Spectral Invariants

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Spectral Invariants, one of the two programs at MSRI in the Spring of 2001, is a field loosely encompassing a number of areas of study which have grown out of the basic problems of spectral geometry, suggested by Mark Kac's (too often quoted) question: can one hear the shape of a drum? The term spectral invariants was used to emphasize the fact that the unifying feature of the program would be the study of quantities which originally arose in spectral geometry (or index theory) but which have since taken on a life of their own.

To describe some of these developments, first recall that if A is a symmetric n-by-n matrix (with real entries), then its spectrum consists of its set of eigenvalues, $\{\lambda_1, \dots, \lambda_n\}$. These are listed in nondecreasing order and with multiplicity, so there are always exactly n of them. Conversely, this n-tuple determines the matrix A, amongst symmetric matrices, up to conjugacy by an orthogonal matrix. Now suppose (M, g) is an n-dimensional closed Riemannian manifold. Its Laplace-Beltrami operator Δ_g is a self-adjoint, nonnegative elliptic operator acting on $L^2(M, dV_g)$, the space of square-integrable functions. It is classical that this operator may be diagonalized, but now there are an infinite number of eigenvalues. Listing these again in nondecreasing order, with multiplicity, we obtain an infinite list of nonnegative real numbers $\{0 = \lambda_0, \lambda_1, \dots\}$, which is called the spectrum of Δ_g. Loosely speaking, the field of spectral geometry is concerned with the precise relationships between this spectrum of Δ_g and the geometry of the Riemannian manifold (M, g).

There is no reason why one should only look at the spectrum of the scalar Laplacian, and it is also of interest to seek similar relationships for the Hodge Laplacian, acting on differential forms, or the Dirac (or Dirac-type) operators. More generally still, the spectral analysis of (stationary) Schrödinger operators of the form $\Delta + V$ on $\mathbb{R}^n$ is central in some parts of mathematical physics. For simplicity and brevity, we shall mostly focus on the spectrum of the scalar Laplacian here. However, we also include the case where M is a manifold with boundary (e.g. a domain in $\mathbb{R}^n$), and Δ_g acts on functions which satisfy Dirichlet boundary conditions, i.e. vanish at the boundary.

The study of $\operatorname{spec}(\Delta_g)$ naturally divides into two parts, one focussing on the geometric information encoded in the 'low eigenvalues' and the other on the high frequency asymptotics and other properties concerning the statistical distribution of the sequence of eigenvalues. In the first, one seeks to find both upper and lower estimates for the first few eigenvalues in terms of geometric quantities on the manifold. Estimates from above on an eigenvalue are not hard to come by, since eigenvalues may be described by a minimax procedure from the classical Rayleigh quotient $\int |\nabla u|^2 dV_g / \int |u|^2 dV_g$. However, lower estimates are typically much more subtle; one example is the lower bound discovered by Cheeger for the first nonzero eigenvalue λ_1 in terms of an isoperimetric ratio. Geometric estimates on higher eigenvalues become progressively harder to obtain.

The basic result concerning the so-called 'high frequency' asymptotics of the spectrum is the Weyl estimate, concerning the counting function $N(\lambda)$, the number of eigenvalues less than or equal to λ. This states that $N(\lambda)$ grows at the same rate as $c_n \mathrm{Vol}(M) \lambda^{n/2}$, where c_n is a purely dimensional constant. Although this result in its simplest form does indeed trace back to Weyl, more recent developments (since the 1960's) revolve around optimal estimates for the remainder $N(\lambda) - c_n \mathrm{Vol}(M) \lambda^{n/2}$. For the Laplacian, these are due to Avakumovic (and later by Hörmander for more general pseudodifferential operators, in his famous paper introducing the theory of Fourier Integral Operators); Duistermaat and Guillemin and then Ivrii showed how to obtain the sharpest form. One of the beautiful discoveries here is that the dynamics of the geodesic flow on M is closely related to the size of this remainder term. The manifolds for which this remainder is the smallest are those such as the sphere with all geodesics closed, while it is quantitatively larger when the set of closed geodesics has measure zero, in an appropriate sense.

Which nondecreasing lists of numbers can arise as the spectrum of Δ_g for some manifold M and some metric g? Quite remarkably, up to some very mild conditions, any finite list $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ can occur as the first N terms of the spectrum of a Laplace operator for some manifold and some metric. This surprising fact, discovered by Colin de Verdiere, is proved by first establishing the analogous fact for the combinatorial Laplacian on some finite graph, and then considering the manifold obtained by 'thickening up' this graph. In contrast, although the Weyl asymptotic estimate is an obvious constraint on the tail of the sequence, it is far from the only one as we shall see shortly, and it is most likely a hopeless task to characterize the range of this 'spectrum mapping'.

The uniqueness question, concerning how many Riemannian manifolds can have the same spectrum, has also been much studied. At this point there are only a rather small number of examples of manifolds or domains which are know to be 'spectrally determined', i.e. uniquely determined by their spectrum. In particular, Zelditch has established the spectral determination of a large class of real analytic domains in the plane with a single reflection symmetry; these results were announced at the March workshop for this spectral invariants program. On the other hand, there are many counterexamples, and many general constructions which give rise to pairs (or larger families) of 'isospectral manifolds'.

To study each of these questions, it has proved to be extremely fruitful to consider, rather than individual eigenvalues, functions which are built out of the full spectrum. There are three of greatest interest: the heat trace $H(t) =$

$\sum \exp(-\lambda_j t)$, the wave trace $W(t) = \sum \exp(\pm i\sqrt{\lambda_j} t)$ and the zeta function $\zeta(s) = \sum \lambda_j^{-s}$. The first two of these get their names from the fact that they are the (Hilbert space operator-theoretic) traces of the fundamental solutions of the heat and wave equations, respectively, corresponding to the Laplacian. The third is a trace too, of the (complex) powers of the Laplacian. $H(t)$ is defined and smooth on $\mathbb{R}^+$, and its geometric information is contained in the coefficients of its asymptotic expansion as $t \searrow 0$. These coefficients are called the heat invariants of the metric. $W(t)$ is defined as a distribution on the whole line, and it has a singularity at $t = 0$, but also, quite beautifully, at (some subset of) the lengths of the closed geodesics on the manifold M. This famous theorem was proved by Chazarain and in sharper form by Duistermaat-Guillemin. (That this 'length spectrum' is a spectral invariant was obtained somewhat earlier by Colin de Verdiere using the heat equation continued to complex time.) The asymptotic expansion of $W(t)$ as $t \to 0$ contains the same information as the expansion of $H(t)$, but its expansions at these other singularities contain new information, related to the dynamical properties (in particular, the Birkhoff normal form) of the geodesic flow near the corresponding closed geodesics, and is very difficult to obtain. Only in the past few years have some of the secrets of these nonlocal invariants been unlocked, in work of Guillemin and of Zelditch. Finally, using the Weyl asymptotics it is clear that $\zeta(s)$ is holomorphic in the half-plane $\mathrm{Re}\,(s) > n/2$, but it turns out to extend meromorphically to the entire complex plane. Numerical invariants are obtained by evaluating $\zeta(s)$ at special points. Of greatest interest is the value $\zeta'(0)$, which was introduced by Ray and Singer as a viable way to define the logarithm of the determinant of Δ_g.

Each of these functions determines the other two by various of the classical transforms, and thus the geometric information carried in any one of these functions is somehow latent in the other two. However, some types of information may be easier to obtain from one than the other. For example, to derive the expansions for $H(t)$ and $W(t)$, one makes direct use of the associated underlying PDE's (the heat and wave equations) and constructs approximate 'parametrices' to the fundamental solutions for these equations using local geometric methods. The parametrix for the heat equation is obtained by quite elementary means, and provides an easy link between the heat invariants and local geometric data. (Pursued further, this leads to the heat equation proof of the Atiyah-Singer index theorem.) On the other hand, the spectral invariants associated to closed geodesics are most naturally obtained using the wave equation.

These functions all have classical antecedents: theta functions, the Poisson summation formula and number-theoretic zeta functions, respectively. Just as for these arithmetic analogues, the location of singularities and existence of expansions for $H(t)$ and $W(t)$ and the meromorphy of the zeta function are indicative of some of the concealed structure of the eigenvalue sequence.

These functions are the main examples of spectral invariants, and as indicated earlier, the body of results which have emerged from their study goes considerably beyond the original scope of spectral geometry.

I shall describe just a few themes of some of these developments. First, one remarkable and useful feature of the zeta function, and more particularly the determinant, is the existence of 'cut and paste' formulas, which allow some way of partially localizing this global invariant into contributions from the pieces of a decomposition of the manifold. This 'Mayer-Vietoris principle' was explored in great detail by Burghelea, Friedlander and Kappeler and by Hassell and others, and it plays a significant role in some proofs of the Cheeger-Müller theorem on the equality of analytic and Reidemeister torsions. This is a long story, worthy of a separate account, so I turn instead to the well-known results on compactness of isospectral sets of metrics. These give some control on the uniqueness question, and are perhaps the best general results in this direction that could be expected. The basic point is that sets of metrics which share the same spectrum are shown to be compact in an appropriate topology by demonstrating that the eigenvalues, or some of these spectral invariants, control the geometry of the metrics in a strong way. Melrose made the original observation (during his participation in the program in MSRI's first year of operation, 1982-3) that for planar domains, the heat invariants serve essentially as a sequence of nonlinear Sobolev norms of the curvature function of the boundary; since these are all constant on an isospectral set, the curvature functions must lie in a compact set. (A related result had been obtained earlier by Brüning in a somewhat simpler context.) Later, Osgood, Phillips and Sarnak proved that isospectral sets of metrics on Riemann surfaces are compact. To do this they used $\log\det\Delta_g$ as a 'height function' on the space of metrics; they showed moreover that within any conformal class this function attains its unique minimum precisely at the constant curvature metric. This result introduces an important theme: that canonical metrics are closely related to (first variations of) spectral invariants. The key tool here is a formula, coming out of string theory, and due to Polyakov, which computes the variation of this determinant of the Laplacian with respect to any one-parameter family of conformally related metrics. The determinant is a global invariant of the metric, but quite remarkably, this variation turns out to be given by a local expression, i.e. an integral over the surface involving the conformal factor and explicit geometric quantities.

The existence of this Polyakov variational formula is closely tied to the fact that the Laplacian in two dimensions behaves quite simply under conformal changes of metric. Formulas of this type exist in greater generality for natural geometric operators which exhibit a conformally covariance. A naturally geometric operator is one which is associated canonically to a Riemannian metric; this operator is conformally covariant if the two operators corresponding to two conformally related metrics are simply related by pre- and post-multiplication by powers of the conformal factor. The Laplacian in dimensions greater than two does not have this property, but for example, adding a zeroeth order correction term involving the scalar curvature produces a conformally covariant operator known as the conformal Laplacian.

This suggests one line of questions: Can one characterize the the conformally covariant operators in general? Are there good compactness results for isospectral families (within a conformal class) for these? Is there any hope for obtaining similar results for non conformally covariant operators?

To address this last question first, there has been some success in obtaining compactness for sets of conformally related isospectral metrics for the Laplacian in dimensions three and four, by the work of Yang, Gursky and others. In addition, Okikiolu has shown how to analyze the variation of the determinant of the Laplacian across conformal classes, and how canonical metrics, e.g. round metrics on spheres, are often extremals for this function.

Next, what about the existence of other conformally covariant operators, beyond the classical example of the conformal Laplacian, and some newer ones, such as the fourth order Paneitz operator? Within the last several years, substantial progress has been made on understanding when such operators exist and when they do not, through the work of Eastwood, Graham and their collaborators. This subject has close ties to some parts of representation theory, and during this Spring's program, Graham and Zworski showed how some of these conformally covariant operators are closely related to the scattering theory of conformally compact Einstein metrics. (These metrics were originally introduced by Fefferman and Graham for the purpose of systematizing conformal invariant theory; to connect with some of the other focusses of this program, Anderson announced a very satisfactory existence theory for these metrics in four dimensions in the May workshop.)

Finally, Branson and Ørsted proved that the Polyakov formulæ for determinants of conformally covariant operators have a beautiful structure. In four dimensions, the variation of the (log of the) determinant, as always within a conformal class, is given by an expression of the form $\alpha I + \beta II + \gamma III$, where I, II and III are explicit functionals involving the conformal factor and local geometric quantities of the metric. Quite remarkably, these functionals are *universal*, by which I mean that different conformally covariant operators always lead to linear combinations of these same functionals; the coefficients α, β, γ alone depend on the specific operator!

Obviously, unlike the two-dimensional case, one does not hope for very simple characterizations of the extremals of these functionals, if for no other reason than that geometry in four dimensions is much more subtle. However, in the work of Chang, Yang, Gursky and their collaborators, some interesting geometric structure has arisen from the study of some of the individual functionals I, II and III. In particular, an analysis of their extrema has led to some valuable geometric results, such as the existence of metrics with positive Ricci curvature in certain conformal classes. This direction of research is still in its infancy, and there remain many interesting open problems concerning these new classes of curvature equations coming from this variational theory, and their geometric uses.

There are many other canonical metric constructions using determinants and their variations. Perhaps most famous is the Quillen metric on the determinant line bundle; Bismut and his collaborators have made an extensive and penetrating exploration of its geometry and have shown how to use it (and associated objects and constructions) in complex algebraic geometry. We also note the work of Müller and Wendland relating determinants to extremal Kähler metrics.

I have concentrated on just a few of the themes which were discussed during this Spring's program; most of the results mentioned above were due to mathematicians who took part for shorter or longer periods in this MSRI program.

I have, of necessity omitted mention of many other important topics discussed during the seminars and workshops of this program, including the whole field of geometric scattering theory. To summarize this area very briefly, the spectral analysis of the Laplacian on complete noncompact Riemannian manifolds is fairly hopeless in general, but restricting to manifolds with some sort of regular structure at infinity (e.g. those which are asymptotically modelled at infinity by ends of locally symmetric manifolds), there is hope of developing a good theory. The correct set of objects to study, and a good set of directions to pursue (at least at first) is suggested by classical mathematical scattering theory (traditionally carried out for Schrödinger operators $\Delta + V$ on $\mathbb{R}^n$). The goal in this broader geometric context is to study the fine-scale structure of the spectrum, and other related objects and operators such as the scattering matrix and the resonances, and to establish relationships between these and the underlying geometry. New analytic tools must be developed, and the most successful of these come from Melrose's work on the geometrization of microlocal analysis. There is now a substantial methodology and body of results in this direction.

This article gives at least a small hint of the flavor of this subject, or rather of this collection of interrelated subjects. The problems and directions of research discussed here constitute an important branch of geometric analysis, one which has deep connections and applications to many other fields. This Spectral Invariants semester provided a crucial forum for an appraisal of the current state of this subject.