.

The universal enveloping algebra of $\mathcal{V}_e$ is the ring $\mathrm{Diff}_e^*(X)$ of differential operators of edge type, and consists of operators which may be expressed locally as finite sums of products of elements of $\mathcal{V}_e$. Using (2.1)

$$L \in \mathrm{Diff}_e^*(X) \Longleftrightarrow L = \sum_{j+|\alpha|+|\beta|\le m} a_{j,\alpha,\beta}(x,y,z)(x\partial_x)^j(x\partial_y)^\alpha\partial_z^\beta \tag{2.3}$$

with all coefficients smooth. $\mathrm{Diff}_e^*(X)$ is filtered by the subspaces $\mathrm{Diff}_e^m(X)$ of operators of order less than or equal to m. There is a symbol map on $\mathrm{Diff}_e^*(X)$. For L as in (2.3) set

$$
{}^e\sigma_m(L)(x,y,z;\xi,\eta,\zeta) \;=\; \sum_{j+|\alpha|+|\beta|=m} a_{j,\alpha,\beta}(x,y,z)\xi^j\eta^\alpha\zeta^\beta. \tag{2.4}
$$

This is an invariantly defined homogeneous polynomial of degree m on the fibres of ${}^eT^*X$, see (3.6). As usual, L is said to be elliptic if ${}^e\sigma_m(L)(x,y,z;\xi,\eta,\zeta) \neq 0$ if $(\xi,\eta,\zeta) \neq 0$.

The key examples of edge operators are the Laplacians of metrics in the class

$$
\mathcal{G}_e = \{g \in \mathcal{C}^\infty(\mathrm{Sym}^2({}^eT^*X)) : g \gg 0 \text{ on } {}^eTX\}. \tag{2.5}
$$

Each $g \in \mathcal{G}_e$ is geodesically complete and of uniformly bounded geometry, that is, the curvature tensor of g and all its covariant derivatives are bounded and the injectivity radius of g is positive. In fact, all sectional curvatures tend strongly to asymptotic limits at infinity. The following is easy to check:

(2.6) Proposition. *For any $g \in \mathcal{G}_e$ the Laplacian Δ_g is an elliptic element of $\mathit{Diff}_e^2(X)$*

This is also true for the Hodge Laplacian acting on sections of the exterior powers of ${}^eT^*X$, i.e. the edge form bundles, and also for the trace Laplacian on any edge tensor bundle. Dirac-type operators corresponding to these metrics are elements of $\mathrm{Diff}_e^1(X)$ acting between sections of the appropriate bundles.

The goal of this paper is the construction of parametrices for elliptic edge operators. Because such an operator is elliptic in the ordinary sense in the interior of X a parametrix may be constructed there by classical methods. Recall [Hö] that the Schwartz kernel of an interior parametrix of an elliptic operator is a distribution on the product $X \times X = X^2$ with a conormal singularity along the diagonal. However, any suitable parametrix must have extra singularities at the submanifold of the corner $(\partial X)^2$ where the diagonal hits the boundary. The point of view we adopt is that one should regard this Schwartz kernel as the pushforward of a simpler distribution from a slightly more complicated manifold X_e^2 which covers X^2, and which we define below.

The corner of X^2 contains a submanifold S consisting of all fibres of the product fibration $\pi^2 : (\partial X)^2 \to Y^2$ which intersect the diagonal of $(\partial X)^2$. The 'edge double product' X_e^2 is the union of $X^2\backslash S$ with the interior spherical normal bundle of S in X^2, $X_e^2 = (X^2\backslash S) \sqcup (N^+(S)/\mathbf{R}^+)$, endowed with the unique minimal differential structure with respect to which smooth functions on X^2 and polar coordinates on X^2 around S are smooth. This process is called blowing up X^2 around the submanifold S, and as such is also denoted $X_e^2 = [X;S]$. X_e^2 has three boundary hypersurfaces. Two of them, the 'left' and 'right' boundaries, correspond to the two hypersurface boundaries of X^2. The third corresponds to the interior spherical normal bundle of S and is called the front face of X_e^2.

We now introduce coordinate systems on X_e^2 and discuss the geometry of the blow-up using them. First, let (x,y,z) and $(\tilde{x},\tilde{y},\tilde{z})$ be identical sets of coordinates as described earlier on the left and right (first and second) copies of X in X^2. Note that the left and right faces $\partial X \times X$ and $X \times \partial X$ of X^2 are defined by $x = 0$ and $\tilde{x} = 0$, respectively, and the diagonal Δ is the set $\{x = \tilde{x},\ y = \tilde{y},\ z = \tilde{z}\}$. Δ intersects the corner $x = \tilde{x} = 0$ at $\partial\Delta = \{x = \tilde{x} = 0,\ y = \tilde{y},\ z = \tilde{z}\}$. Also the 'fibre-diagonal' $S = \{x = \tilde{x} = 0, y = \tilde{y}\}$. From the definition of the differential structure of X_e^2, polar coordinates around S in X^2 lift to a nonsingular coordinate chart $(r, \theta, \tilde{y}, z, \tilde{z})$ on X_e^2 around the front face. Here

$$(2.7) \qquad r = \sqrt{x^2 + |y - \tilde{y}|^2 + \tilde{x}^2}, \quad \theta = (x, y - \tilde{y}, \tilde{x})/r = (\theta_0, \theta', \theta_{k+1}).$$

These are smooth and independent on X^2 away from S, and when lifted to X_e^2 they become smooth and independent everywhere. In particular X_e^2 is locally diffeomorphic near its front face to $\mathbf{R}^+ \times S_{++}^{k+1} \times F_z \times \partial X_{\{\tilde{y}, \tilde{z}\}}$, where S_{++}^{k+1} is the quarter sphere $\{\theta_0,\ \theta_{k+1} \geq 0\}$. There is a blow-down map $\beta_{(2)} : X_e^2 \to X^2$ which is the identity on $X^2 \backslash S$ and the natural projection from the spherical normal bundle of S to S; it can be written

$$(2.8) \qquad \beta_{(2)}((r, \theta, \tilde{y}, z, \tilde{z})) = (r\theta_0, \tilde{y} + r\theta', z, r\theta_{k+1}, \tilde{y}, \tilde{z}).$$

Following $\beta_{(2)}$ by the projection maps from X^2 onto either the left or right factors of X yields two other maps

$$(2.9) \qquad \beta_L : X_e^2 \to X, \qquad \beta_R : X_e^2 \to X.$$

The boundary faces of X_e^2 will be labelled by a pair of indices ij, $i, j \in \{0, 1\}$ as follows: $B_{10}(X_e^2)$ is the left face $\{\theta_0 = 0\}$, $B_{01}(X_e^2)$ the right $\{\theta_{k+1} = 0\}$, and $B_{11}(X_e^2)$ is the front face $\{r = 0\}$. The left and right boundaries of X^2 will be similarly labelled $B_{10}(X^2)$ and $B_{01}(X^2)$. The lift of the diagonal to X_e^2 is the set $\{\theta_0 = \theta_{k+1},\ \theta' = 0,\ z = \tilde{z}\}$; we call it Δ_e. Finally, the functions ρ_{10}, ρ_{01}, and ρ_{11} will often denote defining functions for B_{10}, B_{01}, and B_{11}, respectively. Of course, we may take these functions to be θ_0, θ_{k+1}, and r, and in particular, r will usually be preferred over ρ_{11}.

We also introduce two systems of projective coordinates on X_e^2 which are quite useful for calculations, but have the drawback that they are not smooth on all of X_e^2. These are the charts $(s, u, \tilde{x}, \tilde{y}, z, \tilde{z})$ and $(x, y, t, v, z, \tilde{z})$, where

$$(2.10) \qquad s = x/\tilde{x},\ u = \frac{y - \tilde{y}}{\tilde{x}} \qquad t = \tilde{x}/x,\ v = \frac{\tilde{y} - y}{x}.$$

s and u are smooth away from B_{01} and t and v are smooth away from B_{10}. Observe that $\tilde{x} = 0$ and $x = 0$ define the front face in either of these two systems. The lifts of elements of $\mathcal{V}_e$ to X_e^2 are rather messy to compute in polar coordinates, but much easier in either of these projective coordinate systems. Thus, a brief calculation shows that the lifts of the vector fields (2.1) (through the left factor of X) are given by

$$(2.11) \qquad \begin{aligned} x\partial_x &= s\partial_s = x\partial_x - t\partial_t - v\partial_v \\ x\partial_y &= s\partial_u = x\partial_y - \partial_v. \end{aligned}$$

(The coordinates $x,\ y$ on the left and far right of these equalities are of course different, but since the second set of projective coordinates are rarely used this should cause no confusion.) From these expressions the following is immediate.

(2.12) Lemma. *Elements of $\beta_L^*(\mathcal{V}_e)$ are smooth vector fields on X_e^2 tangent to all boundary faces, hence restrict to vector fields on the front face $B_{11}(X_e^2)$. The kernel of this restriction map consists of lifts of vector fields which restrict to 0 as sections of ${}^eT_{\partial X}X$.*

This Lemma implies that the edge tangent bundle ${}^eTX$ is naturally isomorphic to the normal bundle of the diagonal in X_e^2, and similarly for the edge cotangent bundle

$$ {}^eTX = N\Delta_e, \quad {}^eT^*X = N^*\Delta_e. \tag{2.13} $$

This could have been used to define ${}^eTX$ in the first place. Closely related to (2.12) is another result which justifies the introduction of X_e^2 as the correct space on which to construct parametrices.

(2.14) Lemma. *If $L \in \mathrm{Diff}_e^*(X)$ is elliptic, then its lift to X_e^2 is transversely elliptic to Δ_e uniformly down to the front face.*

Proof. Suppose L is expressed as in (2.3). We are free to use either of the two projective coordinate systems above to calculate the lift, since they are both nonsingular near Δ_e, and from (2.11) it is clearly preferable to use the system $(s, u, \tilde{x}, \tilde{y}, z, \tilde{z})$. Then

$$ L = \sum_{j+|\alpha|+|\beta|\le m} a_{j,\alpha,\beta}(s\tilde{x}, \tilde{y} + \tilde{x}u, z)(s\partial_s)^j(s\partial_u)^\alpha \partial_z^\beta. \tag{2.15} $$

The conclusion is now clear because L is an elliptic combination of the vector fields $s\partial_s$, $s\partial_u$, ∂_z and the function s equals 1 on Δ_e (hence is nonvanishing nearby) uniformly down to $\tilde{x} = 0$.

The first step of the parametrix construction involves finding a parametrix with the correct singularity along the diagonal. Lemma (2.14) reduces this, as we demonstrate later, to a standard symbolic construction. However, further steps are required to account for its additional singularity. These involve the use of two simpler operators modelling L which we now describe. The first model is suggested by Lemma (2.13) and appears directly on X_e^2.

(2.16) Definition. *For $L \in \mathrm{Diff}_e^*(X)$ the normal operator $N(L)$ is defined to be the restriction to the front face B_{11} of the lift of L to X_e^2.*

In terms of (2.15)

$$ N(L) = \sum_{j+|\alpha|+|\beta|\le m} a(0, \tilde{y}, z)(s\partial_s)^j(s\partial_u)^\alpha \partial_z^\beta. \tag{2.17} $$

Notice that the sum is over $j + |\alpha| + |\beta| \le m$. Also, implicit in this expression is the fact that $(s, u, \tilde{y}, z, \tilde{z})$ are coordinates for B_{11} away from $B_{01} \cap B_{11}$. $N(L)$ is always independent of $\tilde{z}$ in these coordinates, and its dependence on y is only parametric. That $N(L)$ appears to be 'constant coefficient' in the vector fields $s\partial_s$ and $s\partial_u$ is no accident. $B_{11}(X_e^2)$ has a natural group structure on the interior of the quarter-sphere factors in its fibres, and another definition of the normal operator which we omit here defines it directly as translation invariant in these directions. Its dilation invariance in (s,u) and translation invariance in u are the keys to its analysis.

The other model is simpler than $N(L)$ but carries rather less information. It is most directly defined as the operator induced by the action of L on series expansion:

(2.18) Definition. *The indicial family $I_\zeta(L)$ of $L \in \mathit{Diff}_e^*(X)$ is the family of operators given by*

$$L(x^\zeta(\log x)^p f(x,y,z)) \equiv x^\zeta(\log x)^p I_\zeta(L) f(0,y,z) + O(x^\zeta(\log x)^{p-1}), \quad \forall f \in \mathcal{C}^\infty(X),\ \zeta \in \mathbf{C},\ p \in \mathbf{N}_0. \tag{2.19}$$

(If $p=0$ the error term in (2.19) may be replaced by $O(x^{\zeta+1})$.) This is a family of operators on ∂X holomorphic in ζ and in which y enters only as a parameter. Hence $I_\zeta(L)$ restricts to an operator on each F_y. There exists a unique dilation invariant operator $I(L)$ on $\mathbf{R}^+ \times F$ such that $I(L)(y,z,s\partial_s,\partial_z)x^\zeta f(y,z) = x^\zeta I_\zeta(L) f(y,z)$. This is called the indicial operator of L. In local coordinates

$$I(L) = \sum_{j+|\beta|\le m} a_{j,0,\beta}(0,y,z)(s\partial_s)^j \partial_z^\beta. \tag{2.20}$$

$I(L)$ is conjugated to $I_{i\zeta}(L)$ by the Mellin transform.

The indicial family determines a set of values in the complex plane fundamental to the analysis of the mapping properties of L.

(2.21) Definition. *If $L \in \mathit{Diff}_e^*(X)$ is elliptic, the boundary spectrum of L, $\mathit{spec}_b(L)$, is the set of values $\zeta \in \mathbf{C}$ for which the operator $I_\zeta(L)$ fails to be invertible on $L^2(F)$ for some $y \in Y$.*

For a fixed $y \in Y$ the operator $I_\zeta(L)$ fails to be invertible only at a discrete set $\{\zeta_i\} \subset \mathbf{C}$. Furthermore, the inverse $I_\zeta(L)^{-1}$ is meromorphic in $\mathbf{C}$ and the singular Laurent coefficients at its poles are finite rank projectors onto spaces of smooth functions in $L^2(F)$. This is a consequence of the analytic Fredholm theorem and is proved in [Me-Me]. However, these values ζ_i may depend on y, so the boundary spectrum in general is the countable union of images of Y under a sequence of smooth complex-valued maps. In this generality the overall analysis of elliptic edge operators is rather more complicated, cf. [S]. However, the edge operators which arise most frequently in geometry tend to have y-independent indicial roots, so unless explicitly stated we shall henceforth always make the

(2.22) Hypothesis. *The set $\mathit{spec}_b(L)$ is discrete in $\mathbf{C}$, and furthermore, the inverse $I_\zeta(L)^{-1}$ is jointly smooth in $y \in Y$ and holomorphic in $\zeta \notin \mathit{spec}_b(L)$.*

The inverse is automatically meromorphic in ζ for fixed y, so part of this hypothesis is made to ensure smoothness of the inverse in y.

We also define a refinement of $\mathrm{spec}_b(L)$ which takes into account multiplicities of the poles of $I_\zeta(L)^{-1}$. This is the set

$$\widetilde{\mathrm{spec}}_b(L) = \{(\zeta, p) \in \mathbf{C} \times \mathbf{N}_0 : \exists\, \phi(y,z) \text{ s.t. } I(L)(x^\zeta (\log x)^p \phi) = 0\}. \tag{2.23}$$

If $(\zeta_0, p) \in \widetilde{\mathrm{spec}}_b(L)$ then $\zeta_0 \in \mathrm{spec}_b(L)$ and p is the order of the pole of $I_\zeta(L)^{-1}$ there.

§2A. Appendix: Polyhomogeneity

This appendix contains a brief summary of the definitions and main properties of polyhomogeneous conormal distributions. These distributions are central to the whole paper: parametrices for elliptic edge operators (with y-independent indicial roots) turn out to have only polyhomogeneous singularities on X^2_e and consequently their construction is essentially reduced to formal (Taylor series) computations. This structure for parametrices reflects the basic regularity statement that "good" solutions of $Lu = 0$ are polyhomogeneous (although, as we shall see in §7, this is not true for arbitrary solutions), which in turn is a replacement for the usual smoothness up to the boundary for solutions of elliptic boundary problems.

We work somewhat generally in this appendix in the category of manifolds with corners. The foundations of analysis in this setting is the subject of [Me2], and a complete survey (without proofs) is contained in [E-M]. The contents of this appendix are a distillation from these sources. All notations for coordinates, etc., are special to this appendix and sometimes different from similar notations in the main body of the text. Let X be any manifold with corners. This means that at each point there exists a nonnegative integer k such that X is modelled diffeomorphically near that point by a neighbourhood of the origin in the product $(\mathbf{R}^+)^k \times \mathbf{R}^{n-k}$. $\{M_i\}_{i=1}^J = M(X)$ is a labelling of the codimension one boundary faces. (It is useful to remember that in the rest of the paper boundary faces are frequently labelled by pairs or triplets of integers.) We assume for simplicity that all boundary faces are embedded. Denote by $\mathcal{V}_b$ the space of smooth vector fields on X which are tangent to all boundaries. As above, any boundary point p is contained in a corner of maximal codimension k. Choose coordinates $x^1, \dots, x^k, y$ near p, where the x^i are defining functions for the boundary hypersurfaces $M_{i^1}, \dots, M_{i^k}$ intersecting the corner at p, and y is a set of coordinates along this (codimension k) corner. Then $\mathcal{V}_b$ is spanned over $\mathcal{C}^\infty(X)$ near p by $\{x^1\partial_{x^1}, \dots, x^k\partial_{x^k}, \partial_{y^\alpha}\}$. The basic conormal space is

$$\mathcal{A}^0(X) = \{u : V_1 \cdots V_\ell u \in L^\infty(X),\ \forall\, V_i \in \mathcal{V}_b,\ \text{and } \forall\, \ell\}. \tag{A.1}$$

If $s_1, \dots, s_J \in \mathbf{C}$ and $p_1, \dots, p_J \in \mathbf{N}_0$, $\mathbf{N}_0 = \{0, 1, 2, \cdots\}$ then (using multi-index notation) let $\mathcal{A}^{s,p}(X) = x^s (\log x)^p \mathcal{A}^0$. If all $p_i = 0$ then we denote this space simply by $\mathcal{A}^s(X)$. $\mathcal{A}^*(X)$ is the union of these spaces over all s (and p).

We are mostly interested in spaces of conormal functions with classical asymptotic expansions. Suppose for the moment that X has only one codimension one boundary, and let x be a defining function for it. Then any function $x^s(\log x)^p a(y)$ is in $\mathcal{A}^{s,p}(X)$ if

s is any complex number, $p \in \mathbf{N}_0$ and $a(y)$ is smooth. $\mathcal{A}^*_{phg}(X)$ consists of all conormal distributions admitting asymptotic expansions of infinite sums of terms of this simple form:

$$\text{(A.2)} \qquad \mathcal{A}^*_{phg}(X) \ni u \Longrightarrow u \sim \sum_{\Re s_j \to \infty} \sum_{p=0}^{p_j} x^{s_j} (\log x)^p a_{j,p}(x,y), \quad a_{j,p} \in \mathcal{C}^\infty.$$

By conormality, all coefficients $a_{j,p}(x,y)$ are smooth functions on X. Finer spaces of polyhomogeneous conormal distributions are fixed by prescribing an index set which indicates which powers of x and $\log x$ can arise in the expansion (A.2). An index set is a discrete subset $E \subset \mathbf{C} \times \mathbf{N}_0$ such that

$$\text{(A.3)} \qquad (s_j, p_j) \in E, \ |(s_j,p_j)| \to \infty \Longrightarrow \Re(s_j) \to \infty.$$

Then $\mathcal{A}^E_{phg}(X)$ consists of those distributions $u \in \mathcal{A}^*_{phg}$ with polyhomogeneous expansions of the form (A.2) with $(s_j,p_j) \in E$. Property (A.3) is important to ensure that the asymptotic expansions have the usual meaning. There are a few special cases of this notation worth pointing out separately. Functions on X smoothly extendible across ∂X correspond to the space $\mathcal{A}^E_{phg}$ with $E = \{(0,0)\}$, which will usually be written $\mathcal{A}^0_{phg}$. Similarly, the set of smooth functions vanishing to order r along ∂X belong to the polyhomogeneous space $\mathcal{A}^r_{phg}$, and functions vanishing to all orders on M belong to the polyhomogeneous space with empty index set, which will be written $\mathcal{A}^\emptyset_{phg}$. We shall say that an index set E satisfies $\Re(E) \ge C$, for some $C \in \mathbf{R}$ if $\Re s \ge C$ for every s such that $(s,p) \in E$ for some p, and similarly for strict inequality.

When X is now allowed to have many possibly intersecting codimension one boundary components, then we must specify an index set E_i corresponding to each boundary face M_i, $i = 1, \dots, J$. Let $\mathcal{E}$ be the J-tuple of index sets $(E_1, \dots, E_J)$; then $\mathcal{A}^{\mathcal{E}}_{phg}$ will denote the set of distributions with expansions (A.2) at the interior of the face M_i with index set E_i, and with product type expansions at the corners. The rigorous definition of this space involves proceeding inductively from the highest codimension corners, which have no boundaries. In the inductive step, where we assume the definition given for boundaries with corners up to codimension $\ell - 1$, on a face with corners up to codimension ℓ we take the coefficients $a_{j,p}$ in an expansion (A.2) at a boundary hypersurface M_i to lie in the space $\mathcal{A}^{\mathcal{E}^i}_{phg}$, where the collection of index sets $\mathcal{E}^i$ is the subset of the collection $\mathcal{E}$ corresponding to those boundary faces M_j intersecting M_i, so that $M_j \cap M_i \in M(M_i)$. Since M_i has boundary with corners only up to codimension $\ell - 1$, this space is defined by the inductive hypothesis and the definition is complete. To check that a function u lies in a particular space $\mathcal{A}^{\mathcal{E}}_{phg}(X)$, it is often simpler to apply the following criterion, which is proved in [Me2]:

(A.4) Lemma. *Suppose that $u \in \mathcal{A}^*(X)$, and that at each $M_i \in M(X)$ u has an expansion of the form (A.2) with exponents (s_j, p) lying in E_i, but with coefficients $a_{j,p}$ merely required to lie in $\mathcal{A}^*(M_i)$. Then $u \in \mathcal{A}^{\mathcal{E}}_{phg}(X)$.*

For any $M_i \in M(X)$ choose a diffeomorphism from a collar neighbourhood of M_i in X to the inward pointing normal bundle $N^+(M_i)$ of M_i in X, the differential of which induces the identity map on $N^+(M_i)$ (using the obvious identifications). For any collection of index sets $\mathcal{E}$ as above, set

$$\mathcal{E} + 1_i = \{E_1, \dots E_{i-1}, E_i + 1, E_{i+1}, \dots E_J\}, \tag{A.5}$$

where

$$E_i + 1 = \{(s+1, p) : (s, p) \in E_i\}. \tag{A.6}$$

Then consider the quotient space

$$\mathcal{A}_{phg}^{\{\mathcal{E}\}_i}(X) \equiv \mathcal{A}_{phg}^{\mathcal{E}}(X) / \mathcal{A}_{phg}^{\mathcal{E}+1_i}(X). \tag{A.7}$$

Brackets around an index set E always denotes the quotient by the space with elements vanishing one order faster. The subscript $\{\,\}_i$ denotes the boundary face at which this quotient is taken. Using the diffeomorphism described above, we may define a symbol map, σ_i, which assigns to any polyhomogeneous u its leading singularities at M_i and such that the following sequence is exact:

$$0 \to \mathcal{A}_{phg}^{\mathcal{E}+1_i}(X) \to \mathcal{A}_{phg}^{\mathcal{E}}(X) \xrightarrow{\sigma_i} \mathcal{A}_{phg}^{\{\mathcal{E}\}_i}(N^+M_i) \to 0. \tag{A.8}$$

In terms of the expansion (A.2), σ_i corresponds to setting $x = 0$ in each of the coefficients $a_{j,p}$. There are obvious consistency conditions between these symbols at adjoining faces.

Specialize, for this paragraph, to the following situation. Consider the lift of an elliptic edge operator L (satisfying (2.22)) through the left factor of X in X^2 to X_e^2. From Lemma (2.12) this lift, which we still denote L, acts tangentially to all boundary faces, hence preserves any space $\mathcal{A}_{phg}^{\mathcal{E}}(X_e^2)$, where the triplet of index sets $\mathcal{E} = (E_{10}, E_{01}, E_{11})$ correspond to the left, right and front boundaries of X_e^2. We shall be most interested in its action on functions polyhomogeneous at the left boundary B_{10}. Directly from the definition (2.18) its induced action on $\mathcal{A}_{phg}^{\{E\}_{10}}(N^+B_{10})$ is given by the indicial operator $I(L)$. This action is an isomorphism (in a sense we will not make topologically precise) except when there exists $\zeta \in \operatorname{spec}_b(L)$ such that the pair $(\zeta, p) \in E_{10}$ for some $p \in \mathbf{N}_0$. However, given any index set E_{10} there exists a new index set E'_{10}, obtained by increasing some of the integers p corresponding to the logarithmic exponents, so that

$$I(L) : \mathcal{A}_{phg}^{E'_{10}}(N^+B_{10}) \to \mathcal{A}_{phg}^{E_{10}}(N^+B_{10}) \tag{A.9}$$

is surjective. (This is equivalent to the well-known procedure of adding extra logarithmic factors to solve regular singular equations when the right hand side decays exactly like an indicial root.)

We also need to consider distributions which are conormal along certain interior submanifolds as well. The simplest example of an interior conormal singularity is in

$\mathbf{R}^n$ at the origin. A (compactly supported) distribution in $I^m(\mathbf{R}^n, \{0\})$ is one whose Fourier transform is a symbol of order $m - \frac{1}{4}n$, cf. [Hö]. This space is invariant under local diffeomorphisms, hence is defined on any manifold around an isolated point. In keeping with our point of view here we note that symbols may be viewed as distributions conormal at infinity, i.e. at the boundary induced by stereographic projection.

Now let $Y \subset X$ be an embedded *p-submanifold*. This means that in some neighbourhood $\mathcal{U}$ of any point of Y X can be locally decomposed as a product $X \cap \mathcal{U} = X' \times X''$ where X'' has no boundary and so that $Y \cap \mathcal{U} = X' \times \{p''\}$, $p'' \in X''$. Then the space $I^m(X, Y)$ is defined (locally) as the space of smooth functions on X' with values in $I^{m+\frac{1}{4}\dim X'}(X'', \{p''\})$. (The normalization is chosen to give pseudodifferential operators their expected orders.) All distributions in this space are restrictions of distributions which are conormal to any smooth extension of Y across the boundaries of X. We may also consider distributions such as these which are conormal at all boundary faces. Choosing index sets for each boundary face of X as usual we define a space $\mathcal{A}^{\mathcal{E}}_{phg} I^m(X, Y)$. Elements of this space are conormal to Y in the interior as usual, and at all boundary faces have expansions as in (A.2) with coefficients conormal to the intersection of Y with each boundary face. There are various symbol maps for these spaces which we will not make explicit.

Elementary operations on polyhomogeneous conormal distributions, such as addition or multiplication, correspond to elementary operations on their index sets. For example, addition corresponds to set-theoretic union and multiplication corresponds to addition:

$$u \in \mathcal{A}^{\mathcal{E}}_{phg}, \quad v \in \mathcal{A}^{\mathcal{F}}_{phg} \Longrightarrow u + v \in \mathcal{A}^{\mathcal{E} \cup \mathcal{F}}_{phg}, \quad u \cdot v \in \mathcal{A}^{\mathcal{E} + \mathcal{F}}_{phg} \tag{A.10}$$

where $\mathcal{E} \cup \mathcal{F}$ and $\mathcal{E} + \mathcal{F}$ have the obvious meanings.

There are other operations that one might want to perform on these distributions such as pull-back and pushforward under some map $f : X_1 \to X_2$ between manifolds with corners. Of course pushforward is defined quite generally for distributions under the mild assumption that the map f be proper (which is automatic if we assume that the X_i are compact), and pull-back is defined under standard wave-front set assumptions. However these operations do not preserve conormality or polyhomogeneity in general unless further restrictions are made concerning the map f.

(A.11) Definition. *Let r_j and ρ_i be defining functions for the boundary hypersurfaces M_j of X_1 and N_i of X_2, respectively. Then $f : X_1 \to X_2$ is called a b-map if for every i there exist nonnegative integers $e(i,j)$ and a smooth nonvanishing function h such that $f^*(\rho_i) = h \prod r_j^{e(i,j)}$.*

This means that the image of the interior of each M_j is either contained in or disjoint from each N_i, and that the order of vanishing of the differential of f is constant along each M_j. The matrix $(e(i,j))$ is called the lifting matrix for f. Now, if $\mathcal{F}$ is a collection of index sets for the faces of X_2, define $f^\flat(\mathcal{F}) = \mathcal{E}$ where

$$E_j = \{(S, P) : S = \sum_i e(i,j) s_i, \ P = \sum_{e(i,j) \neq 0} p_i, \ (s_i, p_i) \in F_i\}. \tag{A.12}$$

Then the proof of the following is elementary.

(A.13) Proposition. *Let $f : X_1 \to X_2$ be a b-map, and $u \in \mathcal{A}^{\mathcal{F}}_{phg}(X_2)$. Then $f^*(u) \in \mathcal{A}^{\mathcal{E}}_{phg}(X_1)$ where $\mathcal{E} = f^\flat(\mathcal{F})$.*

There is a similar result for the interior conormal spaces described earlier. If Y_2 is contained in the interior of X_2 and the map f is transversal to it, then f^* transforms $I^m(X_2, Y_2)$ to $I^M(X_1, Y_1)$ where $M = m + \frac{1}{4}(\dim X_1 - \dim X_2)$ and $Y_1 = f^{-1}(Y_2)$. At boundary points one must also impose the condition of b-transversality (see below), and then the analogue of (A.13) holds.

For pushforward by f to preserve polyhomogeneity stronger conditions must be placed on the map. To phrase these restrictions efficiently we introduce some notation. Associate to the space of vector fields $\mathcal{V}_b$ on a manifold with corners the totally characteristic tangent and cotangent bundles ${}^bTX$ and ${}^bT^*X$. A map $f : X_1 \to X_2$ induces (by continuous extension from the interior) maps ${}^bf_* : {}^bTX_1 \to {}^bTX_2$ and ${}^bf^* : {}^bT^*X_2 \to {}^bT^*X_1$.

(A.14) Definition. *The b-map $f : X_1 \to X_2$ is called a b-submersion if the associated maps ${}^bf_*$ at each $p \in \partial X_1$ are surjective (or equivalently, if the maps ${}^bf^*$ are injective). f is called a b-fibration if in addition the associated lifting matrix $(e(i,j))$ has the property that for each j there is at most one i such that $e(i,j) \neq 0$. This means that f does not map any boundary hypersurface of X_1 to a corner of X_2.*

The b-fibration condition is sufficient to guarantee that pushforward by f preserves polyhomogeneity. (This is reasonable: if f did map a boundary hypersurface to a corner, then distributions polyhomogeneous to that hypersurface would push forward to distributions polyhomogeneous conormal to that corner, hence would not have product type expansions there.) The exact result may be described as follows. Given a collection of index sets $\mathcal{E}$ corresponding to the boundary faces of X_1, define a pushforward index set $\mathcal{F} = f_\flat(\mathcal{E})$ by

$$F_i = \{(S,P) : S = \sum_j s_j,\ P + 1 = \sum_j (p_j + 1) \text{ where } \exists\, (s_j, p_j) \in E_j \text{ for those } j \text{ s.t. } e(i,j) \neq 0\}. \tag{A.15}$$

Supposing that just two hypersurfaces, with corresponding index sets E_1 and E_2 are mapped to a boundary hypersurface in the range, we call the resulting index set F given by (A.15) the extended union and write

$$F = E_1 \overline{\cup} E_2. \tag{A.16}$$

We may of course take the extended union of more than two index sets. Finally, it is simplest to phrase the result for pushforwards of sections of the b-density bundle ${}^b\Omega$ associated to ${}^bT^*X$

$${}^b\Omega = (\prod x_i)^{-1}\, \Omega. \tag{A.17}$$

Then the precise result is

(A.18) Proposition. *Let $f : X_1 \to X_2$ be a b-fibration. Then for any collection of index sets $\mathcal{E}$ for X_1 such that $\Re(E_j) > 0$ for each j such that $e(i,j) = 0$ for all i (so that the corresponding face M_j maps to the interior of X_2), then pushforward gives a continuous map*

$$f_* : \mathcal{A}^{\mathcal{E}}_{phg}(X_1, {}^b\Omega) \to \mathcal{A}^{f_\flat(\mathcal{E})}_{phg}(X_2, {}^b\Omega).$$

A proof is given in [Me2].

As before, a version of this is required concerning interior conormal singularities. We refer to [E-M-M] for a precise discussion of the following. If $f : X_1 \to X_2$ is a b-submersion and Y_1', $Y_1'' \subset X_1$ are p-submanifolds meeting b-transversally at Y_1, and f embeds Y_1 as a submanifold Y_2 of X_2 then we consider the operation of multiplying distributions conormal at Y_1' and Y_1'' and pushing the result forward to X_2. By standard wave-front set arguments the multiplication is well-defined. The result is that if $u \in \mathcal{A}^{\mathcal{E}}_{phg} I^m(X_1, Y_1')$ and $v \in \mathcal{A}^{\mathcal{E}'}_{phg} I^{m'}(X_1, Y_1'')$ then $f_*(u \cdot v) \in \mathcal{A}^{f_\flat(\mathcal{E}+\mathcal{E}')}_{phg} I^{m+m'}(X_2, Y_2)$.

The last topic we wish to discuss here is the Mellin transform and its relationship to the polyhomogeneity. The simplest form of the Mellin transform is its interpretation as the Fourier transform written in exponential coordinates. Thus for $u \in \mathcal{C}^\infty_0(\mathbf{R}^+)$ define

$$u_M(\zeta) = \int_0^\infty u(x) x^{i\zeta - 1}\, dx. \tag{A.19}$$

Given that u is compactly supported, $u_M(\zeta)$ is an entire function of ζ which decays rapidly along each line $\Im\zeta =$ constant. For convenience set

$$\zeta = \xi + i\eta.$$

There is a Plancherel relationship which allows us to extend this definition to an isomorphism

$$M : L^2(\mathbf{R}^+; dx) \xrightarrow{\sim} L^2(\{\eta = -\tfrac{1}{2}\}; d\xi), \tag{A.20}$$

and more generally, taking the domain to be a weighted space, as an isomorphism

$$x^\delta L^2(\mathbf{R}^+; dx) \xrightarrow{\sim} L^2(\eta = \delta - \tfrac{1}{2}; d\xi). \tag{A.21}$$

Also familiar from the Fourier tranform is the fact that the Mellin transform intertwines differentiation by $x\partial_x$ and multiplication by $-i\zeta$:

$$(x\partial_x u)_M = -i\zeta u_M. \tag{A.22}$$

From this follows immediately our earlier claim that the Mellin transform conjugates the indicial operator of $L \in \mathrm{Diff}^*_e(X)$ to its indicial family.

The inverse Mellin transform is given by integrating along a horizontal line:

$$M^{-1}(u_M(\zeta)) = \int_{\eta = C} u_M(\zeta) x^{-i\zeta} d\xi, \tag{A.23}$$

provided this integral exists. Of course from (A.21) if $u \in x^\delta L^2$ and $C = \delta - \frac{1}{2}$, then $M^{-1}(u_M(\zeta)) = u(x)$.

Suppose $\phi \in \mathcal{C}_0^\infty(\mathbf{R})$ equals 1 near $x = 0$. Then the Mellin transform of $x^s(\log x)^p\phi(x)$ is holomorphic in a half-plane $\{\eta < \Re s\}$, since the function itself lies in $x^\delta L^2$ for any $\delta < \Re s + \frac{1}{2}$. However, an integration by parts shows that it extends to be a meromorphic function in the whole plane, with its only singularity a pole of order $p+1$ at $\zeta = is$, and L^2 along every line $\{\eta = C,\ C \neq s\}$. Now, applying this same argument to functions $u \in \mathcal{A}^E_{phg}(X)$ where X is a manifold with boundary ∂X, if ϕ is a smooth function identically equal to 1 near ∂X, then the function $(\phi u)_M(\zeta, y)$, $y \in \partial X$, extends to a meromorphic with poles of order $p+1$ at points $\zeta = i(s+\ell)$, $\ell \in \mathbf{N}$, for $(s,p) \in E$ with values in $\mathcal{C}^\infty(\partial X)$. The extension of this result to more intersecting boundary components is clear. This relationship is key in proving various facts concerning polyhomogeneity. For example, the extended union operation (A.16) corresponds exactly to multiplication of meromorphic functions.

§3. The Edge Calculus

Using the constructions and notation of the last section the introduction of the edge calculus is now rather straightforward. This edge calculus consists initially of a ring $\Psi_e^*(X)$ of pseudodifferential operators which contains $\mathrm{Diff}_e^*(X)$ as a subring and also the basic parametrices of elliptic differential edge operators. As indicated earlier, the main point in this definition is that the distributional Schwartz kernels of edge operators on X^2 are pushforwards of distributions on X_e^2 with very simple structure. There are two levels of the edge calculus: the small calculus contains those operators with kernels vanishing to all order at the side boundary faces $B_{10}(X_e^2)$ and $B_{01}(X_e^2)$ while the large calculus incorporates operators with kernels having polyhomogeneous conormal singularities at these faces. Elements of the small calculus are used in the first step of the elliptic parametrix construction and for the crude regularity theory. In a sense which should become obvious, the estimates obtained using them are equivalent to the scale-invariant interior Schauder estimates found, for example, in [G-T]. The full parametrices for elliptic edge operators are elements of the large calculus.

To motivate the definition below consider the lifts of elements of $\mathrm{Diff}_e^*(X)$ to X_e^2. The simplest such element is the identity operator I (i.e. multiplication by the constant function 1). Its Schwartz kernel is the delta function along the diagonal. However, since our operators act on half-densities, this kernel must be regarded as a distributional section of the $\Omega^{\frac{1}{2}}(X^2)$. In terms of standard coordinates near the boundary choose a canonical nonvanishing half-density $\mu = \sqrt{dx\,dy\,dz\,d\tilde{x}\,d\tilde{y}\,d\tilde{z}}$. (This is determined up to a smooth nonvanishing function; this factor would cause no difference in anything below.) Then the identity operator corresponds to the distributional density

$$K_I = \delta(x - \tilde{x})\,\delta(y - \tilde{y})\,\delta(z - \tilde{z})\,\mu. \tag{3.1}$$

(Strictly speaking we can only identify this object as a nonvanishing delta section of the half-density bundle.) Now lift this distributional density to X_e^2 using the polar coordinates (2.7). Since the Jacobian determinant of $\beta_2 : X^2_e \to X^2$ is r^{k+1}, the half-density μ lifts to $r^{\frac{k+1}{2}}\nu$ where ν is a nonvanishing standard half-density on X^2_e (for example, we may take $\nu = \sqrt{dr\, d\theta\, d\tilde{y}\, dz\, d\tilde{z}}$ locally). Since the delta function at the origin in $\mathbf{R}^\ell$ is homogeneous of degree $-\ell$, (3.1) lifts to

$$\beta^* K_I = \delta(\theta_0 - \theta_{k+1})\, \delta(\theta')\, \delta(z - \tilde{z})\, r^{-\frac{k+1}{2}}\, \nu. \tag{3.2}$$

We prefer that the identity operator corresponds to a simple delta function with no additional vanishing or singular factors, so it and the kernels below will be regarded as distributional sections of the singular half-density bundle $r^{-\frac{k+1}{2}}\Omega^{\frac{1}{2}}(X^2_e)$. The lift of any $L \in \mathrm{Diff}^*_e(X)$ to X^2_e (through the left factor) may be expressed using the projective coordinates (2.9). The kernel K_L corresponding to L and normalized as above is then of the form $\kappa_L\, r^{-\frac{k+1}{2}}\nu$ where κ_L is obtained by applying L to (3.2), hence is a sum of smooth multiples of derivatives of the delta function along the diagonal.

This calculation motivates the definition of the small calculus.

(3.3) Definition. *Using notation from §2A*

$$\Psi^*_e(X; \Omega^{\frac{1}{2}}) = \mathcal{A}^{\mathcal{E}}_{phg} I^*(X^2_e, \Delta_e; r^{-\frac{k+1}{2}}\Omega^{\frac{1}{2}})$$

where $\mathcal{E} = \{\emptyset, \emptyset, (0,0)\}$.

Every $A \in \Psi^*_e$ corresponds, after factoring out the singular half-density, to a distribution κ_A conormal along the lifted diagonal of X^2_e, vanishing to infinite order at the side boundaries and smooth across the front face. As mentioned earlier, Ψ^*_e is a ring, i.e. the composition of any two elements $C = A \circ B$ is defined and again an operator of the same type. It is filtered by the subspaces Ψ^m_e consisting of elements A of order m which have kernels $\kappa_A \in \mathcal{A}^{\mathcal{E}}_{phg} I^m(X^2_e, \Delta_e)$ for the same $\mathcal{E}$ as above.

Before discussing how these operators act on half-densities, we note that although the definition of Lu is obvious when $L \in \mathrm{Diff}^*_e(X)$ and u is a function, there is no completely natural action on half-densities. We must fix a nonvanishing section $\gamma \in \mathcal{C}^\infty(X, \Omega^{\frac{1}{2}})$ to give an isomorphism between the spaces of distributions and distributional half-densities. If $f \in C^{-\infty}(X)$ then we decree that $L \in \mathrm{Diff}^*_e(X)$ acts on the half-density $f\,\gamma$ by $L(f\,\gamma) \equiv (Lf)\,\gamma$. Henceforth we always assume that local coordinates are chosen so that $\gamma = \sqrt{dx\, dy\, dz}$, but a different choice would only result in conjugating everything by a nonvanishing smooth function. Notice that such a conjugation does not change the indicial roots of L, hence in particular does not affect their constancy.

Edge pseudodifferential operators already have these half-densities built in, so if $A \in \Psi^*_e$, when we define its action by

$$g\,\gamma = A(u\,\gamma) = (\beta_L)_*(\kappa_A\, r^{-\frac{k+1}{2}}\nu\, (\beta_R)^*(u\,\gamma)), \tag{3.4}$$

where $\beta_L : X^2_e \to X$ and $\beta_R : X^2_e \to X$ are blow-down $X^2_e \to X^2$ followed by projection $X^2 \to X$ onto the left and right factors of X, respectively. To examine (3.4) more explicitly, multiply both sides by γ, and on the right side use the notation that $\gamma_L = (\beta_L)^*\gamma$ and $\gamma_R = (\beta_R)^*\gamma$. Then

$$\begin{aligned} g\,\gamma^2 = g\,dx\,dy\,dz &= (\beta_L)_*(\kappa_A\,(\beta_R)^* u\, r^{-\frac{k+1}{2}}\,\gamma_L\gamma_R\,\nu) \\ &= (\beta_L)_*(\kappa_A\,(\beta_R)^* u\,\nu^2), \end{aligned}$$

since $\gamma_L\,\gamma_R = r^{\frac{k+1}{2}}\nu$. Using the projective coordinates $(s, u, z, \tilde{x}, \tilde{y}, \tilde{z})$ this becomes

$$g(x,y,z) = \int \kappa_A\left(\frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}}, z, \tilde{x}, \tilde{y}, \tilde{z}\right) u(\tilde{x},\tilde{y},\tilde{z})\, r^{-k-1}\, d\tilde{x}\, d\tilde{y}\, d\tilde{z}. \tag{3.5}$$

ν^2 has been rewritten as $r^{-k-1}dx\,dy\,dz\,d\tilde{x}\,d\tilde{y}\,d\tilde{z}$ (with r the usual polar distance function) and the common factor $dx\,dy\,dz$ has been divided from both sides. (3.5) leads to the interpretation of edge operators as variable coefficient convolution operators in the variables (x, y) on the group H which is the semi-direct product of $\mathbf{R}^+$ with $\mathbf{R}^k$.

The symbol map (2.4) for edge differential operators extends to a principal symbol map for edge pseudodifferential operators. For $A \in \Psi_e^m$ the symbol of the Schwartz kernel of A as a conormal section of the singular half-density bundle is an element of

$$S^{\{m\}}(N^*\Delta_e) \otimes \Omega_{\text{fibre}}(N^*(\Delta_e)) \otimes r^{-\frac{k+1}{2}}\Omega^{\frac{1}{2}}.$$

The bracket in the order of the symbol denotes, as in §2A, the quotient by symbols of one order lower. $\Omega_{\text{fibre}}(N^*\Delta_e)$ is the bundle of translation invariant densities along each fibre of $N^*\Delta_e$, which arises from duality when the invariant Fourier transform transverse to the diagonal is used to transform a conormal distribution to its symbol. Now the density bundle of X_e^2 restricted to Δ_e splits into the density bundle of Δ_e, which is naturally equivalent to the density bundle $\Omega(X)$ of X, and the fibre-densities on $N\Delta_e$, so that

$$r^{-\frac{k+1}{2}}\Omega^{\frac{1}{2}}(X_e^2)\Big|_{\Delta_e} \cong r^{-\frac{k+1}{2}}\Omega^{\frac{1}{2}}(X) \otimes \Omega_{\text{fibre}}^{\frac{1}{2}}(N\Delta_e).$$

$r^{-k-1}\Omega(X)$ is equivalent to the edge-density bundle ${}^e\Omega(X)$ associated to ${}^eT^*X$, and recalling (2.13) we see that it is also identified with $\Omega_{\text{fibre}}(N^*\Delta_e)$. Putting these natural isomorphisms together shows that we may regard edge symbols as functions on ${}^eT^*X$. This finally leads to an exact sequence

$$0 \to \Psi_e^{m-1}(X) \longrightarrow \Psi_e^m(X) \xrightarrow{{}^e\sigma_m} S^{\{m\}}({}^eT^*X) \to 0. \tag{3.6}$$

A pseudodifferential operator A of order m is said to be elliptic if its symbol a is an invertible element of $S^{\{m\}}({}^eT^*X)$, i.e. if there exists a symbol $b \in S^{\{-m\}}({}^eT^*X)$ such that $a\cdot b - 1,\ b\cdot a - 1 \equiv 0$. We shall prove later in this section that Ψ_e^* is closed under composition. From this proof and the results of §2A it follows that

$${}^e\sigma_{m+m'}(AB) = {}^e\sigma_m(A)\cdot{}^e\sigma_{m'}(B) \tag{3.7}$$

The following is proved by the standard iterative inversion scheme using the exact sequence (3.6) and the property (3.7):

(3.8) Theorem. *If $A \in \Psi_e^m$ is elliptic then there exists $B \in \Psi_e^{-m}$ such that $A\,B - I \in \Psi_e^{-\infty}$ and $B\,A - I \in \Psi_e^{-\infty}$. This parametrix B is well defined up to an element of $\Psi_e^{-\infty}$.*

The two remainder terms $R_1 = I - A\,B$, $R_2 = I - B\,A$, although smoothing in the interior, are not compact operators because their Schwartz kernels are smooth only on X_e^2 and not on X^2. We will show later that this construction may be carried further, and eventually leads in favourable circumstances to parametrices for which one or both error terms are compact, so that A is semi-Fredholm or Fredholm. This is where the large calculus comes in. Elements of this large calculus are obtained by adding to elements of the small calculus operators with kernels smooth in the interior of X_e^2 and polyhomogeneous conormal at all boundary faces. X_e^2 has three boundary faces, so we shall let $\mathcal{E} = (E_{10}, E_{01}, E_{11})$ be a triplet of index sets describing conormal singularities at the boundary faces B_{10}, B_{01}, B_{11}.

(3.9) Definition. *For any $m \in \mathbf{R}$ and $\mathcal{E}$ an ordered triple of index sets, let*

$$\Psi_e^{m,\mathcal{E}}(X) = \{C = A + B : A \in \Psi_e^m,\ B \leftrightarrow \kappa_B r^{-\frac{k+1}{2}}\nu,\ \kappa_B \in \mathcal{A}_{phg}^{\mathcal{E}}(X_e^2)\}.$$

We shall also use 'very residual operators' having nothing to do with the edge structure which correspond to Schwartz kernels on X^2 smooth in the interior and polyhomogeneous conormal at the two boundaries. If $\mathcal{F} = \{F_{10}, F_{01}\}$ is a pair of index sets for these two boundary faces, we set

$$\Psi^{-\infty,\mathcal{F}}(X) = \mathcal{A}_{phg}^{\mathcal{F}}(X^2, \Omega^{\frac{1}{2}}(X^2)). \tag{3.10}$$

It will be convenient to use the notation that if $\mathcal{E} = (E_{10}, E_{01}, E_{11})$ is a collection of index sets for the boundaries of X_e^2 then $\mathcal{E}' = (E_{10}, E_{01})$ is a pair of index sets corresponding to the boundaries of X^2.

The remainder of this section is devoted to the composition and boundedness properties of these operators. The most natural proof of composition for the large calculus uses an auxiliary construction analogous to the one in [E-M-M], cf. also [Me2], called the edge triple product X_e^3. As with the edge double product X_e^2, this is a space obtained from the ordinary triple product $X^3 = X \times X \times X$ by a sequence of blow-ups. It is equipped not only with a blow-down map $\beta_{(3)} : X_e^3 \to X^3$, but also with partial blow-down maps followed by projection $\beta_{ij} : X_e^3 \to X_e^2 \times X \to X_e^2$, $i,j = L, M, R$, where L,M,R stand for the left, middle, and right factors of X^3, and the image of the map β_{ij} is the edge double product on the i-j pair of factors of X. The reason for using three factors of X at all is that in composing two operators A, $B \to A \circ B$, B acts from functions on the third factor to functions on the second, and A acts from functions on the second to functions on the first, so that $A \circ B$ acts from functions on the third to functions on the first. The Schwartz kernels we are dealing with live on the space X_e^2 rather than X^2, so X_e^3 is introduced because it is a natural place for the two Schwartz kernels on different factors, as above, to live.

A functorial way to define the kernel of the composition is to lift the kernel of A from the first two factors of X to X^3 and that of B from the second two factors, multiply them and push forward off the middle factor to the first and third factors. Symbolically, if $\pi_i,\ i = L,\ M,\ R$ are the projections $X^3 \to X^2$ off the left, middle or right factors of X, then $\kappa_{A\circ B} = (\pi_M)_*(\pi_R^*\kappa_A\, \pi_L^*\kappa_B)$. But of course the extra structure of the kernels of A and B forces us to consider an analogous formula using the space X_e^3 and the 'projections' β_{ij}. Recall that r will always denote the defining function (2.7) for the boundary face $B_{11}(X_e^2)$. Whenever we need local coordinates, we shall use identical sets (x, y, z), (x', y', z') and (x'', y'', z'') on the three factors. Taking density factors into account, we arrive at the (formal) definition:

$$\kappa_{A\circ B}\, r^{-\frac{k+1}{2}}\nu = (\beta_{LR})_*\left(\beta_{LM}^*(\kappa_A\, r^{-\frac{k+1}{2}}\nu)\, \beta_{MR}^*(\kappa_B\, r^{-\frac{k+1}{2}}\nu)\right). \tag{3.11}$$

We now define X_e^3 to make sense of (3.11). Recall the submanifold S in the corner of X^2 that is blown up in the definition of X_e^2. In each of the three pairs of copies of X in X^3 there is a copy of S which will be called S_{ij}, $i, j = L, M, R$. In order that the maps β_{ij} described earlier exist, we will have to blow up the S_{ij}. However, they intersect at a submanifold which we call T in such a way that the order in which we blow them up matters. For X_e^3 to be completely symmetric T must be blown up first. So X_e^3 is the space obtained by first blowing up T, which is a codimension $2k+3$ submanifold defined by $\{x = x' = x'' = 0,\ y = y' = y''\}$, and in the resulting space blowing up the lifts from X^3 of the submanifolds S_{ij}. Figure I should help in picturing this. Recall from [Me2], cf. also [E-M], the notation for iterated blow-ups. If Y and Z are two p-submanifolds in X, then $[X; Y; Z]$ denotes the space obtained by first blowing up Y in X, lifting Z to this space and then blowing up the resulting submanifold. This notation presupposes that the lift of Z to $[X; Y]$ is a p-submanifold. In terms of this notation

$$X_e^3 = [X^3; T; S_{LM} \cup S_{MR} \cup S_{LR}]. \tag{3.12}$$

X_e^3 has seven boundary hypersurfaces, and these will be labelled by a scheme generalizing the one introduced earlier for the boundary faces of X_e^2. Each boundary hypersurface shall be labelled by an ordered triplet of the integers {0,1}. B_{111} is the front face obtained by blowing up T, B_{110}, B_{101} and B_{011} correspond to the blow-ups of (the lifts of) S_{LM}, S_{LR} and S_{LR}, respectively, and B_{100}, B_{010} and B_{001} correspond to the lifts of the left, middle and right faces $\partial X \times X^2$, $X \times \partial X \times X$ and $X^2 \times \partial X$ of X^3. By convention B_{000} corresponds to the interior of X_e^3. For each triplet $ij\ell$, let $\rho_{ij\ell}$ be a defining function for the corresponding face.

The existence of the maps $\beta_{ij} : X_e^3 \to X_e^2$, $i, j = L, M, R$ must now be verified. By symmetry it suffices to check the case $i, j = L, R$. β_{LR} will be given as a composition of blow-down maps followed at the last step by a projection from $X_e^2 \times X$ to X_e^2. Since the blow-downs are performed in a different order than the blow-ups in the definition of X_e^3, this process is not completely trivial. The first step is to blow down the two faces B_{110} and B_{011}. The image is close to the product $X_e^2 \times X$, but has an additional boundary face corresponding to B_{111}. To proceed further we must appeal to the following result.

(3.13) Proposition. *Let $X \supset Y \supset Z$ be a nested sequence of closed manifolds without boundaries. Then there is a natural diffeomorphism (obtained as the unique continuous extension from the interior) between the manifolds with corners obtained by first blowing up Y and then blowing up the lift of Z or first blowing up Z and then blowing up Y:*

$$[X; Y; Z] \cong [X; Z; Y].$$

The proof is a straightforward verification in local coordinates (w, w', w'') on X in which $Y = \{w = 0\}$ and $Z = \{w = w' = 0\}$. We need this result when some number of each of the collections of coordinates w, w' or w'' live in half-lines; this is equally easy to prove. A general result of this type is proved in [Me2], see also [E-M-M] and [E-M].

Returning to our construction, after the first step described above the space is equal to $[X^3; T; S_{LR}]$, which by (3.13) is equivalent to $[X^3; S_{LR}; T]$. If we now reverse the last step of this iterated blow-up, i.e. blow down T, we obtain $[X^3; S_{LR}] = X_e^2 \times X$, where the edge double product is taken on the first and third (left and right) copies of X. Composing these blow-downs with the projection $X_e^2 \times X \to X_e^2$ yields the map β_{LR}. The two other maps β_{LM} and β_{MR} are defined similarly. Figure II is a schematic representation of this process. The following is easy to check from the definitions

(3.14) Lemma. *The maps β_{ij} are b-fibrations in the sense of Definition (A.14).*

In the interests of brevity we prove the composition formulæ for the large and small calculi simultaneously. The following notation is helpful. Recall that μ and ν are nonvanishing smooth half-densities on X^2 and X_e^2, respectively. They are related (up to multiplication by smooth nonvanishing factors, which we always ignore) by $\beta_{(2)}^* \mu = r^{\frac{k+1}{2}} \nu$, where as always r is a defining function for $B_{11}(X_e^2)$. Any distribution or density adorned with the pair of subscripts ij, $i, j = L, M, R$, denotes the lift to X_e^3 of some density on X_e^2 by the map β_{ij}; thus $\nu_{ij} = \beta_{ij}^* \nu$, $r_{ij} = \beta_{ij}^* r$, etc. We also let μ_3 and ν_3 be nonvanishing smooth half-densities on X^3 and X_e^3, respectively, and recall that $\beta_{(3)} : X_e^3 \to X^3$ is the blow-down.

(3.15) Theorem. *Let $A \in \Psi_e^{m,\mathcal{E}}$ and $B \in \Psi_e^{m',\mathcal{F}}$. Then provided $\Re(E_{01}) + \Re(F_{10}) > -1$, the composition $C = A \circ B$ is defined. If this is the case, then $C \in \Psi_e^{m+m',\mathcal{G}}$, where*

$$G_{10} = (E_{11} + F_{10}) \overline{\cup} E_{10}, \quad G_{01} = (E_{01} + F_{11}) \overline{\cup} F_{01},$$
$$G_{11} = (E_{11} + F_{11}) \overline{\cup} (E_{10} + F_{01} + k + 1).$$

Proof. Formula (3.11) will be used extensively. First multiply both sides of (3.11) by $r^{-\frac{k+1}{2}} \nu$ to obtain

$$\kappa_C \, r^{-k-1} \nu^2 = (\beta_{LR})_* \left\{ (\kappa_A)_{LM} (\kappa_B)_{MR} (r_{LM} \, r_{MR} \, r_{LR})^{-\frac{k+1}{2}} \nu_{LM} \, \nu_{MR} \, \nu_{LR} \right\}.$$

A simple computation shows that

$$\nu_{LM} \, \nu_{MR} \, \nu_{LR} = (r_{LM} \, r_{MR} \, r_{LR})^{-\frac{k+1}{2}} \beta_{(3)}^* (\mu_3^2).$$

Using this we rewrite the last formula as

$$\kappa_C\, r^{-k-1}\, \nu^2 = (\beta_{LR})_* \left\{ (\kappa_A)_{LM}\, (\kappa_B)_{MR}\, (r_{LM}\, r_{MR}\, r_{LR})^{-k-1} \beta^*_{(3)}(\mu_3^2) \right\}. \tag{3.16}$$

Now using the usual transformations of densities in polar coordinates we compute

$$\beta^*_{(3)}\mu_3^2 = (\rho_{110}\rho_{101}\rho_{011})^{k+1}\rho_{111}^{2k+2}\nu_3^2,$$

and also note how the defining functions r_{ij} on X_e^2 relate to the various $\rho_{ij\ell}$:

$$r_{LM} = \rho_{011}\rho_{111},\ r_{MR} = \rho_{110}\rho_{111},\ r_{LR} = \rho_{101}\rho_{111}.$$

Inserting these into (3.16) implies that

$$\kappa_C\, r^{-\frac{k+1}{2}}\, \nu^2 = (\beta_{LR})_* \left\{ (\kappa_A)_{LM}\, (\kappa_B)_{MR}\, \rho_{111}^{-k-1}\, \nu_3^2 \right\}.$$

This is almost in the form that we require. However, the pushforward theorem (A.18) requires the quantity to be pushed forward to be a section of the density bundle ${}^b\Omega$ of (A.17). A non-vanishing smooth density ν^2 on X_e^2 is related to a nonvanishing smooth section ${}^b\nu^2$ of ${}^b\Omega$ by $\nu^2 = r_{10}r_{01}r_{11}\,{}^b\nu_2$ (as always, up to multiplication by a smooth nonvanishing function), and similarly $\nu_3^2 = \rho_{100}\rho_{010}\rho_{001}\rho_{110}\rho_{101}\rho_{011}\rho_{111}\,{}^b\nu_3^2$. Using these in the formula above we finally arrive at

$$\kappa_C\,{}^b\nu^2 = (\beta_{LR})_* \left\{ (\kappa_A)_{LM}\, (\kappa_B)_{MR}\, \rho_{101}^{k+1}\, \rho_{010}\,{}^b\nu_3^2 \right\}. \tag{3.17}$$

In order to interpret the push-forward in (3.17) we must take into account not only the conormal singularities along all the boundaries of X_e^3 but also the structure of the interior singularities of the distribution to be pushed forward. Recalling the notation of §2A, $\kappa_A \in \mathcal{A}^{\mathcal{E}}_{phg}I^m(X_e^2,\Delta_e)$ and $\kappa_B \in \mathcal{A}^{\mathcal{F}}_{phg}I^{m'}(X_e^2,\Delta_e)$. (We neglect the density factors now since they have already been taken into account.) By (A.13) their pullbacks via β_{LM} and β_{MR}, respectively, are well-defined and conormal to $\beta_{LM}^{-1}(\Delta_e)$ and $\beta_{MR}^{-1}(\Delta_e)$. These partial diagonals meet b-transversally at the triple diagonal of X_e^3, and β_{LR} maps this submanifold onto the edge diagonal of X_e^2 on the left and right pair of factors. Hence by the discussion following (A.18) their product is well-defined by the standard wave-front set result, and by Proposition (A.18) their pushforward is an element of $\mathcal{A}^{\mathcal{G}}_{phg}I^{m+m'}(X_e^2,\Delta_e)$, where $\mathcal{G} = (\beta_{LR})_\flat((\beta_{LM})^\flat(\mathcal{E}) + (\beta_{MR})^\flat(\mathcal{F}) + \mathcal{H})$; here $\mathcal{H}$ is the collection of index sets for the faces of X_e^3 induced by the extra density factors in (3.17) and is given by $H_{101} = \{(k+1,0)\}$, $H_{010} = \{(1,0)\}$ and all other $H_{ij\ell} = \emptyset$. This collection $\mathcal{G}$ is exactly the one in the statement of the theorem.

Proved in a similar, but simpler, fashion is the fact that the space of 'very residual' operators (3.10) is a left and right module over $\Psi_e^{*,\mathcal{E}}$, provided the index sets are composable. We state the result for left multiplication and indicate the proof briefly.

(3.18) Theorem. *Let $A \in \Psi_e^{m,\mathcal{E}}(X)$ and $B \in \Psi^{-\infty,\mathcal{F}}(X)$, for collections of index sets $\mathcal{E}$ and $\mathcal{F}$. Then provided $\Re(E_{01}) + \Re(F_{10}) > -1$ the composition $C = A \circ B$ is defined and an element of $\Psi^{-\infty,\mathcal{G}}$, where $G_{10} = E_{10} \overline{\cup} (E_{11} + F_{10})$ and $G_{01} = F_{01}$.*

The operator A corresponds to the kernel $\kappa_A \, r^{-\frac{k+1}{2}} \nu$ on X_e^2 and B corresponds to $\kappa_B \, \mu$ on X^2. As in (3.11) we interpret the composition functorially:

$$\begin{aligned} \kappa_C \, \mu =& (\beta_{LR})_* \left\{ (\beta_{LM})^* (\kappa_A \, r^{-\frac{k+1}{2}} \nu) \, (\beta_{MR})^* (\kappa_B \, \mu) \right\} \\ =& (\beta_{LR})_* \left\{ (\kappa_A)_{LM} \, (\kappa_B)_{MR} \, \rho_{011}^{-\frac{k+1}{2}} \, \nu_{LM} \, \mu_{MR} \right\}. \end{aligned} \tag{3.19}$$

The notation is slightly different than before. The edge triple product is no longer required because the kernels of B and C live on the ordinary double product X^2. We require a 'smaller' space obtained by blowing up X^3 and for which there exist b-fibrations β_{LM} to X_e^2 and β_{MR} and β_{LR} to X^2. A good candidate is simply the product $X_e^2 \times X = [X^3; S_{LM}]$. $\beta_{LM} : X_e^2 \times X \to X_e^2$ is the obvious projection, and β_{MR} and β_{LR} are obtained by composing the blowdown $X_e^2 \times X \to X^3$ with the projection onto the product X^2 of the second and third or first and third copies of X. Pullbacks of distributions and densities by β_{ij} are once again labelled by the subscripts ij. Multiply both sides of (3.19) by μ to get

$$\kappa_C \, \mu^2 = (\beta_{LR})_* \left\{ (\kappa_A)_{LM} \, (\kappa_B)_{MR} \, (\rho_{011})^{-\frac{k+1}{2}} \, \nu_{LM} \, \mu_{MR} \, \mu_{LR} \right\}.$$

Noting the formula

$$\nu_{LM} \, \mu_{MR} \, \mu_{LR} = (\rho_{011})^{\frac{k+1}{2}} \nu_3^2,$$

where ν_3 is a nonvanishing smooth density on $X_e^2 \times X$, and rewriting everything in terms of nonvanishing smooth $\mathcal{V}_b$ densities we arrive at

$$\kappa_C \, {}^b\mu^2 = (\beta_{LR})_* \left\{ (\kappa_A)_{LM} \, (\kappa_B)_{MR} \, \rho_{010} \, {}^b\nu_3^2 \right\}.$$

Proceeding as before, using Proposition (A.18), we deduce the stated conclusion.

We conclude this section by proving boundedness of edge pseudodifferential operators on various spaces. The primary spaces we consider, since this is essentially an L^2-based theory, are the weighted Sobolev spaces $x^\delta H_e^s(X; \Omega^{\frac{1}{2}})$ based on $\mathcal{V}_e$ defined below. We also prove boundedness on weighted degenerate Hölder spaces as this requires no additional effort and is sometimes useful, cf. [M-S], and also indicate how the theory goes on weighted degenerate L^p spaces. However, since this last topic requires a substantial change of notation we do not spell out details too carefully. We also discuss the action of edge operators on polyhomogeneous conormal spaces and prove compactness and commutation results.

By definition, elements of the large calculus may be decomposed into elements of the small calculus, and residual elements of the large calculus, i.e. elements of $\Psi_e^{-\infty,\mathcal{E}}$, it suffices to prove boundedness individually for each of these. We treat the small calculus first.

(3.20) Proposition. *Fix real numbers δ and $m \le 0$. Then if $A \in \Psi_e^m(X)$,*

$$A : x^\delta L^2(X;\Omega^{\frac{1}{2}}) \to x^\delta L^2(X;\Omega^{\frac{1}{2}})$$

is bounded.

Proof. It suffices to show that $\|Au\|^2 \le \|u\|^2$ for $u \in \dot{\mathcal{C}}^\infty(X;\Omega^{\frac{1}{2}})$, the space of half-densities vanishing to all orders at ∂X, where $\|\;\|$ is the norm on $x^\delta L^2$. First conjugate A by x^δ. Because of the rapid vanishing of κ_A on B_{10} and B_{01}, the kernel corresponding to $x^\delta A x^{-\delta}$ also vanishes rapidly there, hence is another element of Ψ_e^m. Thus it suffices to prove boundedness of an operator A of order $m \le 0$ on L^2. We use Hörmander's symbolic method to reduce this to proving boundedness on L^2 of a residual operator $R \in \Psi_e^{-\infty}$. First use the symbol calculus to choose an operator $B \in \Psi_e^0$ such that for some constant $c > 0$, $A^*A + B^*B = cI + R$ (${}^e\sigma_0(B)$ is chosen as a square root of $c - |{}^e\sigma_m(a)|^2$ and so on). Apply this to u and integrate by parts (using that $u \in \dot{\mathcal{C}}^\infty$) to obtain

$$\|Au\|^2 \le \|Au\|^2 + \|Bu\|^2 = c\|u\|^2 + \langle Ru, u\rangle.$$

Thus we only need to show that $|\langle Ru, u\rangle| \le C\|u\|^2$, or equivalently, that $|\langle Ru, v\rangle| \le C\|u\|\|v\|$. Now if $u = \overline{u}\gamma$ and $v = \overline{v}\gamma$ where γ is a smooth nonvanishing half-density on X, then

$$\begin{aligned} |\langle R(\overline{u}\gamma), \overline{v}\gamma\rangle| &= \left|\int \kappa_R\, \overline{u}\, \overline{v}\, \nu^2\right| \qquad \text{since } r^{-\frac{k+1}{2}}\gamma_L\,\gamma_R\,\nu = \nu^2 \\ &\le \left|\int |\kappa_R|\, |\overline{u}|^2 \nu^2\right| \; \left|\int |\kappa_R|\, |\overline{v}|^2 \nu^2\right| \\ &\le C\|u\|\,\|v\|. \end{aligned}$$

For the last inequality we use the (singular) coordinate systems (x, y, z, θ) or $(\tilde{x}, \tilde{y}, \tilde{z}, \theta)$ on X_e^2 to rewrite ν^2 as

$$\nu^2 = \beta_L^*(\gamma^2)\, d\theta/\rho_{10} = \beta_R^*(\gamma^2)\, d\theta/\rho_{01}, \tag{3.21}$$

($\beta_L,\ \beta_R : X_e^2 \to X$ are blow-downs followed by projection onto the left and right copies of X, and θ is the angular coordinate from (2.7)). Then apply Fubini's theorem and use the rapid vanishing of κ_R on the side boundary faces to ensure convergence of the integrals with respect to θ. This completes the proof.

When $\ell \in \mathbf{N}_0$ let

$$x^\delta H_e^\ell(X;\Omega^{\frac{1}{2}}) = \{u = x^\delta v; V_1 \cdots V_j v \in L^2(X;\Omega^{\frac{1}{2}}),\ \ \forall\, j \le \ell,\ V_i \in \mathcal{V}_e\}. \tag{3.22}$$

The definition when ℓ is any real number can be obtained by interpolation and duality.

There is another argument to prove (3.20) which generalizes easily to prove the full boundedness results for the spaces (3.22). These spaces all have an approximate dilation invariance in directions normal to the fibres of ∂X. Choose a Whitney decomposition of X into a union of 'boxes' B_i each having fixed length in the z directions and diameter in x and y directions comparable to its distance to ∂X, e.g. to the value x at any point in that box. For each B_i choose an affine map f_i (in any local coordinate system) which carries a 'standard box' B to B_i. Let $u \in x^\delta H_e^\ell$ and u_i its restriction to B_i. Then the $x^\delta H_e^\ell$ norm of u is comparable to $\sum_i x_i^{-\delta}\|u_i\|_{B_i}$ where x_i is the value of x at any point of B_i and $\|\ \|_{B_i}$ is the H_e^ℓ norm on each B_i. This norm is chosen so that $\|v_i\|_{B_i} \cong \|f_i^* v_i\|_B$. Note that by the duality and interpolation properties of ordinary L^2 Sobolev spaces, this is true for any real ℓ. The boundedness of $A \in \Psi_e^m$ for $m \in \mathbf{R}$ is proved by computing the $H_e^{\ell-m}$ norm of Au in this fashion and noting that $f_i^*(Av)_i = A_i(f_i^* v_i)$, where A_i is the pullback of A from B_i to B using f_i. By the approximate dilation invariance of A the A_i are a uniformly bounded family of pseudodifferential operators of order m. The standard boundedness properties of ordinary pseudodifferential operators can then be applied to estimate each term by the H_e^ℓ norm of $f_i^* v_i$ on a slightly larger box, or equivalently of v_i on a slight enlargement of B_i. Summing these inequalities, using the uniformity of the A_i, yields the following:

(3.23) Corollary. *If $A \in \Psi_e^m$ then*

$$A : x^\delta H_e^\ell(X;\Omega^{\frac{1}{2}}) \longrightarrow x^\delta H_e^{\ell-m}(X;\Omega^{\frac{1}{2}})$$

is bounded.

Conversely we could also define $x^\delta H_e^m$ as the set of all Au for $A \in \Psi_e^m$ and $u \in x^\delta L^2$. The previous argument shows the consistency of this definition.

These arguments using dilations in such a direct manner were used in [G-L] and [A], and are the simplest approach when proving L^p and Hölder boundedness.

Before proceeding further we note the following simple consequences of Corollary (3.23) and Theorem (3.8):

(3.24) Corollary. *Suppose $A \in \Psi_e^m$ is elliptic, and $u \in x^\delta H_e^\ell$ for some δ, ℓ. If $Au = f \in \dot{\mathcal{C}}^\infty(X)$ then $u \in x^\delta H_e^\infty = x^\delta \bigcap_\ell H_e^\ell$.*

Proof. Use Theorem (3.8) to construct an element $B \in \Psi_e^{-m}$ such that $BA = I - R$ for some $R \in \Psi_e^{-\infty}$. Applying this to u yields $u = Ru + Bf$. (3.20) and (3.22) imply that Bf and Ru each lie in $x^\delta H_e^\infty$.

Boundedness for elements of the large calculus is slightly more subtle, but we have already done most of the work.

(3.25) Theorem. *Let $A \in \Psi_e^{-\infty,\mathcal{E}}$ for some collection of index sets $\mathcal{E}$. Then $A : x^\delta H_e^s(X;\Omega^{\frac{1}{2}}) \to x^{\delta'} H_e^\ell(X;\Omega^{\frac{1}{2}})$ is bounded for any $s, \ell \in \mathbf{R}$ provided $\Re(E_{10}) > \delta' - \frac{1}{2}$, $\Re(E_{01}) + \delta > -\frac{1}{2}$, and $\delta + \Re(E_{11}) \geq \delta'$.*

Proof. It suffices, either by scaling arguments or commuting $\mathcal{V}_e$ vector fields through A, to prove boundedness only when $s = \ell = 0$. Furthermore, by replacing A by $x^{\delta'} A \tilde{x}^{-\delta}$

we can also reduce to the case $\delta = \delta' = 0$. Proceeding as in the proof of (3.20), we shall demonstrate the estimate

$$|\langle Au, v\rangle| \le C\|u\|\|v\|$$

for L^2 half-densities $u = \overline{u}\gamma$, $v = \overline{v}\gamma$. Lift the integration in this inequality to an integration over X_e^2 and combine $\gamma_L\,\gamma_R$ with the singular density factor $\rho_{11}^{-\frac{k+1}{2}}$ from A to obtain ν^2 as before. The left hand side becomes

$$\left|\int \kappa_A\,\overline{u}_L\,\overline{v}_R\,\nu^2\right|.$$

Use (3.21) again and apply the Cauchy-Schwarz inequality and Fubini's theorem successively to get

$$\begin{aligned}\int |\kappa_A|\,|\overline{u}_L|\,|\overline{v}_R|\,\nu^2 &\le \left\{\int |\kappa_A|\,|\overline{u}_L|^2(\rho_{10}/\rho_{01})^{\frac{1}{2}}\nu^2\right\}^{\frac{1}{2}}\left\{\int |\kappa_A|\,|\overline{v}_R|^2(\rho_{01}/\rho_{10})^{\frac{1}{2}}\nu^2\right\}^{\frac{1}{2}}\\ &\le \left\{\int |\kappa_A|\,|\overline{u}|^2(\rho_{10}\rho_{01})^{-\frac{1}{2}}\gamma_L d\theta\right\}^{\frac{1}{2}}\left\{\int |\kappa_A|\,|\overline{v}|^2(\rho_{10}\rho_{01})^{-\frac{1}{2}}\gamma_R d\theta\right\}^{\frac{1}{2}}\\ &\le C\|u\|\,\|v\|.\end{aligned}$$

The last inequality holds because the integral with respect to $d\theta$ is convergent because of the hypotheses on $\mathcal{E}$. This completes the proof.

The boundedness on Hölder spaces is even simpler. First we define the appropriate spaces.

(3.26) Definition. *For $\ell \in \mathbf{N}_0$, $\alpha \in (0,1)$, and $\delta \in \mathbf{R}$, let*

$$x^\delta\Lambda^{\ell,\alpha}(X;\Omega^{\frac{1}{2}}) = \{u = x^\delta\,v\,\gamma : V_1\dots V_j v \in \Lambda^{0,\alpha},\ \forall\ V_i \in \mathcal{V}_e,\ j \le k\}$$

where $\Lambda^{0,\alpha,0}(X)$ is the space of half-densities $u = v\gamma$ such that

$$\|v\|_{0,\alpha,0} \equiv \sup|v| + \sup\ \frac{(x+\tilde{x})^\alpha\,|v(x,y,z) - v(\tilde{x},\tilde{y},\tilde{z})|}{|x-\tilde{x}|^\alpha + |y-\tilde{y}|^\alpha + (x+\tilde{x})^\alpha|z-\tilde{z}|^\alpha} < \infty.$$

(3.27) Proposition. *Let $A \in \Psi_e^{m,\mathcal{E}}$. Suppose $\ell \ge m$ are integers and the index sets satisfy $\Re(E_{01}) + \delta > -1$ and $\Re(E_{10}) \ge \delta$, $\Re(E_{11}) \ge 0$, with strict inequality in at least one of the last two conditions and $p = 0$ in any term (s,p) in either of the index sets realizing equality. Then $A : x^\delta\Lambda^{\ell,\alpha} \to x^\delta\Lambda^{\ell-m,\alpha}$ is bounded.*

Proof. As before, divide $A = B + C$, where $B \in \Psi_e^m$, $C \in \Psi_e^{-\infty,\mathcal{E}}$; we need only check that B and C individually are bounded.

The boundedness of B may be checked using the scaling method outlined above, and using that ordinary pseudodifferential operators are bounded on ordinary Hölder spaces, cf. [M-S].

Since C is residual, Cu should lie in $x^\delta \Lambda^{s,\alpha}$ for any s. Write $u = v\gamma$ and $Cu = f\gamma$. By applying arbitrarily many $\mathcal{V}_e$ vector fields to Cu, all of which land on the kernel of C and do not change its character nor its index sets we see that it suffices to prove that $|f| \le c_2 x^\delta$ assuming that $|v| \le c_1 x^\delta$. Represent Cu by the usual pushforward formula. After the usual reductions we arrive at the relationship

$$f\, {}^b\gamma^2 = (\beta_L)_*(\kappa_C\, \beta_R^* v\, \rho_{01}\, {}^b\nu^2),$$

where as usual ${}^b\gamma^2$ and ${}^b\nu^2$ are nonvanishing smooth b-densities. Take absolute values of both sides of this equation, and pass the absolute values into the integration on the right, changing the equality into an inequality. By assumption we can estimate $|\beta_R^* v|$ by $c_1 \tilde{x}^\delta = c_1(\rho_{11}\, \rho_{01})^\delta$. Furthermore, although $|\kappa_C|$ itself may no longer be conormal, it is estimated by a polyhomogeneous conormal kernel $K \in \mathcal{A}^{\mathcal{F}}_{phg}(X_e^2)$, where $F_{ij} = \{(s,p) :\ s = \Re(\sigma),\ (\sigma, p) \in E_{ij}\}$. Therefore

$$|f| \le c(\beta_L)_*(K\, (\rho_{11})^\delta\, (\rho_{01})^\delta\, \rho_{01}\, {}^b\nu^2).$$

Now apply (A.18). The hypothesis $\Re(E_{01}) + \delta > -1$ is necessary in order that the pushforward be defined. The right hand side is a polyhomogeneous distribution $g \in \mathcal{A}^{\mathcal{G}}_{phg}(X)$ with $\mathcal{G} = \Re(E_{10})\, \overline{\cup}\, (\Re(E_{11}) + \delta)$. The hypotheses imply that the leading term in the asymptotic expansion for g is $a(y,z)x^s(\log x)^p$ where $s \ge \delta$ and $p = 0$ if $s = \delta$. Thus $|g|$ can be estimated by $c_3 x^\delta$ as required.

We now make a few remarks on the general L^p situation. Clearly the right spaces to use are degenerate weighted L^p spaces, $1 < p < \infty$, analogous to (3.22) which behave nicely under dilations in the (x,y) directions. We also must use sections of the density bundle $\Omega^{\frac{1}{p}}(X)$ rather than half-densities. This in turn forces a different normalization of the singular density bundle on X_e^2 of which edge pseudodifferential operators are sections. These modifications are of course trivial. With these definitions, boundedness of the small calculus on weighted L^p spaces is proved using the dilation method sketched above, and using the fact that ordinary pseudodifferential operators are bounded on ordinary L^p spaces. Boundedness of the large calculus is proved exactly as in (3.25), the only difference being that the Cauchy-Schwarz inequality is applied slightly differently. We do not develop these ideas further because this extension is straightforward but would take more space than seems warranted because of the differences in normalization.

We conclude this section with a few auxiliary results of a general nature.

(3.28) Proposition. *Suppose* $u \in \mathcal{A}^F_{phg}(X; \Omega^{\frac{1}{2}})$ *and let* $A \in \Psi_e^{m,\mathcal{E}}$. *Then provided* $\Re(E_{01}) + \Re(F) > -1$, *we have* $Au \in \mathcal{A}^G_{phg}(X; \Omega^{\frac{1}{2}})$ *where* $G = E_{10}\, \overline{\cup}\, F$.

Proof. Represent Au as a pushforward. Then the product $\kappa_A\, (\beta_R)^*(u\,\gamma)\, r^{-\frac{k+1}{2}}\, \nu$ is an element of $\mathcal{A}^*_{phg} I^m(X_e^2, \Delta_e; r^{-\frac{k+1}{2}}\nu)$. By the discussion following (A.18) the pushforward annihilates the interior singularity (if you like since Y_1'' is empty here), and the result is polyhomogeneous. The induced index sets is computed as $(\beta_L)_\flat((\beta_R)^\flat(F)+\mathcal{E})$, i.e. $E_{10}\overline{\cup}F$.

The following is an application of Theorems (3.20), (3.25) and the Theorem of Arzela-Ascoli.

(3.29) Proposition. *Suppose $A \in \Psi_e^{m,\mathcal{E}}$ where $m < 0$, $\Re(E_{10}) > -\frac{1}{2}$, $\Re(E_{01}) > -\frac{1}{2}$, and $\Re(E_{11}) > 0$. Then A is compact as a mapping on $x^\delta H_e^\ell$ and $x^\delta \Lambda^{\ell,\alpha}$.*

Finally we establish a commutation result. The symbol calculus implies that if $V \in \mathcal{V}_e$ and $A \in \Psi_e^{m,\mathcal{E}}$ then their commutator $[V, A]$ is in $\Psi_e^{m,\mathcal{E}}$ again, but this is also true when V is only totally characteristic.

(3.30) Proposition. *For $V \in \mathcal{V}_b$ and $A \in \Psi_e^{m,\mathcal{E}}$, we have $[V, A] \in \Psi_e^{m,\mathcal{E}}$.*

Proof. Let $u \in \dot{\mathcal{C}}^\infty(X)$, and compute $(VA - AV)u$ as follows

$$\begin{aligned}(VA - AV)u &= (\beta_L)_* \left\{\beta_L^*(V)(\kappa_A\, r^{-\frac{k+1}{2}}\,\nu)\,\beta_R^* u - \kappa_A\, r^{-\frac{k+1}{2}}\,\nu \beta_R^*(Vu)\right\}\\ &= (\beta_L)_* \left\{(\beta_L^*(V) + \beta_R^*(V^t))\,(\kappa_A\, r^{-\frac{k+1}{2}}\,\nu)\,\beta_R^* u\right\}.\end{aligned}$$

The last equality is 'integration by parts', and V^t is the adjoint of V with respect to the given measure. Now the result follows by observing that for any $V \in \mathcal{V}_b$, $\beta_L^* V + \beta_R^* V^t$ is the sum of a vector field tangent to all boundaries of X_e^2 and vanishing along Δ_e and a smooth function. It transforms the distributional half-density to $\kappa_B\, r^{-\frac{k+1}{2}}\,\nu$ where B has the same index sets as A. This is obviously true when $V \in \mathcal{V}_e$ by (2.12), and the only remaining case is when, in local coordinates, V contains some nonvanishing multiple of the ∂_y's. The assertion is easy to check in this case, for examply by computing in polar coordinates and noting that these polar coordinates are always functions of $y - \tilde{y}$.

The point of this Proposition is that ∂_y lifts as a singular vector field to X_e^2: it is tangent to the side boundary faces but blows up to order one on the front face. However, the difference in the commutator cancels this singularity. This result can be used in establishing higher tangential regularity of solutions of equations like $Lu = f$ when $L \in \mathrm{Diff}_e^*(X)$ is elliptic and f has some tangential regularity, cf. [M3] where it is used in establishing regularity for a nonlinear equation of edge type.

§4. Totally Characteristic Operators

The two extremal types of edge structures exist on any manifold with boundary and correspond to the cases when either the base space Y or the fibres F of the fibration of ∂X are points. Then $\mathcal{V}_e$ consists of all vector fields either tangent to ∂X or vanishing on ∂X, and is denoted $\mathcal{V}_b$ or $\mathcal{V}_o$. These are the space of b- (or totally characteristic) vector fields or uniformly degenerate vector fields, respectively. $\mathcal{V}_b$ was the first 'boundary fibration structure' (in the sense of [Me2]) to be systematically explored [Me1], [Me-Me]; $\mathcal{V}_o$ was discussed later in [M1]. Of the two, the calculus associated to $\mathcal{V}_b$ is the more elementary.

Use coordinates (x,z) as in §2 (with y omitted). Then

$$x\partial_x, \partial_{z^1}, \cdots, \partial_{z^q} \tag{4.1}$$

generate $\mathcal{V}_b$ over $\mathcal{C}^\infty(X)$. From $\mathcal{V}_b$ the rings $\mathrm{Diff}_b^*(X)$ and $\Psi_b^*(X)$ of b-differential and b-pseudodifferential operators on X are defined, exactly as in §2 and §3. We replace all e's by b's to indicate objects associated to $\mathcal{V}_b$.

In this section we construct parametrices for elliptic elements of $\mathrm{Diff}_b^*(X)$. This a relatively straightforward introduction to the constructions later on, and furthermore, parametrices for b-operators are used in the construction for more general edge operators later on. We also discuss mapping properties of elliptic b-operators, at least when X is compact.

The set $\mathrm{spec}_b(L)$ is defined in (2.21). Now it is automatically discrete, so (2.22) is no longer necessary. Thus for each $\ell, \delta \in \mathbf{R}$ consider the mapping

$$L : x^\delta H_b^{\ell+m}(X, \Omega^{\frac{1}{2}}X) \to x^\delta H_b^\ell(X, \Omega^{\frac{1}{2}}X) \tag{4.2}$$

where $L \in \mathrm{Diff}_b^m(X)$ is elliptic, and as usual the function u is identified, when necessary, with the half-density $u\sqrt{dx\,dz}$. The Sobolev spaces in (4.2) are defined as in (3.22), but with respect to the $\mathcal{V}_b$. Set

$$\Lambda = \{\Re(\zeta) + 1/2 : \zeta \in \mathrm{spec}_b(L)\}. \tag{4.3}$$

The main result of this section is

(4.4) Theorem. *For $\delta \notin \Lambda$ the mapping (4.2) is Fredholm.*

Theorem (4.4) is proved by constructing parametrices for L which are bounded between suitable spaces, and which are inverses up to compact error. Note that in contrast to results below for the general edge case, the kernel and cokernel here are finite dimensional when the operator has closed range. These spaces can be infinite dimensional only when $L^2(Y)$ is infinite dimensional, which is clearly not the case here.

To fix notation let

$$L \;=\; \sum_{j+|\beta|\le m} a_{j,\beta}(x,z)(x\partial_x)^j\partial_z^\beta \tag{4.5}$$

be the elliptic operator in question. Notice that its normal operator and indicial operator coincide, and both may be expressed as in (2.20), but of course with the parameter y absent.

The first approximation to a good parametrix for L uses the small calculus (3.3), and is the operator obtained in Theorem (3.8). Recall how this operator, which we call A_0, is obtained. The lift of L through the left to X_b^2 is transversally elliptic to the lifted diagonal Δ_b. Thus, factoring out the singular half-densities, we may choose an element $A_0 \in \Psi_b^{-m}(X)$ such that

$$LA_0 \equiv I \mod \Psi_b^{-\infty}(X). \tag{4.6}$$

For convenience we can assume that κ_{A_0} is supported in a small neighbourhood of Δ_b, i.e. in some set $C^{-1} \le s \le C$. Writing (4.6) more explicitly,

$$LA_0 = I - R_0, \tag{4.7}$$

where $R_0 \in \Psi_b^{-\infty}(X)$. By Corollary (3.23) A_0 is bounded between $x^\delta H_b^{\ell-m}$ and $x^\delta H_b^\ell$ for every ℓ and δ, while R_0 is bounded between $x^\delta H_b^\ell$ and $x^\delta H_b^s$ for every ℓ, s and δ.

R_0 is not a compact operator since it does not vanish on the front face, so we must proceed further to prove (4.4). In fact we have used nothing about the value of the weight parameter δ. Thus we require a second step which does. In this step we obtain an operator $A_1 \in \Psi_b^{-\infty,*}$ such that

$$L(A_0 + A_1) = I - R_1 \tag{4.8}$$

where $R_1 \in \Psi_b^{-\infty,\mathcal{F}}$ for some collection of index sets $\mathcal{F}$ with $F_{11} = 1$, i.e. so that R_1 vanishes to first order at the front face of X_b^2. This is accomplished by solving the equation

$$(LA_1)\big|_{B_{11}} \equiv I(L)(A_1\big|_{B_{11}}) = R_0\big|_{B_{11}}, \tag{4.9}$$

recalling that $N(L) = I(L)$. All terms in (4.9) are boundary values of the bundle $r^{-\frac{1}{2}}\Omega^{\frac{1}{2}}(X_b^2)$.

(4.9) has many solutions, which arise in the following way. Since $I(L)$ is $\mathbf{R}^+$ invariant it is natural to tranform (4.9) via the Mellin transform to

$$I_\zeta(L)(A_1\big|_{B_{11}})_M = (R_0\big|_{B_{11}})_M. \tag{4.10}$$

Since $R_0\big|_{B_{11}}$ is compactly supported, the right side of (4.10) is entire and rapidly decreasing along each line $\Im(\zeta) = C$. Now, it is proved in [Me-Me], and is essentially equivalent to standard properties of the resolvent of the elliptic operator $I_0(L)$ on the compact space ∂X, that $I_\zeta(L)$ is invertible whenever $|\Re(\zeta)| \gg 0$ along each line $\Im(\zeta) = C$. By analytic Fredholm theory $I_\zeta(L)^{-1}$ exists and is meromorphic on the complement of a discrete set, which is simply $\operatorname{spec}_b(L)$. Thus we may apply $I_\zeta(L)^{-1}$ to both sides of (4.10) to obtain an expression for the Mellin transform of the restriction of A_1 to the front face B_{11} as a meromorphic function on $\mathbf{C}$. The indeterminacy arises in how we take the inverse Mellin transform. Following the discussion in §2A, this inversion is obtained by integrating $x^{-i\zeta}I_\zeta(L)^{-1}(R_0\big|_{B_{11}})_M$ along a line $\{\Im\zeta = C\}$ containing no points of $\operatorname{spec}_b(L)$. The value of C determines which points of $\operatorname{spec}_b(L)$ contribute to terms in the expansion of the solution of the equation at $s = 0$ and which contribute to terms in the expansion at $s = \infty$, and by the Cauchy integral formula the solution obtained is locally constant in $C \in \mathbf{R}\backslash\Lambda$. All solutions of (4.9) which lie in some space $x^\delta L^2$ are obtained in this way by varying the interval of $\mathbf{R} - \Lambda$ in which C lies.

Now fix $\delta \in \mathbf{R} - \Lambda$ and take inverse Mellin transform along $\Im(\zeta) = \delta - \frac{1}{2}$. Note that in terms of the identification of $\mathbf{R}^+ \times \partial X$ with B_{11}, $s = 0$ corresponds to $B_{10} \cap B_{11}$ and $s = \infty$ to $B_{01} \cap B_{11}$. The solution to (4.9) is polyhomogeneous conormal at these boundaries because $I_\zeta(L)^{-1}(R_0|_{B_{11}})_M$ is meromorphic in ζ and depends smoothly on z. The solution has index sets E_{10} and E_{01} at these boundaries, where

$$
\begin{aligned}
E_{10} &= \{(\zeta, p) : (\zeta, p) \in \widetilde{\operatorname{spec}}_b(L),\ \Im\zeta > C\} \\
E_{01} &= \{(-\zeta, p) : (\zeta, p) \in \widetilde{\operatorname{spec}}_b(L),\ \Im\zeta < C\}.
\end{aligned}
\tag{4.11}
$$

The change in sign in E_{01} arises because the index set describes polyhomogeneity as $1/s \to 0$ rather than $s \to \infty$. Now extend this solution smoothly off the front face. The resulting operator A_1 is an element of $\Psi_b^{-\infty,\mathcal{E}}$, where the E_{10} and E_{01} are as in (4.11) and $E_{11} = 0$. Furthermore, the remainder term R_1 in (4.8) is an element of $\Psi_b^{-\infty,\mathcal{E}(1)}$, where we introduce the notation that for any triplet of index sets $\mathcal{F}$

$$
\mathcal{F}(k) = (F_{10}, F_{01}, F_{11} + k) \tag{4.12}
$$

for $k \in \mathbf{N}$. The 1 in the third slot of the collection $\mathcal{E}(1)$ of index sets for R_1 is by construction, while the other two index sets remain the same because L differentiates tangentially to all boundaries on X_b^2.

We have now constructed a right parametrix A_1 for L which is bounded and is a right inverse up to an error R_1 which vanishes simply on B_{11}. A duality argument which we describe in detail below gives a left parametrix and a similar error term in the same way. The existence of these, and in particular the compactness by (3.29) of these errors implies straight away that (4.2) is indeed Fredholm. The precise regularity theory for solutions of $Lu = 0$, i.e. that solutions are polyhomogeneous, may also be deduced quite easily. However, this regularity requires iteration arguments because the error term vanishes only to first order on the front face, and it is perhaps neater to carry these arguments out on the parametrix itself. Namely, we find parametrices with error terms vanishing to successively higher orders on the front face, and the final parametrix is an asymptotic limit of these approximations. This is accomplished by composing with the Neumann series for $I - R_1$. However, in order for this argument to work, and in particular in order that this series be asymptotically summable, it is first necessary to modify $A_0 + A_1$ again by adding on a term A_2 to make the error term $R_2 = I - L(A_0 + A_1 + A_2)$ vanish to infinite order on the left boundary. Thus we choose A_2 so that the polyhomogeneous expansion of LA_2 on B_{10} matches that of R_1. This may be done using the boundary symbol σ_{10} at the face B_{10}, see (A.9), to solve away all terms in the expansion for R_1 at this face. Assume also that the kernel of A_2 is supported near B_{10}. Since all coefficients in the expansion for R_1 vanish at B_{11} those for the kernel for A_2 vanish there too. As a result the operator

$$
A = A_0 + A_1 + A_2 \tag{4.13}
$$

solves

$$
LA = I - R \tag{4.14}
$$

where $A \in \Psi_b^{-m,\mathcal{E}}$ and $R \in \Psi_b^{-\infty,\emptyset,E_{01},1}$, with $\mathcal{E}$ as in (4.11) (though possibly modified by increasing some of the logarithmic exponents p because of A_2).

Now form the Neumann series for $(I-R)^{-1}$. Theorem (3.15) implies that for each j, $R^j \in \Psi_b^{-\infty,\emptyset,F_j,j}$, with the F_j defined inductively by $F_j = E_{01} \overline{\cup} (F_{j-1}+1)$, $F_0 = \emptyset$, so that by Borel's lemma there exists a $P \in \Psi_b^{-\infty,\emptyset,F,1}$ with F the union of all the F_j and such that

$$I - (I-R)(I+P) \in \Psi_b^{-\infty,\emptyset,F,\emptyset} = \mathcal{A}_{phg}^{\emptyset,F}(X^2;\Omega^{\frac{1}{2}}).$$

(3.15) again implies that

$$\overline{A} = A(I+P) \in \Psi_b^{-m,E_{10},G_{01},G_{11}}$$

where

$$G_{01} = E_{01} \cup (F \,\overline{\cup}\, (E_{01}+1)), \quad G_{11} = 0 \cup (1 \,\overline{\cup}\, (E_{10}+F+1)).$$

The exact form of the index sets G_{ij} is of no concern except that $G_{01} \geq E_{01}$ and $G_{11} \geq 0$. Using this and the definition of E_{10} and E_{01} in (4.11) we see that $\Re(E_{10}) > \delta - \frac{1}{2}$ and $\Re(E_{01}) > -\delta + \frac{1}{2} > -\delta - \frac{1}{2}$ so by Proposition (3.23)

$$\begin{aligned} \overline{A} &: x^\delta H_b^\ell \to x^\delta H_b^{\ell+m} \\ \overline{R} &: x^\delta H_b^\ell \to \dot{\mathcal{C}}^\infty(X) \end{aligned} \tag{4.15}$$

are bounded for every $\ell \in \mathbf{R}$. It is also important that the error term is 'very residual' as in (3.10). The existence of a parametrix in the large calculus with very residual error term will imply that the actual generalized inverse for L is in the large calculus, with index sets determined by the equation (hence many of the terms in the index sets above are extraneous).

As noted earlier, the same sort of argument yields by duality a left parametrix as we now make explicit. Let L^t denote the adjoint of L with respect to the measure $dx\,dz$ and $L^* = x^{2\delta} L^t x^{-2\delta}$ the adjoint in $x^\delta L^2$, i.e. with respect to the measure $x^{-2\delta}\,dx\,dz$; these are also elliptic in $\mathrm{Diff}_b^m(X)$ and have boundary spectra

$$\begin{aligned} \mathrm{spec}_b(L^t) &= \{-\zeta - 1 : \zeta \in \mathrm{spec}_b(L)\} \\ \mathrm{spec}_b(L^*) &= \{-\zeta - 2\delta - 1 : \zeta \in \mathrm{spec}_b(L)\}. \end{aligned} \tag{4.16}$$

The dual of $x^\delta L^2(dx\,dz)$ is $x^{-\delta} L^2(dx\,dz)$, so perform the entire construction above on L^t with respect to $x^{-\delta}L^2$. This yields operators B, Q which satisfy

$$L^t B = I - Q, \quad B \in \Psi_b^{-m,\mathcal{H}}, \; Q \in \Psi^{-\infty,\emptyset,H_{01}},$$

for some index sets $\mathcal{H}$ corresponding to L^t in the same way as those earlier corresponded to L. B and Q are bounded as in (4.15) with δ replaced by $-\delta$. Thus duality with respect to the natural pairing of $x^\delta L^2$ with $x^{-\delta}L^2$ corresponds to the passage from L to L^t. This means that

$$B^t L = I - Q^t \tag{4.17}$$

as operators on the spaces $x^\delta H_b^\ell$ for any ℓ; here B^t and Q^t are those b-operators obtained from B and Q using the natural involution of X_b^2 which is the lift of interchanging factors on X^2. The relationship (4.17) is obtained by duality, so it is only valid for functions for which the pairing and integration by parts make sense. B^t enjoys the same boundedness as $\overline{A}$ in (4.15) and

$$Q^t : x^\delta H_b^\ell \to \mathcal{A}_{phg}^{H_{01}}. \tag{4.18}$$

An immediate consequence is

(4.19) Corollary. *If $u \in x^\delta L^2$ solves $Lu \in \mathcal{A}^*_{phg}$ then $u \in \mathcal{A}^*_{phg}$.*

Once u is known to be polyhomogeneous, the equation $Lu = 0$ implies by a completely formal argument that $u \in \mathcal{A}_{phg}^{E_{10}}$ with E_{10} as in (4.11) (except for a possible increase in some of the exponents p).

All parametrices thus far have been local near ∂X, but gluing these boundary parametrices to interior parametrices leads to the main result of this section.

(4.20) Theorem. *For $\delta \notin \Lambda$ the map (4.2) is Fredholm. Therefore, if P_1 and P_2 are the orthogonal projector in $x^\delta L^2$ onto the kernel and orthogonal complement of the range of L, then there exists a generalized inverse G, bounded between the appropriate spaces, satisfying*

$$GL = I - P_1, \quad LG = I - P_2 \tag{4.21}$$

as operators on $x^\delta H_b^\ell$ for any $\ell \in \mathbf{R}$. $P_1 \in \Psi^{-\infty,\mathcal{E}}$, $P_2 \in \Psi^{-\infty,\mathcal{F}}$ and $G \in \Psi_b^{-m,\mathcal{H}} + \Psi^{-\infty,\mathcal{H}'}$, where

$$\begin{aligned}
&E_{10} = \{(s,p) : (s,p) \in \widetilde{spec}_b(L),\ \Re s > \delta - \frac{1}{2}\},\\
&E_{01} = \{(s,p) : (s + 2\delta, p) \in E_{10}\},\\
&F_{10} = \{(s,p) : (s - 2\delta, p) \in F_{01}\},\\
&F_{01} = \{(s,p) : (s,p) \in \widetilde{spec}_b(L^*),\ \Re s > -\delta - \frac{1}{2}\},\\
&E_{11} = F_{11} = 0,\\
&H_{10} = E_{10} \,\overline{\cup}\, F_{10}, \quad H_{01} = E_{01} \,\overline{\cup}\, F_{01}, \quad H_{11} = 0
\end{aligned} \tag{4.22}$$

and $\mathcal{H}' = (H_{10}, H_{01})$. Therefore

$$\begin{aligned}
&G : x^\delta H_b^\ell \to x^\delta H_b^{\ell+m},\\
&P_i : x^\delta H_b^\ell \to x^\delta H_b^s
\end{aligned} \tag{4.23}$$

are bounded for every $\ell, s \in \mathbf{R}$.

Proof. The maps (4.2) are Fredholm because of the existence of the right and left parametrices $\overline{A}$ and B and the compactness of the remainder terms on the appropriate spaces. This guarantees on abstract grounds the existence of a generalized inverse G satisfying (4.23).

What remains is to show that G and the P_i lie in the edge calculus, and satisfy (4.22). Since we know that L is Fredholm, its kernel and cokernel are finite dimensional, and so both P_1 and P_2 are finite rank projectors. Thus they may be written as a finite sum of products in each factor of basis elements of the kernel of L or L^*. Elements of either of these nullspaces are polyhomogeneous by (4.19). Thus $P_1 \in \Psi^{-\infty,\mathcal{E}}$ and $P_2 \in \Psi^{-\infty,\mathcal{F}}$ for some index sets $\mathcal{E}, \mathcal{F}$. E_{10} and F_{01} correspond to subsets of $\widetilde{\mathrm{spec}}_b(L)$ and $\widetilde{\mathrm{spec}}_b(L^*)$, respectively, because of the equations $LP_1 = 0$, $L^*P_2 = 0$. The relationships $E_{01} = E_{10} - 2\delta$, $F_{10} = F_{01} + 2\delta$ follow from self-adjointness: $P_i = P_i^*$. Finally, the the boundedness criterion in (3.25) determines the precise form of E_{10} and F_{01}.

To establish that G is in the edge calculus we use (4.21) in conjunction with the existence of left and right parametrices, which for simplicity we now write as $A_1 L = I - R_1$, $LA_2 = I - R_2$. Computing the compositions $A_1 LG$ and GLA_2 in two different ways results in the equalities

$$G = A_1 + R_1 G - A_1 P_2 = A_2 + GR_2 - P_1 A_2. \tag{4.24}$$

Inserting the second expression for G in the first implies that

$$G = A_1 + R_1 G R_2 + R_1 A_2 - R_1 P_1 A_2 - A_1 P_2. \tag{4.25}$$

Since the P_i are already established to be very residual in the edge calculus, all but the first term in this expression are very residual and the first is an element of the large calculus. Note the important role here of the fact (3.18) that the very residual calculus is an ideal in the large calculus, and that $R_1 K R_2$ is very residual whenever R_1 and R_2 are very residual and K is any bounded operator. Finally, now that we know G is the sum of a large b-pseudodifferential operator and a very residual element, the index set of the first summand at $B_{11}(X_b^2)$ must be $(0,0)$ and the index sets at the other boundaries and for the very residual summand is determined by the equations (4.21) and Proposition (3.28).

Since G and the P_i are bounded on the weighted degenerate Hölder spaces (3.26) we also obtain

(4.26) Theorem. *For L as above, $\delta \notin \{\Re(\zeta) : \zeta \in spec_b(L)\}$, $\ell \in \mathbf{N}_0$, $\ell \geq m$ and $0 < \alpha < 1$, the mapping*

$$L : x^\delta \Lambda^{\ell,\alpha} \longrightarrow x^\delta \Lambda^{\ell-m,\alpha}$$

is Fredholm.

Of course, a similar statement also holds for L as a mapping between weighted degenerate L^p spaces.

§5. The Model Problem

The key step in constructing a 'good' parametrix (e.g. one with compact error) for an elliptic $L \in \mathrm{Diff}_e^*(X)$ is the detailed analysis of its normal operator $N(L)$. This analysis includes its mapping properties and the structure of its generalized inverse and projectors as in our analysis of totally characteristic operators in the last section. Using the translation and dilation invariance of $N(L)$, these properties are deduced readily from the analogous properties of an associated family of b-operators with an extra Bessel structure which we define below. This section is devoted to the analysis of $N(L)$. We shall always assume the hypothesis (2.22).

Recall that $N(L)$ is defined as the restriction to the front face of the lift through the left of L to X_e^2. The front face $B_{11}(X_e^2)$ fibres naturally over ∂X, and $N(L)$ restricts to an elliptic operator on the interior of each leaf of this fibration. These leaves are diffeomorphic to the product $S_{++}^{k+1} \times F$, and can also be identified in a natural way as a compactification of the inward pointing normal bundle N^+F to some fibre F of ∂X. To see this recall that the front face is the inward-pointing spherical normal bundle of the submanifold S in $(\partial X)^2$. Thus the interior of the quarter-sphere factor of the fibre of B_{11} over some point $(\tilde{y}, \tilde{z}) \in \partial X$ is naturally identified stereographically with the tangent space at the point $\theta = (0,0,1)$, which in turn is naturally identified with the inward pointing normal bundle of the fibre $F_{\tilde{y}}$, which is a half-space $\mathbf{R}_+^{k+1}$. The projective coordinates (s,u) from (2.10) determine a splitting $\mathbf{R}_+^{k+1} = \mathbf{R}^+ \times \mathbf{R}^k$ which is, of course, not natural. Note that the boundary of this half-space is identified canonically with the tangent space $T_{\tilde{y}}Y$.

$N(L)$ acts on the product $\mathbf{R}_+^{k+1} \times F$, and is invariant under the linear translations and dilations on the first factor. Using this invariance we can reduce it to to a family of differential b-operators as follows. First conjugate by the Fourier transform in the $\mathbf{R}^k$ directions. Letting η be the dual variable (i.e. is the linear coordinate in $N_{\tilde{y}}^*Y$), then starting with the expression (2.17) for $N(L)$ we arrive at

$$\widehat{N(L)} = \sum_{j+|\alpha|+|\beta| \le m} a_{j,\alpha,\beta}(s\partial_s)^j (is\eta)^\alpha \partial_z^\beta. \tag{5.1}$$

The $a_{j,\alpha,\beta}$ depend on $\tilde{y}$ and z, as in (2.17). Now exploit the special structure of this operator again by rescaling. That is, define a new variable $t = s|\eta|$ and $\hat{\eta} = \eta/|\eta| \in S_{\tilde{y}}^*Y$ and substitute it into (5.1) to get

$$L_0 = \sum_{j+|\alpha|+|\beta| \le m} a_{j,\alpha,\beta}(t\partial_t)^j (it\hat{\eta})^\alpha \partial_z^\beta \tag{5.2}$$

This is a family of b-operators on $\mathbf{R}^+ \times F$ depending smoothly on the parameters $(\tilde{y}, \hat{\eta}) \in S^*Y$.

The operator L_0 is more complicated than the indicial operator $I(L)$ because it has an extra 'Bessel structure', and this significantly alters its behaviour near infinity. We now discuss the ramifications of this additional structure.

(5.3) Definition. $L_0 \in$ *$Diff_b^m(\mathbf{R}^+ \times F)$ is said to be of Bessel type if it has the form*

$$L_0 = \sum_{j+\ell+|\beta|\le m} a_{j,\alpha,\beta}(z)(t\partial_t)^j (it)^\ell \partial_z^\beta,$$

and is called elliptic if the associated symbol

$$\sum_{j+\ell+|\beta|=m} a_{j,\alpha,\beta}(z)\tau^j\sigma^\ell\zeta^\beta$$

is elliptic.

The prototypical example on $\mathbf{R}^+$ is $L_0 = (t\partial_t)^2 - t^2$. The elements of the nullspace of this particular operator either grow or decrease exponentially as $t \to \infty$. This is a general phenomenon for elliptic Bessel operators. To prove this, introduce the family of spaces on $\mathbf{R}^+ \times F$

$$\mathcal{H}^{r,\delta,\ell} = \{u : \phi(s)u \in t^\delta H^r,\ (1-\phi(t))u \in t^{-\ell}H^r\} \tag{5.4}$$

where H^r is the usual L^2 Sobolev space based on the vector fields ∂_t, ∂_z and the measure $dt\,dz$, and $\phi(t) \in \mathcal{C}_0^\infty(\mathbf{R}^+)$ equals 1 near $t = 0$. Recall the set Λ from (4.3).

(5.5) Lemma. $L_0 : \mathcal{H}^{r+2,\delta,\ell} \to \mathcal{H}^{r,\delta,\ell-m}$ *is Fredholm provided $\delta \notin \Lambda$.*

Proof. Fix $\delta \notin \Lambda$. As usual, we prove Fredholm properties by constructing right and left parametrices for L_0, which are bounded on the appropriate spaces, and which are inverses up to compact error. It suffices to construct only a right parametrix since a left parametrix is obtained by the duality arguments explained in the last section. This parametrix H is obtained by patching together local parametrices for L_0 near 0 and ∞. Near $t = 0$ only the b-structure of L_0 is relevant, and a parametrix in this case was constructed in §4. It is bounded between $t^\delta H_b^r$ and $t^\delta H_b^{r+m}$ locally near $t = 0$. Near infinity we proceed as follows. By the ellipticity assumption of (5.3), the partial principal symbol

$$\tilde{\sigma}(L_0) = (i)^m(a_{m,0,0}t^m\tau^m + a_{0,m,0}t^m) + \sum_{|\beta|=m} a_{0,0,\beta}\partial_z^\beta$$

(with τ dual to t) satisfies $\langle\tilde{\sigma}(L_0)u, u\rangle \ge Ct^m(1+\tau^m)\|u\|^2$ for $t \ge s_0$. Hence the operator norm of $\tilde{\sigma}(L_0)^{-1}$ is bounded by $Ct^{-m}(1+\tau)^{-m}$. Define the parametrix H_∞ near ∞ by

$$H_\infty(u) = \int e^{it\tau}\,\tilde{\sigma}(L_0)^{-1}\hat{u}(\tau, z)\,dz.$$

The remarks above show that

$$(1-\phi(t))H_\infty : \mathcal{H}^{r,\delta,\ell-m} \to \mathcal{H}^{r+m,\delta,\ell}$$

is bounded. Patching H_∞ to the parametrix near 0 we obtain an operator H which satisfies $LH = I - K$, for some error term K. Now K is a finite sum of terms, each of which is both smoothing of order at least one, decaying at least like t^{-1} as $t \to \infty$ and like t^ϵ for some $\epsilon > 0$ as $t \to 0$. Hence both maps in

$$K : \mathcal{H}^{r,\delta,\ell} \to \mathcal{H}^{r+1,\delta+\epsilon,\ell+1} \hookrightarrow \mathcal{H}^{r,\delta,\ell} \tag{5.6}$$

are bounded. Since the second inclusion is compact, K itself must be compact. This finishes the proof.

We can also deduce the rapid decrease of any temperate solution to $L_0 u = 0$.

(5.7) Corollary. *If $u \in \mathcal{H}^{r,\delta,\ell}$ for some ℓ and $L_0 u = 0$, then $u \in \mathcal{H}^{r,\delta,\ell}$ for every ℓ and r.*

Proof. By duality from the construction above we can find a left parametrix H' such that $H'L_0 = I - K'$, where K' has the same boundedness properties as in (5.7). Then $L_0 u = 0$ implies that $u = K'u$, and the result follows by iteration.

Also, of course, u is polyhomogeneous near $t = 0$.

Fix δ. Now that Fredholm properties for L_0 have been established we shall only consider the case $\ell = -\delta$, so that $\mathcal{H}^{0,\delta,-\delta} = s^\delta L^2$. It now follows as before that there exists a generalized inverse and projectors onto the kernel and cokernel of L_0, G_0 and P_{0i}; we choose the P_{0i} to be orthogonal projectors when $r = 0$. These satisfy

$$\begin{aligned} G_0 L_0 &= I - P_{01} \\ L_0 G_0 &= I - P_{02}. \end{aligned} \tag{5.8}$$

By definition,

$$\begin{aligned} G_0 &: \mathcal{H}^{r,\delta,\ell-m} \to \mathcal{H}^{r+m,\delta,\ell} \\ P_{0i} &: \mathcal{H}^{r,\delta,\ell-m} \to \mathcal{H}^{r',\delta,\ell'} \end{aligned} \tag{5.9}$$

are bounded for any r, r', ℓ, ℓ'.

The proof of (4.20) implies that

$$G_0 \in \Psi_b^{-m,\mathcal{H}} + \Psi^{-\infty,\mathcal{H}}, \;\; P_{01} \in \Psi^{-\infty,\mathcal{E}}, \;\; P_{02} \in \Psi^{-\infty,\mathcal{F}} \tag{5.10}$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$ are the collections of index sets (4.22).

This description of these operators is not complete: we must also determine their behaviour near ∞.

(5.11) Lemma. *$G_0(t, z, \tilde{t}, \tilde{z})$ and $P_{0i}(t, z, \tilde{t}, \tilde{z})$ are rapidly decreasing in t locally uniformly in $(\tilde{z}, \tilde{t}, \tilde{z})$ and rapidly decreasing in $\tilde{t}$ locally uniformly in $(t, z, \tilde{z})$. Furthermore, the coefficient functions for the asymptotic expansion of any of these functions as $t \to 0$ is rapidly decreasing in $\tilde{t}$, locally uniformly in the other variables, and vice versa.*

Proof. Since the P_{0i} are finite sums of products of elements of the nullspace of either L_0 or L_0^*, the lemma follows for them from (5.7). Now use (5.8) to obtain the same decay for G_0.

Proved similarly is

(5.12) Lemma. *As $\lambda \to \infty$, $G_0(\lambda t, z, \lambda \tilde{t}, \tilde{z})$ and $P_{0i}(\lambda t, z, \lambda \tilde{t}, \tilde{z})$ decrease rapidly locally uniformly in $(t, z, \tilde{t}, \tilde{z})$; for G_0 this is only true provided $(t, z) \neq (\tilde{t}, \tilde{z})$.*

This completes the discussion of individual elliptic Bessel operators. We next discuss how these results are affected by parameters. First we recall the

(5.13) Definition. *An operator A is said to have the unique continuation property (UCP) at a boundary B if any solution of $Au = 0$ vanishing to infinite order at B vanishes identically.*

The following seems crucial, especially in Part II of this paper.

(5.14) Hypothesis. *For each value of the parameters $\tilde{y}$ and $\hat{\eta}$ both L_0 and its adjoint L_0^t have the unique continuation property at $\{t = 0\}$.*

It is possible that this hypothesis always holds; it is known to be true in certain cases, e.g. whenever L is uniformly degenerate so that L_0 is an ODE, and also for many second order operators [A-L]. The hypotheses (5.14) and (2.22) are the two basic assumptions we use in obtaining theorems about elliptic edge operators. Because of (5.14) it follows that for each value of $(\tilde{y}, \hat{\eta}) \in S^*Y$, L_0 is surjective on $t^{\delta}L^2$, and injective on $t^{\delta'}L^2$, for δ sufficiently negative and δ'_N sufficiently large. Let $\underline{\delta}(\tilde{y}, \hat{\eta})$ denote the supremum of all such δ for fixed $(\tilde{y}, \hat{\eta})$ and $\overline{\delta}(\tilde{y}, \hat{\eta})$ the infimum of all such δ'. These values must lie in the set Λ of (4.3). As functions on S^*Y they are semi-continuous functions and everywhere finite; therefore they reach minimal, respectively maximal, values on S^*Y. Fix these values now for once and all and label them $\underline{\delta}$ and $\overline{\delta}$. $\underline{\delta}$ is not necessarily less than $\overline{\delta}$, cf. [M-S], in which case the results of this paper show that L is semi-Fredholm if $\delta \notin \Lambda$, and is actually Fredholm provided $\overline{\delta} < \delta < \underline{\delta}$, $\delta \notin \Lambda$. Now suppose $\delta < \underline{\delta}$ and $\delta \notin \Lambda$. Then L_0 is surjective with finite dimensional nullspace on $s^{\delta}L^2$ for every $(\tilde{y}, \hat{\eta})$. Since the index of L_0 is independent of $(\tilde{y}, \hat{\eta})$, the nullspace of L_0 in $s^{\delta}L^2$ must have constant dimension. The analogous discussion applies when $\delta > \overline{\delta}$, $\delta \notin \Lambda$. Standard perturbation theory now implies the rest of the proof of

(5.15) Proposition. *For δ as above, the generalized inverse and orthogonal projectors for L_0 on $t^{\delta}L^2$, G_0 and P_{0i}, vary smoothly in the parameters $(\tilde{y}, \hat{\eta})$ and satisfy (5.10), (5.11) and (5.12) uniformly. $P_{02} = 0$ when $\delta < \underline{\delta}$ and $P_{01} = 0$ when $\delta > \overline{\delta}$.*

A result analogous to Proposition (5.15) for $N(L)$ is now simple to establish. Actually, since it requires no extra effort, we assume below that $\delta \notin \Lambda$ is any weight such that G_0 and P_{0i} vary smoothly in the parameters, but not necessarily chosen so that one of the projectors vanish. Notice though that we are demanding that the rank of the kernel and cokernel of L_0 remain constant. We deduce closed range properties for $N(L)$ acting on the space $s^{\delta}H_e^r(\mathbf{R}^+ \times \mathbf{R}^k \times F; ds\, du\, dz)$ defined as in (3.22) (but with the vector fields V_i there invariant with respect to translations and dilations in (s, u) since we are working globally in these variables) and the existence and structure of the generalized inverse and projectors for this mapping. We also examine the nature of the Schwartz kernels of these operators on the stereographic compactification $S_{++}^{k+1} \times F$ of $\mathbf{R}^+ \times \mathbf{R}^k \times F$.

(5.16) Theorem. *Let the weight parameter δ be as above. Then the mapping $N(L)$: $s^\delta H_e^{r+m} \to s^\delta H_e^r$ has closed range. In particular there exists a generalized inverse $N(G)$ and projectors $N(P_1)$, $N(P_2)$ (which are orthogonal when r=0) onto the kernel and cokernel of $N(L)$ such that*

$$N(G) : s^\delta H_e^r \to s^\delta H_e^{r+m}$$
$$N(P_i) : s^\delta H_e^r \to s^\delta H_e^\ell$$

are bounded for any r, ℓ.

The proof of this theorem follows easily from results earlier in this section, although there are several details to be checked. $N(G)$ and the $N(P_i)$ are recovered from G_0 and P_{0i} in two steps: first by rescaling so as to correspond to $\widehat{N(L)}$ and then by inverse Fourier transformation.

Since L_0 is obtained from $\widehat{N(L)}$ by the substitution $s|\eta| = t$, we only need perform this same substitution in reverse to obtain the generalized inverse and projectors for $\widehat{N(L)}$.

(5.17) Lemma. *The operators defined by $\hat{G}(s, \tilde{s}, z, \tilde{z}, \eta) = G_0(s|\eta|, \tilde{s}|\eta|, z, \tilde{z}, \hat{\eta})|\eta|$ and $\hat{P}_i(s, \tilde{s}, z, \tilde{z}, \eta) = P_{0i}(s|\eta|, \tilde{s}|\eta|, z, \tilde{z}, \hat{\eta})|\eta|$ are bounded from $s^\delta \hat{H}_e^r(ds\, dz)$ to $s^\delta \hat{H}_e^{r+m}$ and $s^\delta H_e^\ell$, respectively, for any r, ℓ and $\eta \neq 0$, with bound independent of η. The spaces $s^\delta \hat{H}_e^l$ are defined as in (3.22) except that differentiations with respect to $s\partial_u$ are replaced by multiplications by $s\eta$.*

Proof. First we prove boundedness of these operators on $s^\delta L^2(ds\, dz)$ with bound independent of η. As always, conjugating by s^δ allows us to assume that $\delta = 0$. Notice that this is the same as conjugating by $(s|\eta|)^\delta$, and the change of variables below respects this reduction. The transformation $u(s, z) \to u_a(s, z) \equiv a^{-1/2}u(s/a, z)$ is an isometry of L^2. Now, with $\|\ \|$ the norm on either $L^2(ds\, dz)$ or $L^2(dt\, dz)$ depending on the context we compute by changing variables

$$\|\hat{G}u\| = \|(\hat{G}u)_{|\eta|}\| = \|G_0(u_{|\eta|})\| \leq C\|u_{|\eta|}\| = C\|u\|$$

The inequality here follows from the boundedness of G_0 on $L^2(dt\, dz)$ and C is its ($|\eta|$-independent) norm. The boundedness of $\hat{G}$ from H_e^r to H_e^{r+m} for r a positive integer follows easily by applying the same argument and using that the operators $t^m G_0$, $(t\partial_t)^i \partial_z^j G_0$, $i + j \leq m$, are also bounded on $L^2(dt\, dz)$. When r is a negative integer the result follows by duality and finally for general r one may use interpolation. The arguments for the $\hat{P}_i$ are identical.

Proof of (5.16). It suffices to construct the operators $N(G)$ and $N(P_i)$. Clearly their correct definition is given by

$$N(G)(s, \tilde{s}, u, \tilde{u}, z, \tilde{z}) = \int e^{i(u-\tilde{u})\eta} G_0(s|\eta|, \tilde{s}|\eta|, z, \tilde{z}, \hat{\eta})|\eta|\, d\eta \tag{5.18}$$
$$N(P_i)(s, \tilde{s}, u, \tilde{u}, z, \tilde{z}) = \int e^{i(u-\tilde{u})\eta} P_{0i}(s|\eta|, \tilde{s}|\eta|, z, \tilde{z}, \hat{\eta})|\eta|\, d\eta.$$

That $N(G)N(L) = I - N(P_1),\ N(L)N(G) = I - N(P_2)$ is obvious from the definitions. Boundedness of these operators follows from Lemma (5.17) and the Plancherel formula. By Lemma (5.12) the Schwartz kernels of these operators are smooth in the interior (excepting of course the singularity of $N(G)$ along the diagonal). The $N(P_i)$ project onto the full kernel and cokernel of $N(L)$ because $\hat{P}_i$ project onto the full kernel and cokernel of $\hat{N}(L)$ for every η.

Let $\Sigma = \mathbf{R}^+ \times \mathbf{R}^k \times F$. It is clear from (5.18) that the functions $N(G)$ and $N(P_i)$ above are homogeneous of degree $-k-1$ in the variables $(s, u, \tilde{s}, \tilde{u})$ on Σ^2. We have been regarding these kernels as corresponding to operators acting on functions rather than half-densities. If we now revert to our previous conventions and multiply them by the standard half-density factors $\mu = \sqrt{ds\, du\, dz\, d\tilde{s}\, d\tilde{u}\, d\tilde{z}}$, we obtain the distributional sections $\kappa_{N(G)} r^{-\frac{k+1}{2}} \nu$ and $\kappa_{N(P_i)} r^{-\frac{k+1}{2}}$ of the singular half-density bundle $r^{-\frac{k+1}{2}} \Omega^{\frac{1}{2}}$ on Σ_e^2 where the distributions $\kappa_{N(G)}$ and $\kappa_{N(P_i)}$ are homogeneous of degree 0. For simplicity we call these distributions $\kappa(G)$ and $\kappa(P_i)$. Their interior smoothness and homogeneity imply that they are smooth down to $B_{11}(\Sigma_e^2)$ (only away from the diagonal for $N(G)$ of course). We now establish that they are polyhomogenous at the side boundaries so that $N(G)$ and $N(P_i)$ are of edge type.

(5.19) Proposition. *With collections of index sets $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{F}$ as in (5.10) and (4.22), we have $N(G) \in \Psi_e^{-m,\mathcal{H}}(\Sigma)$, $N(P_1) \in \Psi_e^{-\infty,\mathcal{E}}(\Sigma)$ and $N(P_2) \in \Psi_e^{-\infty,\mathcal{F}}(\Sigma)$.*

Proof. We prove this only for $N(G)$; the proof for the $N(P_i)$ is identical. As noted above, it suffices to show that the function $\kappa(G)$ is polyhomogeneous at the boundaries B_{10} and B_{01} of Σ_e^2 away from B_{11}.

The polyhomogeneity of G in the interior of either of the side boundary faces, hence at points in the interior of $B_{10} \cap B_{11}$ or $B_{01} \cap B_{11}$, follows from the expression (5.18) and Lemma (5.11). For by this lemma, as $s \to 0$, for example, and $\tilde{s}, u$ remain in a bounded set, with $\tilde{s}$ bounded away from 0 as well, we may replace G_0 in (5.18) by any finite sum of its expansion plus a term vanishing at some high rate. Since the coefficients are rapidly decreasing in $|\eta|$ the integral is smooth in s, u, $\tilde{u}$, and the error term is similarly controlled.

To show that $\kappa(G)$ is polyhomogeneous at the corners we use Lemma (A.4). Replace G_0 in (5.18) by its expansion near $s = 0$, for example; then each of the coefficients in the expansion there is of the form

$$a(\tilde{s}, u, \tilde{u}, z, \tilde{z}) = \int e^{i(u-\tilde{u})\eta}\, F(\tilde{s}|\eta|, z, \tilde{z}, \hat{\eta})|\eta|^{\ell}\, d\eta \tag{5.20}$$

for some $\ell \in \mathbf{R}$. Here F is some function rapidly decreasing in $s|\eta|$, $\tilde{s}|\eta|$, but polyhomogeneous as either of these variables tends to zero, and smooth in the remaining variables. We already know from above that a is smooth for $\tilde{s}$, u, $\tilde{u}$, z, $\tilde{z}$ in a bounded set. Furthermore, we may also restrict to $\{\tilde{s} = 1,\ \tilde{u} = 0\}$. Now we must check that a is of stable regularity with respect to vector fields on $\{u \in \mathbf{R}^k,\ z, \tilde{z} \in F\}$ which transform under stereographic projection to totally characteristic vector fields on $B_{10} \cap B_{11}$.

Clearly a is smooth in z and $\tilde{z}$, so we may restrict attention to behaviour in u. The vector fields $u_j\partial_{u_k}$, when projected stereographically, span $\mathcal{V}_b(B_{10} \cap B_{11})$, so we must check that a is of stable regularity with respect to these. For any multiindices α, β with $|\alpha| = |\beta|$

$$u^\alpha \partial_u^\beta a = \int u^\alpha (i\eta)^\beta e^{iu\cdot\eta} F(|\eta|, z, \tilde{z}, \hat{\eta})|\eta|^\ell \, d\eta. \tag{5.21}$$

Because of the rapid decrease of F the extra factors of η cause no problems, but the extra factors of u do increase growth at infinity. Fortunately these counterbalance one another. To see this, replace $u^\alpha(i\eta)^\beta e^{iu\cdot\eta}$ by $(i\eta)^\beta(-i\partial_\eta)^\alpha e^{iu\cdot\eta} = \eta^\beta \partial_\eta^\alpha e^{iu\cdot\eta}$. Since $|\alpha| = |\beta|$, $\eta^\beta\partial_\eta^\alpha$ is equal to some combination of the vector fields $(\rho\partial_\rho)$ and $\partial_{\hat{\eta}}$, where $\rho = |\eta|$. Now integrate by parts $|\alpha|$ times, throwing all derivates from the exponential onto $F(|\eta|, z, \tilde{z}, \hat{\eta})|\eta|^\ell$. Because of the rapid decrease of F as $\rho \to \infty$, its conormality as $\rho \to 0$ and its smoothness in $\hat{\eta}$, the integration by parts produces a sum of terms, all of which are integrals as above of some functions F_i which decrease rapidly as $\rho \to \infty$, and which have the same order of conormality at $\rho = 0$ as the integrand in (5.20). Taken together, these facts imply that a is of stable regularity as $u \to \infty$. It is clear that the collections of index sets $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{F}$ are the same as for G_0, P_{01} and P_{02}.

§6. Semi-Fredholm Theory

Using the contents of the last three sections the complete analysis of elliptic edge operators in the semi-Fredholm case is now possible. As in §4, the basic idea is to add correction terms to the crude parametrix from Theorem (3.8). However only rarely will it be possible to obtain a two-sided parametrix with compact remainders since this would imply that L is Fredholm. When $N(L)$ is either injective or surjective then correction terms may be added in a manner almost identical to the one explained in detail for totally characteristic operators to produce a one-sided parametrix with compact remainder. This implies that L has closed range. However unless L is actually Fredholm the part of the proof of (4.20) giving the precise structure of the generalized inverse and projectors breaks down. This is because of the difficulty of directly controlling the parametrix so that its range is essentially equal to the orthogonal complement of the nullspace of L. However there is a trick to reduce the semi-Fredholm case to the (self-adjoint) Fredholm case which circumvents this difficulty. This leads to our main result:

(6.1) Theorem. *Suppose $L \in \mathrm{Diff}_e^* m(X)$ is elliptic and satisfies the hypotheses (2.22) and (5.14). Suppose also that a weight parameter $\delta \notin \Lambda$ is chosen so that either $\delta > \overline{\delta}$ or $\delta < \underline{\delta}$. Then $L : x^\delta H_e^{\ell+m}(X) \to x^\delta H_e^\ell(X)$ is either essentially injective or essentially surjective, respectively, and in either case has closed range. Therefore it has a generalized inverse G and orthogonal projectors P_i onto the nullspace and orthogonal* *complement of the range of L which are edge operators:*

$$(6.2)\qquad \begin{aligned} &G \in \Psi_e^{-m,\mathcal{H}}(X) + \Psi^{-\infty,\mathcal{H}'}(X),\\ &P_1 \in \Psi_e^{-\infty,\mathcal{E}}(X) + \Psi^{-\infty,\mathcal{E}'}(X),\\ &P_2 \in \Psi_e^{-\infty,\mathcal{F}}(X) + \Psi^{-\infty,\mathcal{F}'}(X). \end{aligned}$$

Here $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{H}$ are the collections of index sets from (5.10) and the prime means omit the 11 component of the triplet of index sets. P_1 is very residual if $\delta < \underline{\delta}$ and P_2 is very residual if $\delta > \overline{\delta}$.

Proof. By the hypotheses on δ we can always reduce to the case when the normal operator is an isomorphism. For definiteness assume that $\delta < \underline{\delta}$ so that $N(L)$ is surjective. Form the operator $\mathcal{L} = LL^*$. This is an elliptic edge operator of order $2m$ which extends to be self-adjoint. Since $N(\mathcal{L}) = N(L)N(L)^*$ it is an isomorphism. If $N(L)$ is already an isomorphism (i.e. $\delta > \overline{\delta}$ as well) then we do not need to resort to this artifice and the proof proceeds in the same way, except that then self-adjointness plays no role. We shall not compute $\mathrm{spec}_b(\mathcal{L})$, since our immediate aim is only to establish that L has closed range and that G and the P_i are edge operators.

We construct right and left parametrices for $\mathcal{L}$ following closely the steps of Theorem (4.4), however switching the first two steps. Thus first choose $A_0 \in \Psi_e^{-m,\mathcal{H}}(X)$ to be any extension off $B_{11}(X_e^2)$ of $N(G)$. Then $\mathcal{L}A_0 = I - R_0$ where R_0 vanishes to first order on the front face but still has a conormal singularity along Δ_e, but which also vanishes on approach to B_{11}. Next choose A_1 supported near Δ_e so that $\mathcal{L}A_1$ cancels off this conormal singularity. Now $\mathcal{L}(A_0 + A_1) = I - R_1$ where $R_1 \in \Psi_e^{-\infty,\mathcal{H}(1)}(X)$, i.e. $H_{11} = 1$. (A.9) allows us to choose an operator A_2 supported near this boundary and so that $\mathcal{L}A_2 - R_1 = -R_2$ vanishes to infinite order along B_{10}, but still only to first order along B_{11}. Finally choose an asymptotic sum $I + R'$ for the Neumann series corresponding to $(I - R_2)^{-1}$ and postmultiply $\mathcal{L}A_2 = I - R_2$ by this. The final parametrix is $A = (A_0 + A_1 + A_2)(I + R')$ and it satisfies $\mathcal{L}A = I - R$ where $R \in \Psi^{-\infty,\emptyset,I}(X)$ for some I. Since $\mathcal{L}$ is self-adjoint on $x^\delta L^2$ we can take A^* as a left parametrix with corresponding remainder R^*. All operators here are bounded on $x^\delta L^2$. We have been rather brief here because this is exactly as in §4. The only difference is that the analysis of §5 replaces the Mellin transform arguments there. Since R and R^* are both compact operators this proves that $\mathcal{L}$ is Fredholm.

The proof that the generalized inverse $\mathcal{G}$ and projector $\mathcal{P}_1 = \mathcal{P}_2 \equiv \mathcal{P}$ corresponding to $\mathcal{L}$ are in the edge calculus is quite similar to the one in the proof of (4.20). First note that any $u \in \ker(\mathcal{L})$ satisfies $u = R^*u$, hence is polyhomogeneous. Since $\mathcal{P}$ is a finite sum of terms in the nullspace of $\mathcal{L}$ in $x^\delta L^2$ we see that $\mathcal{P} \in \Psi^{-\infty,\mathcal{I}}$ with $\mathcal{I}$ as above. Next write the analogue of (4.24) and (4.25), replacing A_1 and A_2 there by A and A^* and R_1 and R_2 by R and R^*. This expresses $\mathcal{G}$ as a sum of one term in the large edge calculus and other very residual terms. Hence $\mathcal{G}$ is of edge type.

Now we can return to the operator L. Were L to have closed range and a generalized inverse G and orthogonal harmonic projectors P_i then clearly we would have to have

$P_2 = \mathcal{P}$ and $G = L^*\mathcal{G}$. Thus we simply define G and P_2 in this way. Automatically these operators are in the edge calculus and $LG = I - P_2$. To verify that G is the generalized inverse for L on $x^\delta L^2$ we must also check that $GL = I - P_1$ with P_1 and P_2 the orthogonal projectors in $x^\delta L^2$ onto the nullspace and orthogonal complement to the range of L. Clearly the nullspace of $\mathcal{L}$ is equal to the nullspace of L^* which in turn is equal to the orthogonal complement of the range of L. Thus P_2 is the correct projector. On the other hand, define the element Q of the edge calculus by $GL = I - Q$. We must check that Q is the orthogonal projector onto the nullspace of L. Since $GL = L^*\mathcal{G}L$ is self-adjoint, so is Q. Furthermore, computing LGL in two different ways shows that $LQ = 0$. In particular, Q is residual. Next, apply $GL = I - Q$ to Q and use that $LQ = 0$ to get $Q^2 = Q$. Finally, the equation $GL = I - Q$ shows that $Q = I$ on the nullspace of L in $x^\delta L^2$. These properties uniquely characterize the orthogonal projector onto the nullspace of L, so $Q = P_1$. Finally, since P_1 and P_2 are the correct projectors, G must be the generalized inverse for L on $x^\delta L^2$ which maps $x^\delta L^2$ onto the $x^\delta L^2$-orthogonal complement of the kernel of L in $x^\delta H^m_e$. In particular, the existence of G and its boundedness on all spaces $x^\delta H^\ell_e$ implies that $L : x^\delta H^\ell_e \to x^\delta H^{\ell-m}_e$ has closed range for every $\ell \in \mathbf{R}$.

We have now established that G and the P_i are each in the edge calculus, and more particularly, each one of these operators is a sum of an operator in the large calculus and a very residual operator. The self-adjointness of the P_i and the equations $LP_1 = 0$, $L^*P_2 = 0$ imply that these operators have the form (6.2). Finally the equations $GL = I - P_1$ and $LG = I - P_2$ imply that G has the correct form.

Following directly from the boundedness of edge operators on Hölder spaces is

(6.4) Corollary. *For L as in this theorem, $\ell \geq m$ a positive integer and $0 < \alpha < 1$ the mapping $L : x^\nu \Lambda^{\ell,\alpha} \to x^\nu \Lambda^{\ell-m,\alpha}$ is semi-Fredholm provided $\nu = \delta - \frac{1}{2}$ and $\delta \notin \Lambda$ is as in the previous theorem. Furthermore, supposing for convenience that $\delta < \underline{\delta}$ so that L is essentially surjective, $x^\nu \Lambda^{\ell,\alpha}$ topologically splits into the direct sum $P_1(x^\nu L^{\ell,\alpha}) \oplus (I - P_1)(x^\nu L^{\ell,\alpha})$ of the nullspace of L and its topological complement.*

Our eventual goal is to establish a more general theorem of this type, namely that elliptic edge operators with infinite dimensional kernel and cokernel on the same space have closed range. Unfortunately this requires a rather different approach involving Poisson kernels and Calderon projectors. This has been relegated to a separate paper because the full development is rather lengthy. The basic problem is that even if we form $\mathcal{L}$ in the same way, it is self-adjoint but we do not know that its range is closed. However if we knew this already then we could prove that L also has closed range and that the generalized inverse and harmonic projector for both L and $\mathcal{L}$ are of edge type. This is proved by showing that the resolvent $(\mathcal{L} - \lambda)^{-1}$ is of edge type when λ is in the resolvent set for $\mathcal{L}$. This is proved exactly as above. Then the harmonic projector P is given by a contour integral of this resolvent around a small loop surrounding the isolated pole at $\lambda = 0$. Only here do we require that $\mathcal{L}$ have closed range so that this pole is isolated. An argument can then be made to show that this contour integral represents a pseudodifferential edge operator. We shall explain this in Part II.

§7. Asymptotics

We now prove some general facts about the existence of asymptotics of solutions to $Lu = 0$, or more generally of solutions to $Lu = f$ when f vanishes to a much higher order than u. As always we shall assume the hypothesis (2.22). The arguments in this section generalize those in [M2] and similar ideas are to be found in [Me2].

Asymptotics for elliptic edge operators are a generalization of classical results concerning smoothness to the boundary of solutions of elliptic boundary problems. We shall prove here that an arbitrary solution of $Lu = 0$, or more generally $Lu \in \dot{\mathcal{C}}^\infty(X)$, admits an asymptotic development

$$u \sim \sum_{\Re(s_j)\to\infty} \sum_{\ell=0}^{\infty} \sum_{p=0}^{p_j} x^{s_j+\ell}(\log x)^p u_{j,\ell,p}(y,z). \tag{7.1}$$

(The s_j are the indicial roots of L.) In general this expansion has the curious property that the coefficient functions $u_{j,\ell,p}$ are smooth in z but progressively worse as functions of y as $j,\ \ell \to \infty$. Typically $u_{j,\ell,p} \in H^{r_j-\ell}$ for some $r_j \in \mathbf{R},\ r_j \to -\infty$ as $j \to \infty$, as a function of y. We shall refer to asymptotic expansions with this property as weak asymptotic expansions later on. We shall present an explicit example toward the end of the section which exhibits this behaviour clearly.

To put our goals into context, Corollary (3.24) states that if $Lu = 0$ for an elliptic L (possibly with varying indicial roots) and $u \in x^\delta L^2$ for some δ then $u \in x^\delta H_e^\infty(X)$; another way to phrase this is to say that u is of stable regularity with respect to $\mathcal{V}_e$. If u also has a weak asymptotic expansion, it can then be regarded as a 'classical' element of $x^\delta H_e^\infty$. The drops in regularity in its expansion are associated naturally with $\mathcal{V}_e$. On the other hand, if all coefficients are smooth, then u is a classical element in the space of functions of stable regularity with respect to $\mathcal{V}_b$, i.e. u is polyhomogeneous in the usual sense. Polyhomogeneous expansions are to be expected only when the indicial roots of L are constant.

The fact that solutions may have weak asymptotic expansions seems to be at variance with the well known boundary regularity for solutions of elliptic boundary problems, which should be a special case of results here. This is clarified by two results we prove here which give conditions for a given solution u in order that all coefficients in its expansion be smooth. The first, which is the analogue of boundary regularity for the homogeneous Dirichlet problem, states that if u lies in a weighted space $x^\delta L^2$, where $\delta > \overline{\delta}$, where $\overline{\delta}$ is as in §5 so that the normal operator $N(L)$ is injective on $s^\delta L^2$, then u is polyhomogeneous. Another way to phrase this is that all coefficients are $\mathcal{C}^\infty$ provided the coefficients $a_{j,\ell,p}$ for which $\Re(s_j) + \ell \le \overline{\delta} - \frac{1}{2}$ all vanish. Regularity theory for the inhomogeneous Dirichlet problem is contingent upon the regularity of the Dirichlet data. The analogous criterion to ensure that u is polyhomogeneous is to require that all coefficients $a_{j,\ell,p}$ for which $\Re(s_j) + \ell \le \delta' - \frac{1}{2}$ are smooth. This will imply smoothness of all higher coefficients. The former result is obviously a special case of this one.

The basic tool in proving these facts is the Mellin transform (A.19) acting on functions of x with values in some space of distributions in (y, z). We shall use the following notation below. The basic spaces are $x^\delta L^2(dx\,dy\,dz)$; for simplicity we have omitted mention of the domain of the functions in question. The ensuing discussion in this section is local at the boundary, hence we shall always assume that all functions are supported in a small neighbourhood of ∂X where, say, $x < 1$. Actually, all results below are local in the neighbourhood of any fibre $F_y \subset \partial X$ but global on each fibre. Thus we shall often be a bit sloppy about the domain of dependence in y. Notice that $x^\delta L^2(dx\,dy\,dz)$ may be rewritten as $x^\delta L^2(dx; L^2(dy\,dz))$. The Mellin transform carries it isomorphically to the space consisting of all functions from $\eta < \delta - \frac{1}{2}$ with values in $L^2(dy\,dz)$ which are holomorphic in ζ and such that $\int |u_M|^2\, d(\Re\zeta)$ is uniformly bounded independently of $\Im\zeta$. The integrand here is square of the L^2 norm of u_M in y, z for fixed ζ and as usual $\zeta = \xi + i\eta$.

(7.2) Definition. *For any real number t denote by $x^\delta L^2(dx\,dz; H^t(dy))$ the space of functions which are in the space $x^\delta L^2(dx\,dz)$ with values in $H^t(dy)$, for any choice of coordinates. Denote the range under the Mellin transform of this space by $ML^2(\delta - \frac{1}{2}, t)$. It consists of the space of all functions holomorphic in the half-plane $\eta < \delta - \frac{1}{2}$ with values in $L^2(dz; H^t(dy))$, and such that $\int_{\eta=C} \|u_M(y,z)\|^2(\zeta)\, d\xi$ is uniformly bounded independently of $C \le \delta - \frac{1}{2}$.*

Recall from the appendix §2A that functions which have expansions in the variable x have Mellin transforms which extend meromorphically to all of $\mathbf{C}$ with poles at locations determined by the exponents in the expansion. Our technique consists in proving that the Mellin transform of a solution u admits a meromorphic continuation. However, this continuation exists in general only if we allow u_M to take values in any fixed function space in z and the space of arbitrary distributions in y. More precisely, we shall show that if u solves $Lu = 0$, where L is an elliptic edge operator, and $u \in x^\delta L^2$ then u_M continues meromorphically to the half-plane $\eta < \delta + N - \frac{1}{2}$ with values in $L^2(dz; H^{-N}(dy))$.

(7.3) Theorem. *Suppose $u \in x^\delta L^2(dx\,dy\,dz)$ satisfies the equation $Lu = f \in \dot{\mathcal{C}}^\infty(X)$ where $L \in \mathrm{Diff}_e^*(X)$ is elliptic (with constant indicial roots). Then u admits an asymptotic development of the form (7.1) in which the s_j are all in $\mathrm{spec}_b(L)$ and $\Re s_j > \delta - \frac{1}{2}$. The coefficients $a_{j,\ell,p}(y,z)$ are smooth functions in z and lie in the space H^{-r_j} as functions of y, $r_j = \Re s_j + \ell - \delta + \frac{1}{2}$.*

Proof. Suppose $A \in \Psi_e^{-m}$ is a parametrix for L, so that $AL = I - Q$, with $Q \in \Psi_e^{-\infty}$. Then

$$u = Qu + Af. \tag{7.4}$$

Since f vanishes to all orders at ∂X, $Af \in \dot{\mathcal{C}}^\infty$ as well, hence contributes nothing to any (putative) expansion. For the rest of the argument we shall neglect it and simply assume that $f = 0$ and $u = Qu$. In particular, we get that $u \in x^\delta H_e^\infty$.

Rewrite the equation $Lu = 0$ near the boundary as

$$I(L)u = Eu \tag{7.5}$$

where $I(L)$ is the indicial operator of L (in our fixed choice of coordinates near the boundary) and E contains all 'higher order terms' in L, i.e. all terms which have the form $b_{j,\alpha,\beta}\, x^{\ell}\, (x\partial_x)^j (x\partial_y)^{\alpha} \partial_z^{\beta}$, where $\ell \geq 1$ if $j \neq 0$, with $b_{j,\alpha,\beta}$ smooth. Now pass to the Mellin transform of (7.5). From the discussion above we see that $I_{\zeta}(L)u_M \in ML^2(\delta - \frac{1}{2}, 0)$, since $u \in x^{\delta} H_e^{\infty}$ implies that $Eu \in x^{\delta} L^2$. Now recall the discussion following (2.21) concerning the invertibility of $I_{\zeta}(L)$. This inverse exists at all but a discrete set of points, where it has poles. For simplicity we shall assume that there are no poles on $\Im\zeta = \delta - \frac{1}{2} + N$ for any positive integer N. The Laurent coefficients of all singular terms at these poles are finite rank projectors onto smooth functions in z and by (2.22) depend only algebraically and smoothly on y. Furthermore there are only a finite number of these poles in any strip $a < \eta < b$. Hence we conclude that

$$u_M = I_{\zeta}(L)^{-1}(Eu)_M \tag{7.6}$$

as a meromorphic function in $\eta < \delta - \frac{1}{2}$. Since we already know that u_M is holomorphic in this half-plane, the poles of $I_{\zeta}(L)^{-1}$ must be cancelled by zeroes of $(Eu)_M$. Notice also that since ζ appears polynomially in $I_{\zeta}(L)$, $I_{\zeta}(L)^{-1}$ is bounded along horizontal lines in the ζ plane, hence does not affect the L^2 estimates along these lines.

The first terms in the expansion for u are obtained from (7.6) by reinterpreting the term Eu as an element of $x^{\delta+1}L^2(dx\, dz; H^{-1}(dy))$ as follows. Eu is a sum of terms $ax^{\ell}(x\partial_x)^j(x\partial_y)^{\alpha}\partial_z^{\beta}u$, where $\ell \geq 1$ whenever $j > 0$. For any such term in which $\alpha = 0$, $u \in x^{\delta} H_e^{\infty}$ implies that we may regard this term as an element of $x^{\delta+1}L^2(dx\, dz; L^2(dy)) \subset x^{\delta+1}L^2(dx\, dz; H^{-1}(dy))$. However, if $\alpha \neq 0$ then combine the factor of x from any one $x\partial_{y^i}$ into the weight in front and differentiate u with the remaining y derivative. The remaining factors are treated as before and do not affect the regularity. This allows us to raise the weight by one order at the expense of lowering regularity in y by one degree. Now $(Eu)_M \in ML^2(\delta + \frac{1}{2}, -1)$. From (7.6) we conclude that u_M may be meromorphically continued into $\eta < \delta + \frac{1}{2}$ with values in $L^2(dz; H^{-1}(dy))$ with at most a finite number of poles in the strip $\delta - \frac{1}{2} < \eta < \delta + \frac{1}{2}$. Take the inverse Mellin transform of this function by integrating along a contour $\eta = \delta + \frac{1}{2} - \epsilon$ above all poles to conclude that

$$u - \sum x^{s_j} (\log x)^p u_{j,0,p}(y, z) \in x^{\delta+1}L^2(dx\, dz; H^{-1}(dy)), \tag{7.7}$$

where the sum is finite and all $\delta - \frac{1}{2} < s_j < \delta + \frac{1}{2}$ for all j.

A priori we can only conclude that the coefficients $u_{j,0,p}(y, z)$ are in $L^2(dz; H^{-1}(dy))$. However, first of all, since they are the singular Laurent coefficients at the poles of u_M, they are the images under the singular Laurent coefficients of $I_{\zeta}(L)^{-1}$ at these poles. As remarked earlier, these coefficients project onto finite dimensional spaces of smooth functions in z, hence the $u_{j,0,p}(y,z)$ are smooth as functions of z. We can also improve regularity in y somewhat using Calderón's method of complex interpolation, for which the Mellin transform is ideally suited. More specifically, we would like to conclude that $u_{j,0,p}$ is in $H^{-(\Re s_j-\delta+\frac{1}{2})}$ as a function of y. (Note that by assumption on the s_j, these orders all lie between -1 and 0.) Since $I_\zeta(L)$ only depends smoothly and algebraically on y, it will suffice to conclude that, for the holomorphic extension of $(Eu)_M$ into $\eta < \delta+\frac{1}{2}$ described above, with values in $L^2(dz; H^{-1}(dy))$, $(Eu)_M(\zeta)$ actually lies in $L^2(dz; H^{-t}(dy))$ at any point of the horizontal line $\eta = \delta - \frac{1}{2} + t$.

The method of complex interpolation, cf. [T], assigns to any pair of Banach spaces E and F contained in some ambient Banach space V and to any $\epsilon \in (0,1)$ an intermediate space $[E,F]_\epsilon$ in the following manner. Consider the space of all maps $w(z)$ from the strip $0 \le \Re z \le 1$ to V which are holomorphic in the interior of this strip and continuous in its closure. We require also that, writing $z = x+iy$, $w(iy)$ takes values in E with E-norm bounded independently of y, and $w(1+iy)$ takes values in F, again with F-norm bounded independently of y. The space $[E,F]_\epsilon$ then is defined to be the set of all possible values of such maps at the point $z=\epsilon$. In our situation we consider the holomorphic map $w(z) = u_M(iz+\xi+i(\delta-\frac{1}{2}))$, $\xi \in \mathbf{R}$. For any z in the strip this takes uniformly bounded values in the space $L^2(d\xi\, dz; H^{-1}(dy))$, in particular for $\Re z = 1$; furthermore, when $\Re z = 0$ it takes uniformly bounded values in $L^2(d\xi\, dz; L^2(dy))$. Therefore, at $z=\epsilon$, or indeed at any point along $\Re z = \epsilon$, it takes values in the intermediate space $[L^2(d\xi\, dz; L^2(dy)), L^2(d\xi\, dz; H^{-1}(dy))]_t$. By the techniques of Theorem (4.2) of Ch. 1 in [T] this intermediate space is seen to coincide with $L^2(d\xi\, dz; H^{-t}(dy))$. This implies the regularity of the coefficients as stated.

Now it is clear that we may continue in this fashion, extending u_M meromorphically over each successive strip $\delta - \frac{1}{2} + N - 1 \le \eta < \delta - \frac{1}{2} + N$ as a function with values in the space $L^2(dz; H^{-N}(dy))$. The poles of this function occur not only at the points s_j lying in the set of indicial roots of L in this strip, but also at points $s_j + \ell$, for all $\ell \in \mathbf{N}$. This is because Eu now contains smooth multiples of (derivatives of) terms which already arose in the expansion at an earlier stage. The Mellin transform of a term $a(x,y,z)x^{s_j}(\log x)^m$ has poles at all points $\sigma_j + \ell$, reflecting the Taylor expansion of a in x. Arguing as before using complex interpolation also implies the precise regularity of the coefficients. This completes the proof.

(7.8) *Example*: As promised, we now give an example of a solution u to $Lu = 0$ with an explicit weak asymptotic expansion. In this example the operator L is the Laplacian on the two-dimensional hyperbolic space $\mathbf{H}^2 = \{(x,y) \in \mathbf{R}^2 : x > 0\}$: $L = x^2(\partial_x^2 + \partial_y^2)$, and u is the Poisson kernel with pole at $(0,0)$: $u = x/(x^2+y^2)$. It is easy to check that $\mathrm{spec}_b(L) = \{0,1\}$. Theorem (7.3) guarantees that u admits an asymptotic development of the form (7.1) with $s_j \in \{0,1\}$. Actually, since u is smooth up to $x=0$ away from $y=0$, the development contains no logarithmic terms, so that

$$u \sim \sum_{j=0}^{\infty} x^j u_j. \qquad (7.9)$$

Because $Lu = 0$ it is clear that all higher coefficients $u_2, u_3, \cdots$ are formally determined in terms of u_0 and u_1. Thus it suffices to determine these first two coefficients. Note first that since the u_j are distributions, the real meaning of (7.9) is that for any $\phi \in \mathcal{C}_o^\infty(\mathbf{R})$ and for any $N \geq 0$ the distributions $u_0, \cdots, u_N$ satisfy

$$\langle (u - (u_0 + \cdots + x^N u_N)), \phi \rangle = o(x^N) \tag{7.10}$$

as $x \to 0$. These coefficients are obviously uniquely defined, provided they exist, so to check that (7.9) holds for a particular choice of coefficients, it suffices to check that (7.10) holds for arbitrary ϕ.

Now, it is classical that for our choice of u, $u_0 = \pi\delta_0$, the Dirac delta function in y properly normalized at $y = 0$. To determine u_1 we must compute the value of

$$\lim_{x \to 0} \langle \frac{u - u_0}{x}, \phi(y) \rangle. \tag{7.11}$$

For fixed $x > 0$ let ℓ_x denote the functional on the left hand side of (7.11) Using the value of u_0 above it is clear that

$$\ell_x(\phi) = \int \frac{\phi(y) - \phi(0)}{x^2 + y^2}\, dy. \tag{7.12}$$

Expanding $\phi(y)$ in a Taylor series about $y = 0$ shows that at least the limit (7.11) is defined for each ϕ. Furthermore, if ϕ is supported away from $y = 0$ then an integration by parts identifies $\lim_{x \to 0} \ell_x$ with $1/y^2$. It only remains to identify the regularization of this homogeneous distribution at $y = 0$. We omit the details of this computation, which involve only splitting up the integral into pieces and using the fact that $\phi(y) = \phi(0) + y\phi'(0) + y^2\psi(y)$ for some smooth ψ near $y = 0$. The result is that $u_1 = (\log|y|)''$, and in particular that u_1 has no mass at the origin. Now it is a simple matter to check that the further coefficients determined formally from these two satisfy (7.10). The full expansion is

$$u \sim \sum x^{2j} \frac{(-1)^j \pi \partial_y^{2j} \delta_0}{(2j)!} + \sum x^{2j+1} \frac{(-1)^{2j+1} \partial_y^{2j+2} (\log|y|)}{(2j+1)!}. \tag{7.13}$$

The loss of regularity in higher coefficients is clearly exhibited here. Also, $u \in x^\delta L^2$ for any $\delta < 0$ and the coefficient u_j of x^j is an element of $H^t(dy)$ for any $t < -(j + \frac{1}{2})$. For example $\delta_0(y) \in H^t$ for any $t < -\frac{1}{2}$. This agrees with the sharp estimate in the statement of (7.3).

The expansion is (7.13) is universal for this operator L because u is the Poisson kernel. If v is any solution of $Lv = 0$ equal to the Poisson transform of its leading coefficient v_0 then all higher coefficients are obtained as the convolutions of v_0 with u_j. Another perspective yielding the same result is to observe that u is the inverse Fourier transform of the function $e^{-x|\eta|}$, so the coefficients u_j are the inverse Fourier transforms of the coefficients $(-|\eta|)^j/j!$ in its Taylor expansion at $x = 0$. v_j is obtained from v_0 by applying the pseudodifferential operator with this symbol. The drop in regularity of v_j follows because this operator has order j.

There is another result closely related to (7.3) which is quite useful.

(7.14) Theorem. *Suppose that $L : x^\delta H_e^m \to x^\delta L^2$ is surjective, and $u \in x^\delta H_e^m$ is the particular solution of the equation $Lu = f$ given by the right inverse G to L, $u = Gf$, where $f \in x^{\delta'} L^2$ and $\delta' > \delta$. Then u admits a partial expansion*

$$u = \sum x^{s_j+\ell}(\log x)^p a_{j,\ell,p}(y,z) + v, \tag{7.15}$$

where the sum is only over those $s_j \in spec_b(L)$ and $\ell \in \mathbf{N}$ for which $\delta - \frac{1}{2} + \Re s_j + \ell < \delta' - \frac{1}{2}$, and $v \in x^{\delta'} H_e^m$. In this expansion the coefficients $a_{j,\ell,p}(y,z)$ are smooth in z and elements of $H^{\delta' - \frac{1}{2} - \Re s_j - \ell}$ as functions of y.

Proof. Since $G \in \Psi_e^{-m,E} + \Psi^{-\infty,F}$ for some index sets E, F, so that $G = G_1 + G_2 + G_3$ where $G_1 \in \Psi_e^{-m}$, $G_2 \in \Psi_e^{-\infty,E}$ and $G_3 \in \Psi^{-\infty,F}$. Thus we may write $u = u_1 + u_2 + u_3$, where $u_i = G_i f$. Since $u_1 \in x^{\delta'} H_e^m$ by (3.23) and $u_3 \in \mathcal{A}_{phg}^F$, hence has a complete expansion with all coefficients smooth, it suffices to prove that u_2 has a partial expansion with coefficients of the stated regularity.

For this we first assume, without loss of generality by possibly decreasing the original value of δ, that $\delta = \delta' - N$, for some positive integer N. First carry through the Mellin transform argument of (7.3) applied to the equation $Lu = f$. Of course we can now only meromorphically extend u_M to the strip $\eta < \delta' - \frac{1}{2}$ with values in $L^2(dz; H^{-N}(dy))$. Note that $(u_1)_M$ is holomorphic in $\eta < \delta' - \frac{1}{2}$ with values in $L^2(dy\, dz)$, and $(u_3)_M$ extends meromorphically to the whole plane with singular Laurent terms at all poles smooth in both y and z. Thus it suffices to prove that the contributions of u_2 all have the correct regularity. Rewrite $u_2 = G_2 f$ as $u_2 = (G_2 \tilde{x}^N)(g)$, where $g = x^{-N} f \in x^\delta L^2$. Now observe that for any multi-indices α, β, $\partial_y^\alpha \partial_z^\beta u_2 = (\partial_y^\alpha \partial_z^\beta G_2 \tilde{x}^N) g$. Because of the factor $\tilde{x}^N$ $G_2 \tilde{x}^N$ vanishes to order N on the front face B_{11}, and because any $r\partial_y$ lifts to a vector field on X_e^2 which is tangential to all boundary faces and ∂_z lifts smoothly and tangentially, the kernel $\partial_y^\alpha \partial_z^\beta G_2 \tilde{x}^N$ is an element of $\Psi_e^{-\infty, E_{10}, E_{01}, N-|\alpha|}$. In particular, it is bounded on $x^\delta L^2$ provided $|\alpha| \le N$, and for any β. This means that $u_2 \in x^\delta L^2(dx\, dz; H^N(dy))$, hence the meromorphic continuation $(u_2)_M(\zeta)$ actually takes in functions smooth in z and in $L^2(dy)$ for $\eta \le \delta' - \frac{1}{2}$ and in $H^N(dy)$ for $\eta = \delta - \frac{1}{2}$. The complex interpolation method now implies that the coefficients $a_{j,\ell,p}$ have the correct regularity.

(7.16) *Remark:* If we assume that the function f in the statement of (7.14) possesses more tangential regularity, then we can conclude that the given solution u also possesses more tangential regularity. An extreme form of this is Lemma (3.28): if f is polyhomogeneous, then $u = Gf$ is polyhomogeneous as well. It should also be pointed out that, although a general solution to $Lu = f$ possesses a partial expansion of the form (7.15), the regularity of the coefficients tends to be much worse than that proved above. This is because the particular solution $u = Gf$ is in some sense the best one amongs all possible solutions $u + \phi$, $\phi \in \ker(L)$.

As mentioned earlier, in certain cases it is possible to conclude that all coefficients $u_{j,\ell,p}$ in an expansion are $\mathcal{C}^\infty$, so that $u \in \mathcal{A}_{phg}$.

(7.17) Proposition. *Suppose that $\delta \notin \Lambda$ and $\delta > \overline{\delta}$ as in §5 so that the normal operator $N(L)$ is injective on $x^\delta L^2$. Then if $u \in x^\delta L^2$ satisfies $Lu \in \mathcal{A}^*_{phg}$ we must have $u \in \mathcal{A}^*_{phg}$.*

Proof. Rather than working directly with the expansion (7.1) and proving that all coefficients are smooth, we shall instead use the result of Theorem (6.1) which states that under the given hypothesis L is essentially injective, so that $GL = I - P_1$ where G is the generalized inverse for L and P_2 is the (very residual) orthogonal harmonic projector. If $Lu = f$ then this equation implies that

$$u = Gf + P_2 u. \tag{7.18}$$

The result now follows because both terms on the right are polyhomogeneous. Note that this does not work if P_2 is only large residual.

(7.19) Corollary. *Suppose δ' is a value of the weight parameter which satisfies the hypothesis of Proposition (7.17). Suppose that $u \in x^\delta L^2$ for an arbitrary value of δ, and $Lu = 0$, so that u has an expansion of the form (7.1). If all $u_{j,0,p}$ are smooth for every j such that $\Re s_j < \delta' - \frac{1}{2}$ then $u_{j,\ell,p}$ is smooth for every j, ℓ, p and $u \in \mathcal{A}^*_{phg}$.*

Proof. First reduce to the case where all coefficients $u_{j,0,p}$ with $\Re s_j < \delta' - \frac{1}{2}$ vanish. Let w be the sum of all terms $x^{s_j+\ell}(\log x)^p u_{j,\ell,p}$ with $\Re s_j < \delta' - \frac{1}{2}$, suitably cut off near $x = 0$ and let $v = u - w$. Then $Lv = f \in x^{\delta'} L^2 \cap \mathcal{A}^*_{phg}$ and since $w \in \mathcal{A}^*_{phg}$ as well it suffices to show that v is polyhomogeneous. By lowering δ or raising δ' we may also suppose that $\delta = -\delta'$.

Let G be the left inverse for L on $x^{\delta'} L^2$, so that

$$GL = I. \tag{7.20}$$

(L might actually have a finite rank nullspace which would contribute a very residual operator on the right, but since this causes no difficulties we assume that this term is absent for simplicity.) If we could apply (7.20) to v then the equation $v = Gf$ would imply the result. However, recall from §6 that (7.20) is obtained by duality from the distributional equality $L^t G^t = I$ where G^t is the right inverse for L^t on $x^{-\delta'} L^2$, hence as it stands can only be applied to functions in $x^{\delta'} L^2$, which v isn't. Somewhat more generally we may apply (7.20) to any function v for which it is valid to perform the pairings and integration by parts in

$$0 = \langle (L^t G^t - I)\phi, v\rangle = \langle \phi, (GL - I)v\rangle \tag{7.21}$$

for any $\phi \in x^{-\delta'} L^2$. Actually, by density we need only let ϕ range over $x^{\delta'} L^2$. The pairing of v with ϕ then makes sense and we need only justify the equality

$$\langle L^t G^t \phi, v\rangle = \langle \phi, GLv\rangle = \langle \phi, v\rangle. \tag{7.22}$$

Set $\psi = G^t\phi \in x^{-\delta'}L^2$. If we can establish that

$$\langle L^t\psi, v\rangle = \langle \psi, Lv\rangle \tag{7.23}$$

then the transposition of G^t to G follows immediately from Fubini's theorem. Now, $L^t\psi = \phi \in x^{\delta'}L^2$ and $Lv = f \in x^{\delta'}L^2$, so both sides of (7.23) make sense in terms of the natural pairing of $x^{\delta'}L^2$ and $x^{-\delta'}L^2$. The only difficulty is establishing the validity of the integration by parts leading from one side to the other. Rewrite the left side using the Plancherel formula for the Mellin transform as

$$\int_{\eta=\delta'-\frac{1}{2}} (L^t\psi)_M(\zeta)v_M(-i-\zeta)\,d\xi\,dy\,dz \tag{7.24}$$

Since $L^t\psi = \phi \in x^{\delta'}L^2$, both $\phi_M(\zeta)$ and $v_M(-i-\zeta)$ are in L^2 as functions of y along $\eta = \delta' - \frac{1}{2}$, so (7.24) is well-defined (which we already knew). However, now we deform the contour of integration from $\eta = \delta' - \frac{1}{2}$ to $\eta = -\delta' - \frac{1}{2}$. By Theorem (7.3), $(\psi)_M(\zeta)$, and indeed $(T\psi)_M(\zeta)$ for any $T \in \mathrm{Diff}_e^*$, is in $H^{\delta'-\frac{1}{2}-\tau}$ in y along $\eta = \tau$, except for a finite number of possible poles in the strip under consideration. On the other hand, $v_M(-i-\zeta)$ is in $H^{\tau+\frac{1}{2}-\delta'}$ in y along the same line. Furthermore, $v_M(-i-\zeta)$ has zeroes at exactly the same values of ζ at which ψ_M has poles and of the same order too. This follows from the hypothesis of this Theorem. Hence we may integrate each term by parts (recalling that differentiations by $x\partial_x$ have become multiplications by $-i\zeta$ now) to obtain that the left side of (7.23) is equal to

$$\int_{\eta=-\delta'-\frac{1}{2}} \psi_M(\zeta)(Lv)_M(-i-\zeta)\,d\xi\,dy\,dz. \tag{7.25}$$

Since $Lv = f \in x^{\delta'}L^2$ we may apply the Parseval formula again to obtain the right side of (7.23). This now establishes that $v = GLv = Gf \in \mathcal{A}^*_{phg}$ as desired.

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