

The conformal theory of Alexandrov embedded constant mean curvature surfaces in \mathbb{R}^3

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Abstract

We first prove a general gluing theorem which creates new nondegenerate constant mean curvature surfaces by attaching half Delaunay surfaces with small necksize to arbitrary points of any nondegenerate CMC surface. The proof uses the method of Cauchy data matching from [8], cf. also [10]. In the second part of this paper, we develop the consequences of this result and (at least partially) characterize the image of the map which associates to each complete, Alexandrov-embedded CMC surface with finite topology its associated conformal structure, which is a compact Riemann surface with a finite number of punctures. In particular, we show that this ‘forgetful’ map is surjective when the genus is zero. This proves in particular that the CMC moduli space has a complicated topological structure. These latter results are closely related to recent work of Kusner [5].

1 Introduction

In this paper we consider the class of surfaces $\Sigma \subset \mathbb{R}^3$ which have constant mean curvature (CMC) equal to 1, are complete and of finite topology. We shall assume that these surfaces are properly immersed, and in fact satisfy the stronger condition that they are *Alexandrov embedded*. This last condition means that the immersion $\varphi : \Sigma \hookrightarrow \mathbb{R}^3$ extends to a proper immersion $Y \hookrightarrow \mathbb{R}^3$ of a three manifold Y whose boundary is the surface Σ , $\partial Y = \Sigma$. We remark that, except for the sphere, no compact CMC surface, including Wente tori, ever satisfy this condition. The space of all surfaces of this type, of genus g with k ends, will be denoted $\mathcal{M}_{g,k}$. To be definite, we do not identify elements of this space which differ by Euclidean motions.

The simplest examples of surfaces of this type are the rotationally invariant Delaunay surfaces. Up to Euclidean motions, these are parametrized by a single ‘necksize’ parameter τ , and will be denoted D_τ . We discuss these at greater length later. A remarkable theorem of Meeks [14] states that each end of an Alexandrov embedded surface is cylindrically bounded, and using this Korevaar, Kusner and Solomon [6] proved that each end is in fact strongly convergent to some Delaunay surface, and thus each end has an associated asymptotic necksize parameter. This strong control on the asymptotic geometry of these surfaces makes it possible to understand the rudimentary structure of these moduli spaces, and in [7] it is proved that $\mathcal{M}_{g,k}$ is always a locally real analytic space of virtual dimension $3k$. Somewhat remarkably, this dimension only depends on the number of ends and not the genus. Furthermore, if $\Sigma \in \mathcal{M}_{g,k}$ satisfies a certain analytic nondegeneracy condition, which we explain below, then the moduli space is a smooth real analytic manifold of dimension $3k$ in a neighbourhood of the point Σ .

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To describe this nondegeneracy condition, recall that the Jacobi operator L_Σ of a CMC surface Σ is the linearization of the mean curvature operator at Σ . Surfaces which are \mathcal{C}^2 close to Σ may be parametrized as normal graphs, i.e. as the images of the map

$$\Sigma \ni x \longmapsto x + w(x)\nu(x)$$

where ν is the unit normal to Σ at x and w is the (scalar) displacement function. With respect to this representation, L_Σ takes the simple form $\Delta_\Sigma + |A_\Sigma|^2$. Functions in the nullspace of L_Σ are called Jacobi fields.

Definition 1 *The surface $\Sigma \in \mathcal{M}_{g,k}$ is said to be nondegenerate if it has no nontrivial Jacobi fields which decay at all ends of Σ .*

It is natural to investigate the structure of the moduli spaces $\mathcal{M}_{g,k}$ in greater detail. Two basic questions are : first, for which values of g and k is $\mathcal{M}_{g,k}$ nonempty, and second, when does $\mathcal{M}_{g,k}$ contain a nondegenerate element? We now discuss briefly what is known about these questions.

That $\mathcal{M}_{g,1}$ is empty for all g is due to Meeks [14]. Also, $\mathcal{M}_{g,2}$ is empty unless $g = 0$, and in this case this space contains only the Delaunay surfaces. As we indicate later, Delaunay surfaces are always nondegenerate. (As a check on dimensions, note that there is a one dimensional family of Delaunay surfaces with fixed axis which satisfy an additional ‘positioning’ normalization determining their translational location along this axis. The group of Euclidean motions is six dimensional, but the rotations about the fixed axis act trivially on these surfaces. Thus there is a $6 = 3 \cdot 2$ dimensional family of surfaces with 2 ends, as predicted.) To proceed further it is necessary to use transcendental methods, in particular, analytic gluing constructions of these surfaces. The first such surfaces with $k \geq 2$ were constructed by Kapouleas [4], but while his method gives elements in $\mathcal{M}_{g,k}$ for infinitely many values of g and k , it provides little geometric or analytical control on the surfaces themselves, and more specifically it seems hard to determine whether the surfaces he constructs satisfy the nondegeneracy condition. More recently, the first two authors [8] introduced a new method to handle these gluing constructions which involves matching Cauchy data across the gluing interfaces. This technique has many advantages over previous methods : it involves considerably fewer technicalities (and thus makes it possible to approach more complicated geometric problems of this type), it provides very good control of the resulting surfaces, and most importantly, one may often prove that the surfaces obtained in this way are nondegenerate. The main result of [8] is that if M is a complete *minimal* surface of finite total curvature in \mathbb{R}^3 , of genus g with k (asymptotically catenoidal, but not planar) ends which is Alexandrov-embedded and satisfies an analogous nondegeneracy condition, then it is possible to obtain a nondegenerate CMC surface Σ of genus g with k ends by gluing half-Delaunay surfaces onto the boundaries of a truncation of M . This construction yields the existence of nondegenerate elements in $\mathcal{M}_{g,k}$ for infinitely many values of g and k . This result is a basic ingredient in the nice result of Grosse-Brauckman, Kusner and Sullivan [3], who prove that modulo rigid motions, $\mathcal{M}_{0,3}$ is a three-dimensional ball; in particular this space is connected. (Unfortunately, their method does not prove that every element in this space is nondegenerate, although this is quite likely true.) As a further development of this new gluing method, a connected sum construction for compact CMC surfaces with boundary is given in [10]. The forthcoming paper [11] extends this construction to include connected sums of nondegenerate surfaces in $\mathcal{M}_{g,k}$ and moreover shows that for every $g \geq 0$ and $k \geq 3$, $\mathcal{M}_{g,k}$ is nonempty and contains nondegenerate elements.

Beyond these theorems, very little else is known about these moduli spaces. It is of interest to determine even the most basic properties of their topological structure. For example, it is even unknown whether these spaces are ever disconnected. It is also unknown whether the natural Lagrangian structure [7] on these moduli spaces can be used in any significant way.

The present paper has two main parts which we describe now in turn. In the first, which occupies the bulk of the paper, we establish a new gluing theorem :

Theorem 1 *Let $\Sigma \in \mathcal{M}_{g,k}$ be nondegenerate. Then for any point $p \in \Sigma$ there is a one-parameter family of nondegenerate CMC surfaces $\Sigma_\tau(p) \in \mathcal{M}_{g,k+1}$ obtained by gluing a half-Delaunay surface D_τ , with τ sufficiently small, to Σ at p .*

The geometry of Σ away from the point p is perturbed very little in this construction, and in fact as $\tau \rightarrow 0$, $\Sigma_\tau(p)$ converges, on compact subsets of $\mathbb{R}^3 - \{p\}$, to the union of the initial surface Σ and an infinite family of mutually tangent spheres of radius 2 arranged along a ray normal to Σ at p . We actually prove a slightly stronger theorem : we show that if $\Sigma \in \mathcal{M}_{g,k}$ is nondegenerate and $p \in \Sigma$, then there are two distinct one-parameter families of nondegenerate CMC surfaces $\Sigma_\tau^\pm(p)$ with $k+1$ ends; these are obtained by gluing half of either an embedded Delaunay surface (an unduloid) or an immersed Delaunay surface (a nodoid) with very small neck. The surfaces $\Sigma_\tau^-(p)$ obtained by gluing a nodoid on to Σ are no longer Alexandrov embedded, but we see that they behave very much like Alexandrov embedded CMC surfaces. (As an aside, this property of nodoids behaving like Alexandrov embedded CMC surfaces is only true when the necksize is small; the paper [9] shows that this fails rather dramatically when nodoids with large necksizes are considered.) We remark also that this construction was prefigured and motivated by some numerical and computer graphic studies carried out (and brought to our attention) several years ago by Grosse-Brauckmann. There are nice illustrations of these surfaces at <http://www.gang.umass.edu/cmc/>.

An immediate consequence of this gluing theorem is the

Corollary 1 *If $\mathcal{M}_{g,k}$ contains a nondegenerate element, then for any $k' > k$, the moduli space $\mathcal{M}_{g,k'}$ also contains a nondegenerate element.*

The second part of this paper shows how Theorem 1 can be used to obtain some information about the global structure of these CMC moduli spaces. To describe these results, recall from [6] that a complete Alexandrov embedded CMC surface of finite topology is conformally equivalent to the complement of a finite number of points in a compact Riemann surface. Thus any $\Sigma \in \mathcal{M}_{g,k}$ is conformally equivalent to a punctured compact Riemann surface $\bar{\Sigma} - \{p_1, \dots, p_k\}$. This suggests that it might be useful to consider the ‘forgetful map’

$$\mathcal{F}_{g,k} = \mathcal{F} : \mathcal{M}_{g,k} \longrightarrow \mathcal{T}_{g,k}.$$

By definition, $\mathcal{T}_{g,k}$ is the Teichmüller space of conformal structures on a surface of genus g with k punctures, and this map is defined by sending $\Sigma \in \mathcal{M}_{g,k}$ to the marked conformal structure $[\Sigma] \in \mathcal{T}_{g,k}$ determined by its (Alexandrov) embedding. In other words, this map forgets the CMC embedding of this surface and only retains its conformal structure.

To begin, we prove the

Theorem 2 *For each g and k , the map $\mathcal{F}_{g,k}$ is real analytic.*

This statement requires clarification. As already indicated, $\mathcal{M}_{g,k}$ is a locally real analytic space. What this means is that for any $\Sigma \in \mathcal{M}_{g,k}$ there is an open finite dimensional real analytic manifold Y in the space of all Alexandrov-embedded surfaces near to Σ (in an appropriate topology) which contains a neighbourhood \mathcal{U} of Σ in $\mathcal{M}_{g,k}$, such that \mathcal{U} is closed in Y and is given as the zero set of a real analytic function in Y . When Σ is nondegenerate, then Y can be taken as the neighbourhood \mathcal{U} in $\mathcal{M}_{g,k}$ itself. To say that \mathcal{F} is analytic on $\mathcal{M}_{g,k}$ means that it has a real analytic extension to all such real analytic manifolds Y .

One consequence of the real analyticity of these spaces and maps is that $\mathcal{M}_{g,k}$ is stratified by open real analytic manifolds S_j such that on each of these strata \mathcal{F} has constant rank.

We are not claiming that \mathcal{F} is a proper mapping, and indeed, it is clear (from any of the gluing constructions) that this is false. In another paper in this volume [5], Kusner shows that there is a suitable modification of the forgetful map which is proper.

Recall next that the dimension of $\mathcal{M}_{g,k}$ around nondegenerate points is $3k$, while on the other hand, $\mathcal{T}_{g,k}$ is $6g - 6 + 2k$ dimensional. This suggests that \mathcal{F} might conceivably be surjective. This is most likely false, in general, but our next theorems address this issue :

Theorem 3 *Let $g = 0$; then for any $k \geq 3$, the map $\mathcal{F}_{0,k}$ is surjective.*

Kusner has obtained a different proof of this result [5] using his theorem about the properness of \mathcal{F} .

Theorem 4 *Fix $g \geq 1$ and suppose that \mathcal{M}_{g,k_0} contains a nondegenerate element for some $k_0 \geq 3$. Then for any $k \geq k_0$, the image of $\mathcal{F}_{g,k}$ contains an analytic submanifold of codimension $d_{g,k}$ which is uniformly bounded as $k \rightarrow \infty$. In other words, the codimension of the image of $\mathcal{F}_{g,k}$ is bounded as $k \rightarrow \infty$ for each g .*

To explain the statement of this last theorem, recall the stratified structures of these spaces and maps. The image $\mathcal{I}_{g,k}$ of $\mathcal{F}_{g,k}$ is again a stratified real analytic space, and we say that the maximum dimension of any one of these strata in the image is the dimension of $\mathcal{I}_{g,k}$.

Proceeding further, we note that the space $\mathcal{T}_{g,k}$ has rather nontrivial topology. In fact, there is a natural mapping

$$F' : \mathcal{T}_{g,k} \longrightarrow \mathcal{C}_{g,k}$$

which carries the marked Riemann surface $[(\bar{\Sigma}; p_1, \dots, p_k)]$ to the associated element in the configuration space of k distinct ordered points on a surface of genus g . The fundamental group and cohomology ring of this configuration space have been studied intensively, see [1], and are known to be rather complicated. It is (barely) conceivable that the image of $F' \circ \mathcal{F}_{g,k}$ does not see any of this topology, but this is not the case.

Theorem 5 *When $g = 0$ and $k \geq 3$, the map*

$$(F' \circ \mathcal{F}_{g,k})_* : \pi_1(\mathcal{M}_{0,k}) \longrightarrow \pi_1(\mathcal{C}(0,k))$$

is an epimorphism. If $g \geq 1$ and \mathcal{M}_{g,k_0} contains a nondegenerate element for some $k_0 \geq 3$, then for any $k \geq k_0$, the image of the fundamental group $\pi_1(\mathcal{M}_{g,k})$ under the homomorphism $(F' \circ \mathcal{F}_{g,k})_$ contains a finitely generated group with an increasing number of generators as $k \rightarrow \infty$.*

We refer to the final section of this paper for a more detailed statement.

The end-to-end gluing construction of Ratzkin [13], cf. also [11], also implies the topological nontriviality of $\mathcal{M}_{g,k}$, but gives less information than is obtained here.

These results together constitute the first and simplest steps of a more detailed investigation of the topology of the moduli spaces $\mathcal{M}_{g,k}$.

The outline of the rest of the paper is as follows. Sections 2 through 4 contain the proof of the gluing result, Theorem 1. More specifically, §2 contains the analysis of CMC deformations of half-Delaunay surfaces and §3 contains the analysis of CMC deformations of $\Sigma - D$, where Σ is any element in $\mathcal{M}_{g,k}$ and D is a small geometric disk in Σ . These results are then combined in §4, where it is shown how to perform the Cauchy data matching across the gluing interface, and hence how to produce a new CMC surface with $k + 1$ ends. Finally, in §5 we undertake the analysis of the forgetful map and develop its properties and derive the theorems stated above concerning the image of $\mathcal{M}_{g,k}$.

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2 The geometry and analysis of Delaunay surfaces

In this section we recall the family of Delaunay surfaces D_τ , and review some of their basic geometric and analytic properties. We focus particularly on their behavior in the singular limit as $\tau \rightarrow 0$.

2.1 Definition and first properties

To find all CMC surfaces in \mathbb{R}^3 which are rotationally invariant about an axis, one is quickly led to the ODE which the generating curve for such surfaces must satisfy. This ODE has a first integral, and from this one can see that all of its solutions are periodic. Thus we obtain a family of periodic, rotationally invariant surfaces D_τ . With a particular normalization described in the next paragraph, the parameter τ lies in $(-\infty, 0) \cup (0, 1]$; when $\tau > 0$, the surface D_τ is embedded, while when $\tau < 0$, it is only immersed. There is a geometric limit, as $\tau \rightarrow 0^\pm$, consisting of an infinite arrangement of spheres, each tangent to the next, arranged along an axis. We shall not describe this more carefully, but instead refer to [8] for details of the preceding discussion as well as the material in the remainder of this section.

The parametrization given by this cylindrical coordinate description is not very tractable analytically, but fortunately it turns out that there is a parametrization which is much easier to use. This is the isothermal parametrization given by

$$X_\tau : \mathbb{R} \times S^1 \ni (s, \theta) \mapsto \frac{1}{2} \left(\tau e^{\sigma(s)} \cos \theta, \tau e^{\sigma(s)} \sin \theta, \kappa(s) \right), \quad (1)$$

where the functions σ and κ are described as follows (cf. §3 of [8]). For any $\tau \in (0, 1]$, the function σ is defined to be the unique smooth nonconstant solution of the ODE

$$(\partial_s \sigma)^2 + \tau^2 \cosh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) = -\operatorname{arccosh} 1/\tau,$$

while, for any $\tau \in (-\infty, 0)$, the function σ is defined to be the unique smooth nonconstant solution of the ODE

$$(\partial_s \sigma)^2 + \tau^2 \sinh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) = \operatorname{arcsinh} 1/\tau.$$

Again, the definition of κ differs according to whether τ is positive or negative. If $\tau \in (0, 1]$, then we define the function κ by

$$\partial_s \kappa = \tau^2 e^\sigma \cosh \sigma, \quad \kappa(0) = 0,$$

while if $\tau < 0$, we define the function κ by

$$\partial_s \kappa = \tau^2 e^\sigma \sinh \sigma, \quad \kappa(0) = 0.$$

Observe that when $\tau > 0$, κ is monotone increasing, and hence X_τ is an embedding, whereas when $\tau < 0$, this is no longer true and the surfaces are only immersed. The Delaunay surfaces D_τ are known as unduloids and nodoids in these two cases, respectively. The extreme element in the family of unduloids is D_1 , the cylinder of radius $1/2$. The limit of D_τ , either as $\tau \nearrow 0$ or as $\tau \searrow 0$, is an infinite union of tangent spheres of radius 1 , arrayed along a common axis.

As noted above, these surfaces are all periodic. When $\tau > 0$, there is a unique $t_\tau > 0$ such that $\tau \cosh t_\tau = 1$, and if we define

$$s_\tau := \frac{1}{2} \int_0^{t_\tau} \frac{dt}{\sqrt{1 - \tau^2 \cosh^2 t}}, \quad (2)$$

then it is not hard to see that σ has period $8 s_\tau$. On the other hand, when $\tau < 0$, we define $t_\tau > 0$ by the equation $\tau \sinh t_\tau = -1$ and define

$$s_\tau := \frac{1}{2} \int_0^{t_\tau} \frac{dt}{\sqrt{1 - \tau^2 \sinh^2 t}}, \quad (3)$$

once again σ is periodic of period $8 s_\tau$.

2.2 The singular limit of D_τ as $\tau \rightarrow 0$

We now describe some aspects of the surfaces D_τ as $\tau \rightarrow 0$. In this limit, D_τ converges to a singular ‘noded’ surface, which is the infinite union of spheres of radius 1 centered at the points $(0, 0, 2k + 1)$, $k \in \mathbb{Z}$. From (2) or (3) we see that

$$s_\tau = -\frac{1}{4} \log \tau^2 + \mathcal{O}(1) \quad \text{as } \tau \rightarrow 0.$$

To study this limit more closely, notice that the family of rescaled surfaces $\tau^{-2} D_\tau$ converges to a catenoid of revolution around the z -axis. Thus when τ is small, D_τ is well approximated by a sequence of spheres along the z -axis connected by small catenoidal necks. This is central in much of what follows, and so in the remainder of this subsection we make this quantitative by recalling the behavior of the functions σ and κ as $\tau \rightarrow 0$. The expansions below are not hard to derive, but we refer to [8] for detailed proofs.

Definition 2 *The notation $g = \mathcal{O}_{C^\infty}(f)$ means that for any $k \geq 0$ there exists $c_k > 0$ such that*

$$|\partial_s^k g| \leq c_k |f|$$

on the domains of definition of these functions.

To obtain the asymptotics of σ as $\tau \rightarrow 0$, define $r(s) := \tau e^{\sigma(s)}$. This function satisfies the ODE

$$(\partial_s r)^2 = r^2 - \left(\frac{\tau^4}{16} \pm \frac{\tau^2}{2} r^2 + r^4 \right),$$

where the plus or minus sign is chosen according to the sign of τ (+ when $\tau > 0$ and $-$ when $\tau < 0$). Because σ attains its minimum when $s = 0$, $r(s) \geq r(0)$ for all s . Hence we can write $r(s) := r(0) \cosh w(s)$ for some function w . This new function satisfies

$$(\partial_s w)^2 = 1 \mp \frac{\tau^2}{2} - r(0)^2 (\cosh^2 w + 1).$$

Now simply from the definition of $\sigma(0)$ we have

$$r(0) = \frac{\tau^2}{4} + \mathcal{O}(\tau^4) \quad \text{as } \tau \rightarrow 0,$$

and therefore

$$w(s) = \left(1 \mp \frac{\tau^2}{2} \right)^{1/2} s + \mathcal{O}_{C^\infty}(\tau^4 \cosh^2 s).$$

Since $s_\tau \sim -\frac{1}{4} \log \tau^2$, $\tau^4 \cosh^2 s \leq c$ when $|s| \leq 4s_\tau$, and so this estimate is only interesting when $s \in [-4s_\tau, 4s_\tau]$. However, by periodicity we obtain an estimate for w for all $s \in \mathbb{R}$. From here it follows that

$$\kappa(s) = \pm \frac{\tau^2}{2} s + \mathcal{O}_{C^\infty}(\tau^4 \cosh^2 s),$$

according to the sign of τ (with $+$ when $\tau > 0$ and $-$ with $\tau < 0$) but this is only of interest when $s \in [-2s_\tau, 2s_\tau]$. It is possible to refine this argument to obtain a more precise expansion for κ in the larger interval $[-4s_\tau, 4s_\tau]$, but we omit this since it will not be needed.

The image of X_τ restricted to $(0, 2s_\tau] \times S^1$ may also be written as the graph of functions U_τ , which is defined over an annulus in the xy -plane. (The image of X_τ restricted to $[-2s_\tau, 0) \times S^1$ is the graph of $-U_\tau$.) To analyze U_τ as $\tau \rightarrow 0$, we use the function $r = \frac{\tau}{2} e^{\sigma(s)}$, as above and set

$$r_\tau := \frac{\tau}{2} e^{\sigma(-s_\tau)}.$$

For the moment, observe that $r_\tau \sim \tau^{3/2}$ as τ tends to 0. Expanding κ in terms of r near $r = r_\tau$, we find that when $r_\tau/2 \leq r \leq 2r_\tau$,

$$\left| (r \nabla)^k \left(U_\tau(r, \theta) \mp \frac{\tau^2}{4} \log \left(\frac{8r}{\tau^2} \right) \right) \right| \leq c_k \tau^3, \quad \forall k \geq 0, \quad (4)$$

according to the sign of τ (with $-$ when $\tau > 0$ and $+$ with $\tau < 0$). Similar estimates are valid on a larger set, but this behavior near $r = r_\tau$ will be the most crucial later.

2.3 The Jacobi operator on a Delaunay surface

If Σ is a CMC surface, then any surface which is \mathcal{C}^2 close to it may be represented as a normal graph

$$\Sigma_w = \{x + w(x)\nu(x) \mid x \in \Sigma\},$$

where ν is the unit normal vector field and w is a (small) scalar function. Σ_w is itself CMC provided w satisfies a nonlinear second order elliptic equation. This equation will be discussed in more detail later, but it is well-known that its linearization, \mathcal{L} , which is usually called the Jacobi operator, is the sum of the Laplace-Beltrami operator on Σ and the norm squared of the second fundamental form,

$$\mathcal{L} = \Delta_\Sigma + |A_\Sigma|^2.$$

A solution w of the equation $\mathcal{L}w = 0$ is called a Jacobi field on Σ . Ideally, such a function is the tangent vector of a one-parameter family of CMC deformations of Σ . This may not be the case, however, and in general, if w is a Jacobi field, then the elements in any one-parameter family of surfaces $\{\Sigma_{w(\varepsilon)} \mid |\varepsilon| < \varepsilon_0\}$, where $w'(0) = w$, only satisfy the constant mean curvature equation to second order at $\varepsilon = 0$. Nonetheless, an understanding of the Jacobi fields and mapping properties of the Jacobi operator is fundamental to any account of the deformation theory of Σ .

We denote by \mathcal{L}_τ the Jacobi operator associated to the Delaunay surface D_τ . In terms of the isothermal parametrization from §2.1,

$$\mathcal{L}_\tau = \frac{4}{\tau^2 e^{2\sigma}} (\partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma)).$$

We analyze instead the simpler operator

$$L_\tau := \partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma), \quad (5)$$

with the factor $4/(\tau^2 e^{2\sigma})$ removed; it is clear that the mapping properties of one of these operators implies the corresponding properties of the other, and also, their nullspaces are the same. Observe that, since σ is $8s_\tau$ periodic and even, the potential in L_τ is $4s_\tau$ -periodic.

We now undertake a thorough analysis of the operators \mathcal{L}_τ and L_τ .

2.4 Jacobi fields on D_τ

The uniqueness theorem for 2-ended Alexandrov embedded CMC surfaces [6] implies that the only CMC deformations of an entire Delaunay surface are the obvious ones, namely those arising from rigid motions of \mathbb{R}^3 and changes in the Delaunay parameter. The corresponding space of Jacobi fields are either bounded or linearly growing along the Delaunay axis. All other Jacobi fields grow exponentially in one direction or the other along this axis, and hence do not correspond to actual CMC deformations. In the next subsection we describe this former class of ‘geometric’ Jacobi fields, while in the one following that we discuss some features of the latter class. While these do not lead to global CMC deformations, they do correspond to CMC deformations over half of D_τ , and in any case, it is necessary to understand them in order to describe the mapping properties of \mathcal{L}_τ .

2.4.1 Geometric Jacobi fields

As indicated above, there is a special collection of Jacobi fields on D_τ which correspond to explicit geometric deformations of this surface. This family is six-dimensional; there is a three-dimensional space associated to translations in \mathbb{R}^3 , a two-dimensional space associated to the rotations of the Delaunay axis, and a one-dimensional space associated to changing the Delaunay parameter. We now describe six Jacobi fields, which we denote by $\Phi_\tau^{j,\pm}$, $j = 0, \pm 1$, which form a basis for this space. The motivation for this notation will be made apparent in subsection 2.4.2.

We first treat the case where $\tau \neq 1$.

Let us start with $\Phi_\tau^{0,+}$. This Jacobi field corresponds to an infinitesimal translation of D_τ along its axis, and so is obtained by projecting the constant vector field $(0, 0, 1)$ (which is the Killing field associated to this family of translations) along the normal vector field N_τ on D_τ . It is geometrically obvious that $\Phi_\tau^{0,+}$ depends only on s , and is periodic in s , hence is bounded as $s \rightarrow \pm\infty$.

Next, the two Jacobi fields $\Phi_\tau^{\pm 1,+}$ correspond to translations of D_τ in the two directions orthogonal to its axis. These are calculated by projecting the constant vector fields $(1, 0, 0)$ and $(0, 1, 0)$ along N_τ , and hence are once again bounded in s . Moreover,

$$\Phi_\tau^{1,+}(s, \theta) = \phi_\tau^{1,+}(s) \cos \theta, \quad \text{and} \quad \Phi_\tau^{-1,+}(s, \theta) = \phi_\tau^{-1,+}(s) \sin \theta.$$

Continuing, the two Jacobi fields

$$\Phi_\tau^{1,-}(s, \theta) = \phi_\tau^{1,-}(s) \cos \theta, \quad \text{and} \quad \Phi_\tau^{-1,-}(s, \theta) = \phi_\tau^{-1,-}(s) \sin \theta$$

correspond to infinitesimal rotations of D_τ about its axis. These are given by projecting the Killing fields $(z, 0, -x_1)$ and $(0, z, -x_2)$ (corresponding to rotations in the space \mathbb{R}^3 with coordinates (x_1, x_2, z)) along N_τ , and hence grow linearly in s .

Finally, $\Phi_\tau^{0,-}$ corresponds to the derivative of the one parameter family D_τ obtained by varying the Delaunay parameter τ . Since D_τ are surfaces of revolution, this Jacobi field depends only on s . It is linearly growing when $\tau \neq 1$.

The definitions for $\Phi_\tau^{0,-}$, $\Phi_\tau^{\pm 1,-}$ and $\Phi_\tau^{\pm 1,+}$ when $\tau = 1$ are exactly the same, but since D_1 is a cylinder, the definition for $\Phi_1^{0,+}$ above yields 0. There is an intrinsic way to define the Jacobi fields $\Phi_\tau^{0,\pm}$ for all $\tau \in (0, 1]$ by regarding the change of Delaunay parameter and translation along the axis as playing the role of polar coordinates in the space of parameters. More precisely, let (ρ, α) be the ordinary polar coordinates (corresponding to Cartesian coordinates $(y_1, y_2) = (\rho \cos \alpha, \rho \sin \alpha)$). Then there is a smooth map from a small ball in \mathbb{R}^2 into the two-dimensional space of Delaunay surfaces sharing the same axis, given by

$$[0, 1) \times [0, 2\pi) \ni (\rho, \alpha) \longrightarrow D_{1-\rho} + \left(0, 0, \frac{\kappa_{1-\rho}(8s_{1-\rho})}{4\pi} \alpha\right)$$

Then for any value of τ , the Jacobi fields $\Phi_\tau^{0,\pm}$ have the same span as the variations of this family with respect to the Cartesian coordinates.

Further details of these calculations, as well as explicit expressions of these geometric Jacobi fields in terms of σ and κ can be found in [8].

2.4.2 Jacobi fields of exponential type and indicial roots

We now fit the special family of Jacobi fields discussed above into a broader context.

The operator L_τ is invariant with respect to rotations about the Delaunay axis, and is reduced by the eigendecomposition for the cross-sectional operator ∂_θ^2 . This reduces the analysis of L_τ to that of the countable family of operators

$$L_{\tau,j} := \partial_s^2 + (\tau^2 \cosh(2\sigma) - j^2), \quad j = 0, 1, 2, \dots$$

When $j \neq 0$, the operator $L_{\tau,j}$ occurs with multiplicity two, corresponding to the two eigenfunctions $e^{\pm ij\theta}$.

Except in certain exceptional cases described below, for each j there exists a complex number $\zeta_{\tau,j}$, with $\Re \zeta_{\tau,j} \geq 0$, and a basis of solutions $\Psi_\tau^{j,\pm}$ of each of these ordinary differential operators such that

$$\Psi_\tau^{j,\pm}(s + 4s_\tau) = e^{\pm 4\zeta_{\tau,j} s_\tau} \Psi_\tau^{j,\pm}(s).$$

Thus when $\zeta_{\tau,j}$ is real, $\Psi_\tau^{j,\pm}$ grows exponentially as $s \rightarrow \pm\infty$ and decays exponentially as $s \rightarrow \mp\infty$. It happens that either $\zeta_{\tau,j}$ is real or it is pure imaginary. In this case, the corresponding solutions are oscillatory, hence bounded.

The exceptional cases noted above occur when there is one bounded (oscillatory) solution and one solution which grows linearly. For example, we have already seen that this is the case when $j = 0, \pm 1$.

One way to prove these facts is to recall that since the potential in $L_{\tau,j}$ has period $4s_\tau$, there exists a 2×2 matrix $T_{\tau,j}$ such that for any solution w of $L_{\tau,j}w = 0$ on \mathbb{R} ,

$$\begin{pmatrix} w(s + 4s_\tau) \\ \partial_s w(s + 4s_\tau) \end{pmatrix} = T_{\tau,j} \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix}.$$

Using the Wronskian related to the operator $L_{\tau,j}$, it is easy to see that the determinant of this matrix is equal to 1. Since the matrix $T_{\tau,j}$ has real entries, we see that the roots of its characteristic polynomial $\lambda_{\tau,j}^\pm$ satisfy $\lambda_{\tau,j}^+ \lambda_{\tau,j}^- = 1$, and $\lambda_{\tau,j}^+ + \lambda_{\tau,j}^- \in \mathbb{R}$. So we can write them as

$$\lambda_{\tau,j}^\pm = e^{\pm 4\zeta_{\tau,j} s_\tau},$$

where $\zeta_{\tau,j}$ is either real or purely imaginary. In the case where $\lambda_{\tau,j}^\pm = 1$ or -1 it may happen that the matrix $T_{\tau,j}$ cannot be diagonalized and this corresponds to the exceptional cases we were mentioning above. We leave the details of checking the statements in the last paragraph

Definition 3 For each j we define the indicial roots of the operator $L_{\tau,j}$ to be the pair of numbers $\pm\gamma_{\tau,j}$ where

$$\gamma_{\tau,j} := \operatorname{Re} \zeta_{\tau,j} \geq 0.$$

Thus

$$\Gamma_\tau := \{\pm\gamma_{\tau,j} \mid j \in \mathbb{N}\}$$

is the set of all indicial roots of the operator L_τ .

It is not necessary to express these indicial roots exactly (and indeed, it is probably impossible to do so), but the following estimates will suffice for our nefarious purposes.

Proposition 1 *The indicial roots of L_τ satisfy the following properties :*

- (i) *For any $\tau \in (-\infty, 0) \cup (0, 1]$, $\gamma_{\tau,0} = \gamma_{\tau,1} = 0$.*
- (ii) *When $j \geq 2$ and $\tau \in (-\sqrt{j^2 - 2}, 0) \cup (0, 1]$, $\gamma_{\tau,j} > 0$ and $L_{\tau,j}$ satisfies the maximum principle.*
- (iii) *For any $j \geq 2$, $\lim_{\tau \rightarrow 0} \gamma_{\tau,j} = j$.*

Proof: Property (i) follows from the remarks above. Notice that because of the existence of linearly growing solutions, $\zeta_{\tau,j}$, $j = 0, 1$, is identically zero for all τ , instead of just having real part zero.

To prove property (ii), it suffices to show that when τ is in the stated range, the potential in $L_{\tau,j}$ is strictly negative and so this operator satisfies the maximum principle. Hence it cannot have bounded solutions and therefore its indicial roots must be real and nonzero. To show this, first assume that $\tau > 0$. Then

$$\tau^2 \cosh(2\sigma) - j^2 = 2\tau^2 \cosh^2 \sigma - \tau^2 - j^2 = 2 - \tau^2 - j^2 - (\partial_s \sigma)^2,$$

which is strictly negative when $j \geq 2$. On the other hand, when $\tau < 0$,

$$\tau^2 \cosh(2\sigma) - j^2 = 2\tau^2 \sinh^2 \sigma + \tau^2 - j^2 = 2 + \tau^2 - j^2 - (\partial_s \sigma)^2,$$

which is strictly negative when $-\sqrt{j^2 - 2} < \tau < 0$.

Finally, Property (iii) reflects the fact that as τ tends to 0, the potential in L_τ is arbitrarily close to 0 on sets which are arbitrary large. To be more precise, according to Proposition 13 of [8], for any $\eta > 0$ there exist numbers $s_\eta > 0$ and $\tau_0 > 0$ such that when $|\tau| \in (0, \tau_0)$,

$$\tau^2 \cosh(2\sigma) \leq \eta,$$

when $s \in [s_\eta, 4s_\tau - s_\eta]$. It follows from this that the indicial roots of $L_{\tau,j}$ must converge to the indicial roots of the operator $\partial_s^2 - j^2$ as $\tau \rightarrow 0$, see Proposition 20 in [8] for details. \square

This Proposition proves that there exists a number $\tau_* \leq -\sqrt{2}$ such that

$$j \geq 2 \text{ and } \tau \in (\tau_*, 0) \cup (0, 1] \implies \gamma_{\tau,j} > 0, \text{ and } L_{\tau,j} \text{ satisfies the maximum principle.} \quad (6)$$

2.5 Mapping properties of L_τ

We shall call the image by X_τ of $[s_0, \infty) \times S^1$, for any $s_0 \in \mathbb{R}$, a half-Delaunay surface, and sometimes denote it by $D_\tau^+(s_0)$. Our goal in the remainder of this section is to study CMC perturbations of half-Delaunay surfaces with prescribed boundary values. To do this we require rather precise knowledge of the mapping properties of the Jacobi operators, and their inverses, on these surfaces. In fact, we need to understand these mapping properties uniformly as $\tau \rightarrow 0$. In this subsection we give careful statements of the results we need. Results of this type (for fixed τ) were originally proved in [12] for L_τ acting on weighted Sobolev spaces; these were reformulated for the more convenient family of weighted Hölder spaces in [8], and the issue of uniformity in the singular limit was also addressed there.

Let us begin by defining these weighted Hölder spaces on $D_\tau^+(s_0)$.

Definition 4 Let $r \in \mathbb{N}$, $\alpha \in (0, 1)$, and $\mu \in \mathbb{R}$. Then the function space $\mathcal{E}_\mu^{r,\alpha}([s_0, \infty) \times S^1)$ consists of those functions $w \in \mathcal{C}_{loc}^{r,\alpha}([s_0, \infty) \times S^1)$ such that

$$\|w\|_{\mathcal{E}_\mu^{r,\alpha}} := \sup_{s \geq s_0} e^{-\mu s} \|w\|_{C^{r,\alpha}([s, s+1] \times S^1)} < \infty.$$

Here $\|\cdot\|_{C^{r,\alpha}([s, s+1] \times S^1)}$ is the usual Hölder norm on $[s, s+1] \times S^1$. Also, $[\mathcal{E}_\mu^{r,\alpha}([s_0, \infty) \times S^1)]_0$ is the subspace of functions vanishing at $s = s_0$.

Observe that the function $s \mapsto e^{\mu s}$ is in $\mathcal{E}_\mu^{r,\alpha}(\mathbb{R}^+ \times S^1)$.

It is clear that

$$L_\tau : [\mathcal{E}_\mu^{2,\alpha}([s_0, +\infty) \times S^1)]_0 \longrightarrow \mathcal{E}_\mu^{0,\alpha}([s_0, +\infty) \times S^1) \quad (7)$$

is bounded for all $\mu \in \mathbb{R}$. However, it is not Fredholm for every weight. In fact, the existence of a solution of $L_\tau w = 0$ which grows or decays exactly like $e^{\pm\gamma_{\tau,j}s}$ can be used to show that this mapping does not have closed range when $\mu = \pm\gamma_{\tau,j}$ for any j . However, it is proved in [8], cf. also [12], that (7) is Fredholm when $\mu \notin \Gamma_\tau$.

We first consider what happens for τ fixed. For simplicity, denote by $L_\tau(\mu)$ the operator in (7); hence this notation indicates the weighted space on which we are letting L_τ act. A basic observation is that $L_\tau(\mu)$ and $L_\tau(-\mu)$ are essentially dual to one another. (Of course, since we are working on Hölder spaces, this is not quite true. The analogous statement for weighted L^2 spaces is true, however, and this may be used to deduce the following statements.) An important consequence of this is that when $\mu \notin \Gamma_\tau$, then $L_\tau(\mu)$ is surjective if and only if $L_\tau(-\mu)$ is injective; furthermore, if this is the case, then the dimension of the kernel of $L_\tau(\mu)$ is equal to the dimension of the cokernel of $L_\tau(-\mu)$. Summarizing all of this, the precise result is

Proposition 2 Assume that $\mu \in (\gamma_{\tau,j}, \gamma_{\tau,j+1})$ for some $j \in \mathbb{N}$, then $L_\tau(\mu)$ is surjective and has a kernel of dimension $2j + 1$. Thus when $\mu \in (-\gamma_{\tau,j+1}, -\gamma_{\tau,j})$ for some $j \in \mathbb{N}$, then $L_\tau(\mu)$ is injective, and has a cokernel of dimension $2j + 1$.

Notice that the first of these assertions follows from the discussion of Jacobi fields in the last subsection. In fact, we need all $\Psi_\tau^{+,j} e^{\pm ij\theta}$ and $\Psi_\tau^{-,j} e^{\pm ij\theta}$ to lie in $\mathcal{E}_\mu^{r,\alpha}([s_0, \infty))$ in order to take a linear combination of these which vanishes at s_0 .

We shall henceforth assume that $\tau \in (\tau_*, 0) \cup (0, 1]$ and $\mu \in (-\gamma_{\tau,2}, -\gamma_{\tau,1}) = (-\gamma_{\tau,2}, 0)$. Although the previous Proposition states that $L_\tau(\mu)$ is not surjective in this case, there is a way to modify this mapping so as to be surjective by augmenting the domain. In fact, from the comments in the last paragraph, it is clear that if we augment the domain of $L_\tau(\mu)$, for μ in this (negative) range, with a ‘deficiency space’

$$\mathcal{W}_\tau := \text{Span} \{ \Psi_\tau^{j,\pm}(s) e^{ij\theta} \mid j = 0, \pm 1 \},$$

which is simply the span of the geometric Jacobi fields, then the nullspace of $L_\tau(-\mu)$ is contained in this extended domain, i.e.

$$\text{Ker } L_\tau(-\mu) \subset [\mathcal{E}_\mu^{2,\alpha}([s_0, \infty) \times S^1) \oplus \mathcal{W}_\tau]_0.$$

The deficiency subspace \mathcal{W}_τ is 6-dimensional, while from Proposition 2 again, $\dim \text{Ker } L_\tau(-\mu) = 3$. Thus projecting this nullspace onto \mathcal{W}_τ defines a 3-dimensional subspace $\mathcal{N}_\tau \subset \mathcal{W}_\tau$. Choosing any complement \mathcal{K}_τ , so that

$$\mathcal{W}_\tau = \mathcal{N}_\tau \oplus \mathcal{K}_\tau,$$

then

$$L_\tau : [\mathcal{E}_\mu^{2,\alpha}([s_0, \infty) \times S^1) \oplus \mathcal{K}_\tau]_0 \longrightarrow \mathcal{E}_\mu^{0,\alpha}([s_0, \infty) \times S^1) \quad (8)$$

is injective. With slightly more work, as explained in [8], [12], it is possible to show that it is also surjective. Altogether, we have

Proposition 3 For $\tau \in (\tau_*, 0) \cup (0, 1]$, $\mu \in (-\gamma_{\tau,2}, -\gamma_{\tau,1})$ (which is the same as $(-\gamma_{\tau,2}, 0)$ since $\gamma_{\tau,1} = 0$), and any choice of complement \mathcal{K}_τ to \mathcal{N}_τ in \mathcal{W}_τ , then the mapping (8) is an isomorphism.

It is more convenient for us to rephrase this result using the fact that the space of boundary traces $\{w(s_0, \cdot) \mid w \in \mathcal{K}_\tau\}$ lies in the span of $e^{ij\theta}$, $j = 0, \pm 1$:

Proposition 4 Fix $\tau \in (\tau_*, 0) \cup (0, 1]$ and $\mu \in (-\gamma_{\tau,2}, -\gamma_{\tau,1}) = (-\gamma_{\tau,2}, 0)$. Then for any $s_0 \in \mathbb{R}$, there exists a bounded mapping

$$G_{\tau, s_0} : \mathcal{E}_\mu^{0,\alpha}([s_0, \infty) \times S^1) \longrightarrow \mathcal{E}_\mu^{2,\alpha}([s_0, \infty) \times S^1)$$

such that for any $f \in \mathcal{E}_\mu^{0,\alpha}([s_0, \infty) \times S^1)$, the function $w = G_{\tau, s_0}(f)$ satisfies

$$\begin{cases} L_\tau w = f & \text{in } (s_0, \infty) \times S^1 \\ w \in \text{Span}\{e^{-i\theta}, 1, e^{i\theta}\} & \text{on } \{s_0\} \times S^1. \end{cases}$$

Moreover the norm of G_{τ, s_0} is bounded independently of $s_0 \in \mathbb{R}$.

As explained earlier, we must also understand the behaviour of this inverse as $\tau \rightarrow 0$. To explain the statement of this next result, recall that, for all $j \geq 2$, $\gamma_{\tau, j} \rightarrow j$ as $\tau \rightarrow 0$, so that any fixed $\mu \in (-2, -1)$ is eventually in $(-\gamma_{\tau,2}, -\gamma_{\tau,1}) = (-\gamma_{\tau,2}, 0)$ provided τ is chosen small enough.

Proposition 5 Fix any $\mu \in (-2, -1)$. Then, there exists a number $\tau_0 > 0$ such that the norm of G_{τ, s_0} is uniformly bounded, independently of $s_0 \in \mathbb{R}$ and $|\tau| \in (0, \tau_0)$.

The precise range of μ between -2 and -1 is very important here; this result fails when, for example, $\mu \in (-1, 0)$. This is Proposition 21 in [8], and the proof can be found there.

To conclude, we need some facts about the Poisson operator. However, we only require these in the limit as $\tau \rightarrow 0$, and so it suffices to study the Poisson operator for the very simple operator

$$\Delta_0 := \partial_s^2 + \partial_\theta^2$$

on $[0, \infty) \times S^1$.

Lemma 1 For any $h \in \mathcal{C}^{2,\alpha}(S^1)$ such that

$$\int_{S^1} h(\theta) d\theta = \int_{S^1} h(\theta) e^{\pm i\theta} d\theta = 0,$$

(i.e. the function h is orthogonal to 1 and $e^{\pm i\theta}$ in the L^2 sense on S^1) there exists a unique function $w = \mathcal{P}(h) \in \mathcal{E}_{-2}^{2,\alpha}([0, +\infty) \times S^1)$ such that $\Delta_0 w = 0$ on $(0, +\infty) \times S^1$, $w(0, \theta) = h(\theta)$, and moreover, $\|w\|_{\mathcal{E}_{-2}^{2,\alpha}} \leq c \|h\|_{\mathcal{C}^{2,\alpha}}$.

The proof is straightforward.

2.6 CMC surfaces close to a half-Delaunay surface

We shall now study the problem of finding all CMC surfaces near to a given half-Delaunay surface $D_\tau^+(s_0)$. These will be parametrized by their boundary values at $s = s_0$. As before, we require some knowledge of the behavior of this solution as $\tau \rightarrow 0$.

2.6.1 The mean curvature operator on a Delaunay surface

Using the isothermal parametrization (1), the unit normal vector field N_τ on D_τ is given by

$$N_\tau(s, \theta) := (-\tau \cosh \sigma(s) \cos \theta, -\tau \cosh \sigma(s) \sin \theta, \partial_s \sigma(s)), \quad (9)$$

when $\tau \in (0, 1]$ and is given by

$$N_\tau(s, \theta) := (\tau \sinh \sigma(s) \cos \theta, \tau \sinh \sigma(s) \sin \theta, -\partial_s \sigma(s)), \quad (10)$$

when $\tau \in (-\infty, 0)$. Normal graphs over D_τ then admit parametrizations of the form

$$X_w : (s, \theta) \mapsto X_\tau(s, \theta) + w(s, \theta) N_\tau(s, \theta),$$

where w is any function which is suitably small.

The mean curvature of the normal graph of a function w about D_τ is calculated by a fairly complicated nonlinear elliptic expression of w which we shall not write out in full. However, this surface has mean curvature equal to 1 if and only if w is a solution of the corresponding equation, which we write simply as $\mathcal{M}_\tau(w) = 0$. We need to know a bit about the structure of this operator, which is proved in [8] :

Proposition 6 *The equation $\mathcal{M}_\tau(w) = 0$ can be written in the form*

$$L_\tau w = \tau e^\sigma Q_\tau \left(\frac{w}{\tau e^\sigma} \right) \quad (11)$$

where L_τ is the (modified) Jacobi operator (5), and where Q_τ is a nonlinear second order differential operator which satisfies

$$Q_\tau(0) = 0 \quad D_w Q_\tau(0) = 0 \quad \text{and} \quad D_w^2 Q_\tau(0) = 0.$$

The Taylor expansion of Q_τ (in $w/\tau e^\sigma$) has coefficients which are uniformly bounded in s , along with all their derivatives, independently of $\tau \in (\tau_*, 0) \cup (0, 1]$.

We shall specialize now and suppose that $s_0 = -s_\tau$. Then, from the discussion in §2.1, we see that the surface $D_\tau^+(s_\tau)$ (which we now simply call D_τ^+) is nearly flat close to its boundary, and it will be more convenient computationally to use a slightly different parametrization of nearby surfaces, replacing the unit normal N_τ by a small perturbation of it, \bar{N}_τ , which is constant (and in fact vertical, downward pointing) for all s close to $-s_\tau$.

To define \bar{N}_τ , choose a smooth cutoff function χ_τ which is nonnegative and which satisfies $\chi_\tau = 1$ when $s \geq -s_\tau + 2$ and $\chi_\tau = 0$ when $s \leq -s_\tau + 1$. Now set

$$\bar{N}_\tau := \chi_\tau N_\tau - (1 - \chi_\tau)(0, 0, 1).$$

This satisfies

$$|\nabla^k (\bar{N}_\tau \cdot N_\tau - 1)| \leq c_k \tau, \quad \text{for all } k \geq 0 \quad \text{and} \quad s \in [-s_\tau, -s_\tau + 2]. \quad (12)$$

We henceforth use the parametrization

$$\bar{X}_w : (s, \theta) \mapsto X_\tau(s, \theta) + w(s, \theta) \bar{N}_\tau(s, \theta), \quad (s, \theta) \in [-s_\tau, \infty) \times S^1.$$

Denote by $D_\tau^+(w)$ the surface obtained this way.

It follows from (11) and from (12) that $D_\tau^+(w)$ has constant mean curvature equal to 1 if and only if w satisfies a nonlinear equation of the form

$$L_\tau w = \bar{Q}_\tau(w),$$

where

$$\bar{Q}_\tau(w) := \tau \bar{L}_\tau w + \tau e^\sigma \bar{Q}_\tau \left(\frac{w}{\tau e^\sigma} \right).$$

Here \bar{Q}_τ satisfies the same properties as listed for Q_τ in Proposition 6, and in fact these operators agree when $s \geq -s_\tau + 2$. The linear operator $\tau \bar{L}_\tau$ represents the deviation between the linearizations corresponding to the parametrizations X_w using N_τ and \bar{X}_w using \bar{N}_τ . From (12), \bar{L}_τ has coefficients supported in $[-s_\tau, -s_\tau + 2] \times S^1$ which are bounded in any $C^{k,\alpha}$, uniformly in τ . The details of this change of parametrization are contained in [10].

2.6.2 The nonlinear Poisson problem

We are now ready to solve the nonlinear boundary problem $\mathcal{M}_\tau(w) = 0$ on D_τ^+ with the value of w at $s = -s_\tau$ (almost) specified.

Fix $\mu \in (-2, -1)$ and $\delta > 0$. Let h be any element of $C^{2,\alpha}(S^1)$ which is orthogonal to 1 and $e^{\pm i\theta}$, in the L^2 sense, and which satisfies

$$\|h\|_{C^{2,\alpha}} \leq \delta \tau^3.$$

Now define the approximate solution

$$w_h := \mathcal{P}(h)(s + s_\tau),$$

where \mathcal{P} is the Poisson operator for Δ_0 from Lemma 1. It suffices to use this Poisson operator, rather than the one for L_τ , because L_τ is very close to Δ_0 in a long interval around the boundary. Since we are using norms with exponential weight factors, these operators differ by a very small amount in norm. From the bounds in this lemma, the shift by s_τ in the s -variable, and the fact $s_\tau \sim -\frac{1}{4} \log \tau^2$, we have

$$\|w_h\|_{\mathcal{E}_{-2}^{2,\alpha}} \leq c\tau \|h\|_{C^{2,\alpha}}. \quad (13)$$

We shall now search for a solution w , which we write as $w = w_h + v$, of $\mathcal{M}_\tau(w) = 0$. The function v will lie in $\mathcal{E}_\mu^{2,\alpha}([-s_\tau, \infty) \times S^1)$ and should satisfy

$$\begin{cases} L_\tau v = \bar{Q}_\tau(w_h + v) - L_\tau w_h & \text{in } (-s_\tau, \infty) \times S^1 \\ v(-s_\tau, \cdot) \in \text{Span}\{1, e^{\pm i\theta}\}. \end{cases}$$

The reason we are only requiring those eigencomponents of $v(s, \theta) = \sum v_j(s) e^{ij\theta}$ with $|j| \geq 2$ to vanish is that we shall be using the inverse $G_{\tau, -s_\tau}$ from Proposition 4, which does not allow these low eigencomponents to be specified.

We solve this equation by finding a fixed point of the mapping

$$\mathcal{N}_\tau(v) := G_{\tau, -s_\tau} (\bar{Q}_\tau(w_h + v) - L_\tau w_h),$$

From (13) we have

$$\|L_\tau w_h\|_{\mathcal{E}_\mu^{0,\alpha}} \leq c\tau \|h\|_{C^{2,\alpha}}, \quad \|\tau \bar{L}_\tau w_h\|_{\mathcal{E}_\mu^{0,\alpha}} \leq c\tau^{1-\mu/2} \|h\|_{C^{2,\alpha}},$$

and

$$\|\tau e^\sigma \bar{Q}_\tau \left(\frac{w_h}{\tau e^\sigma} \right)\|_{\mathcal{E}_\mu^{0,\alpha}} \leq c\tau^{-(3+\mu)/2} \|h\|_{C^{2,\alpha}}^2.$$

Only the final estimate uses that the norm of h is small, and in fact, it is only really necessary to assume that $\|h\|_{C^{2,\alpha}} \leq c_0 \tau^{3/2}$ for $c_0 > 0$ sufficiently small.

At this point it is straightforward, using Proposition 5, to show that there exists $c_* > 0$ and $\tau_0 > 0$, such that when $|\tau| \in (0, \tau_0)$, the nonlinear mapping \mathcal{N}_τ is a contraction in the ball

$$\mathcal{B} := \left\{ v \mid \|v\|_{\mathcal{E}_\mu^{2,\alpha}} \leq c_* \tau \|h\|_{\mathcal{C}^{2,\alpha}} \right\},$$

and hence \mathcal{N}_τ has a unique fixed point in this ball. Observe that $\tau_0 > 0$ depends on δ while $c_* > 0$ does not depend on δ .

In summary, we have proved

Theorem 6 *Fix $\mu \in (-2, -1)$ and $\delta > 0$. There are numbers $\tau_0 > 0$ and $c > 0$ such that for each τ with $0 < |\tau| < \tau_0$ and for every $h \in \mathcal{C}^{2,\alpha}(S^1)$ which is orthogonal to 1 and $e^{\pm i\theta}$ and which satisfies $\|h\|_{2,\alpha} \leq \delta \tau^3$, there exists an embedded CMC surface $D_\tau(h)$, parameterized by*

$$\bar{X}_w := X_\tau + w \bar{N}_\tau \quad \text{in} \quad [-s_\tau, \infty) \times S^1.$$

The function w here lies in $\mathcal{E}_\mu^{2,\alpha}([-s_\tau, \infty) \times S^1)$ and satisfies $\|w\|_{\mathcal{E}_\mu^{2,\alpha}} \leq c \tau^{-\mu/2} \|h\|_{\mathcal{C}^{2,\alpha}}$ and finally, $w(-s_\tau, \cdot) - h(\cdot) \in \text{Span}\{1, e^{\pm i\theta}\}$.

One of the main reasons for modifying the unit normal vector field N_τ to \bar{N}_τ is because with this definition, the region near the boundary of $D_\tau(h)$ is a vertical graph over an annulus with outer boundary ∂B_{r_τ} in \mathbb{R}^2 . Here $r_\tau = \frac{\tau}{2} e^{\sigma(-s_\tau)} \sim \tau^{3/2}$ is defined in §2.2.

We conclude this subsection with some estimates on the graph function in this representation. To do this, we first define some function spaces :

Definition 5 *For $r \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, we let $\mathcal{C}_\mu^{r,\alpha}(\mathbb{R} - \{0\})$ be the space of functions $w \in \mathcal{C}_{10c}^{r,\alpha}(\mathbb{R} - \{0\})$ such that*

$$\|w\|_{\mathcal{C}_\mu^{r,\alpha}} := \sup_{\rho > 0} \rho^{-\mu} \|w(\rho \cdot)\|_{\mathcal{C}^{r,\alpha}(\overline{B_2} - B_1)} < \infty.$$

If Ω is a closed subset of $\mathbb{R}^2 - \{0\}$, we define the space $\mathcal{C}_\mu^{r,\alpha}(\Omega)$ as the space of restriction of functions of $\mathcal{C}_\mu^{r,\alpha}(\mathbb{R}^2 - \{0\})$ to Ω . This space is naturally endowed with the induced norm.

Notice that the space $\mathcal{C}_\mu^{r,\alpha}(\overline{B_R} - \{0\})$ corresponds precisely to the space $\mathcal{E}_\mu^{r,\alpha}([s_0, \infty) \times S^1)$ where $s_0 = -\log R$, under the elementary change of variables $s = -\log r$. However, we shall not use this obvious change of variables but rather the more complicated change of variables specific to our problem

$$r := \frac{\tau}{2} e^{\sigma(s)},$$

for $s \in [-s_\tau, 0)$ and $\theta \in S^1$. Using the estimates of §2.2, we obtain

$$r \partial_r = (1 + \mathcal{O}(\cosh^{-2} s)) \partial_s, \tag{14}$$

for all $s \in [-s_\tau, 0)$.

We finally translate the surface $D_\tau(h)$ by $\mp \frac{\tau^2}{4} \log\left(\frac{4}{\tau^2}\right)$ along the vertical axis. This surface will still be denoted by $D_\tau(h)$. Using the expansion (4) together with (14), we see that near ∂B_{r_τ} , the surface $D_\tau(h)$ is the graph of the function

$$\overline{B_{r_\tau}} - B_{r_\tau/2} \ni x \mapsto \mp \frac{\tau^2}{4} \log r - W_h(x) + V_{\tau,h}(x), \tag{15}$$

according to the sign of τ (with $-$ when $\tau > 0$ and $+$ with $\tau < 0$). Here W_h denotes the unique harmonic extension of h in the ball B_{r_τ} and $V_{\tau,h}$ is bounded in $\mathcal{C}_0^{2,\alpha}(\overline{B_{r_\tau}} - B_{r_\tau/2})$, which does not depend on δ nor on τ , times τ^3 . This constant τ^3 has its origin in (4).

Observe that, reducing τ_0 if this is necessary, we can assume that the mapping $h \rightarrow V_{\tau,h}$ is continuous and in fact smooth. With little work we also find that

$$\|V_{\tau,\tilde{h}} - V_{\tau,h}\|_{C_0^{2,\alpha}} \leq c\tau^{1+\mu/2} \|\tilde{h} - h\|_{C^{2,\alpha}} \quad (16)$$

for some constant $c > 0$ which does not depend on δ , nor on τ .

3 The geometry and analysis of k -ended CMC surfaces

3.1 Moduli space theory for k -ended CMC surfaces

We now briefly sketch the moduli space theory for k -unduloids as developed in [7]. Because it is no harder to do so, we extend this and consider the deformation theory for complete, finite topology CMC surfaces with k ends, each one of which is asymptotic to a Delaunay unduloid or nodoid with Delaunay parameter $\tau > \tau_*$, where τ_* is defined in (6). The statements and proofs are identical in this slightly broader context.

Definition 6 *The moduli space $\mathcal{M}_{g,k}^{\tau_*}$ consists of the set of all complete constant mean curvature surfaces of finite topology, with genus g and k ends E_1, \dots, E_k such that each end E_j is asymptotic to a half Delaunay surface D_τ^+ with $\tau \in (\tau_*, 0) \cup (0, 1]$.*

Now decompose each $\Sigma \in \mathcal{M}_{g,k}^{\tau_*}$ into a union of a compact component K and ends E_ℓ , $\ell = 1, \dots, k$. For each ℓ choose standard isothermal coordinates (s, θ) for the model Delaunay end D_{τ_ℓ} , so that E_ℓ is parametrized by

$$Y_\ell := X_\ell + w_\ell N_\ell \quad | \quad [0, \infty) \times S^1 \longrightarrow E_\ell.$$

Here X_ℓ is the standard parametrization (1) for D_{τ_ℓ} ; each function w_ℓ decays exponentially and in fact

$$w_\ell \in \mathcal{E}_{-\gamma_{\tau_\ell, 2}}^{2,\alpha}([0, \infty) \times S^1). \quad (17)$$

Definition 7 *For $r \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, let $\mathcal{D}_\mu^{r,\alpha}(\Sigma)$ be the space of functions $v \in C^{r,\alpha}(\Sigma)$ for which*

$$\|v\|_{\mathcal{D}_\mu^{r,\alpha}} := \|v|_K\|_{C^{r,\alpha}} + \sum_{\ell=1}^k \|v \circ Y_\ell|_{E_\ell}\|_{\mathcal{E}_\mu^{r,\alpha}} < \infty.$$

Let $\mathcal{L}_\Sigma = \Delta_\Sigma + |A_\Sigma|^2$ denote the Jacobi operator Σ . Because of the asymptotic structure of the ends of Σ , the various mapping and regularity properties of this operator may be deduced from the analogous properties for \mathcal{L}_{τ_ℓ} . In particular, the set of indicial roots for \mathcal{L}_Σ , given by

$$\Gamma_\Sigma := \{\pm \gamma_{\tau_\ell, j} \quad | \quad j \in \mathbb{N}, \quad \ell = 1, \dots, k\},$$

determine the weighted spaces on which \mathcal{L}_Σ is Fredholm as well as the asymptotic behavior of solutions of the homogeneous equation $\mathcal{L}_\Sigma w = 0$. In particular, from the analysis of §2.5, when $\mu \notin \Gamma_\Sigma$, then

$$\mathcal{L}_\Sigma : \mathcal{D}_\mu^{2,\alpha}(\Sigma) \longrightarrow \mathcal{D}_\mu^{0,\alpha}(\Sigma)$$

is Fredholm. To keep track of the value of the weight parameter, we write this mapping as $\mathcal{L}_\Sigma(\mu)$. As before, the operators $\mathcal{L}_\Sigma(\mu)$ and $\mathcal{L}_\Sigma(-\mu)$ are essentially dual to one another which implies that when $\mu \notin \Gamma_\Sigma$, the operator $\mathcal{L}_\Sigma(\mu)$ is surjective if and only if the operator $\mathcal{L}_\Sigma(-\mu)$ is injective, and moreover $\dim \ker(\mathcal{L}_\Sigma(\mu)) = \dim \operatorname{coker}(\mathcal{L}_\Sigma(-\mu))$. We now give the precise definition of nondegeneracy.

Definition 8 *The surface $\Sigma \in \mathcal{M}_{g,k}^{\tau^*}$ is nondegenerate if $\mathcal{L}_\Sigma(\mu)$ is surjective for all $\mu > 0$ with $\mu \notin \Gamma_\Sigma$, or equivalently, if $\mathcal{L}_\Sigma(-\mu)$ is injective for all $\mu > 0$.*

Thus a surface is nondegenerate if it has no decaying Jacobi fields. Note in particular that from the definition of τ_* , the Delaunay surfaces D_τ are nondegenerate when $\tau > \tau_*$.

The basic result from [7] is

Theorem 7 *Fix any element $\Sigma \in \mathcal{M}_{g,k}^{\tau^*}$. If Σ is nondegenerate, then some neighborhood of Σ in the moduli space $\mathcal{M}_{g,k}^{\tau^*}$ is a real analytic manifold of dimension $3k$. In general, the moduli space is a locally real analytic variety, i.e. there is a neighborhood of $\Sigma \in \mathcal{M}_{g,k}^{\tau^*}$ which is identified, via a real analytic diffeomorphism, with the zero set of a real analytic function defined in a finite dimensional Euclidean space.*

We sketch the main ideas in the proof, but only in the nondegenerate case. The general case is somewhat more intricate, but is based on precisely the same ideas.

The first point is that there exist analogues of the 6 linearly independent geometric Jacobi fields $\Phi_{\tau_\ell}^{j,\pm}$, $j = 0, \pm 1$, on each end E_ℓ , $\ell = 1, \dots, k$ of Σ . These are denoted $\Phi_\ell^{j,\pm}$.

Lemma 2 *Let $\mu \in (-\gamma_{\tau_\ell, 2}, 0)$. Then these geometric Jacobi fields $\Phi_\ell^{j,\pm}$ satisfy*

$$\Phi_\ell^{j,\pm} \circ Y_\ell - \Phi_{\tau_\ell}^{j,\pm} \in \mathcal{E}_\mu^{2,\alpha}(E_\ell).$$

Proof : Except for $\Phi_\ell^{0,-}$, these asymptotics may be deduced from the same constructions as in §2.4.1. The existence and asymptotics of the remaining Jacobi fields $\Phi_\ell^{0,-}$ may be deduced by a perturbation argument using Proposition 4 and (17). \square

The proof of Theorem 7 is based on the implicit function theorem, but to apply this theorem we need to find function spaces on which the nonlinear mean curvature operator acts and on which \mathcal{L}_Σ is surjective. Unfortunately, $\mathcal{L}_\Sigma(\mu)$ is never surjective when $\mu < 0$. This may appear discouraging since the nonlinear operator as given only acts on spaces consisting of functions which decay along the ends. To remedy this, we define the $6k$ -dimensional deficiency space

$$\mathcal{W}_\Sigma := \bigoplus_{\ell=1}^k \text{Span}\{\chi_\ell \Phi_\ell^{j,\pm} \mid j = -1, 0, 1\},$$

where χ_ℓ is a cutoff function equal to 0 near $E_\ell \cap K$ and equal to 1 on $Y_\ell([c, \infty) \times S^1)$ for some $c > 0$.

Proposition 7 *Assume that Σ is nondegenerate and fix $\mu \in (-\inf_\ell \gamma_{\tau_\ell, 2}, 0)$. Then the mapping*

$$\mathcal{L}_\Sigma : \mathcal{D}_\mu^{2,\alpha}(\Sigma) \oplus \mathcal{W}_\Sigma \longrightarrow \mathcal{D}_\mu^{0,\alpha}(\Sigma) \tag{18}$$

is surjective and its nullspace \mathcal{N}_Σ is $3k$ -dimensional. Hence there exists a $3k$ -dimensional subspace $\mathcal{K}_\Sigma \subset \mathcal{W}_\Sigma$ such that

$$\mathcal{L}_\Sigma : \mathcal{D}_\mu^{2,\alpha}(\Sigma) \oplus \mathcal{K}_\Sigma \longrightarrow \mathcal{D}_\mu^{0,\alpha}(\Sigma) \tag{19}$$

is an isomorphism.

The proof can be found in [7].

It is possible to make sense of the mean curvature operator on elements of the domain space in (18) using the fact that elements of \mathcal{W}_Σ correspond to geometric motions as in §2.4.1. Indeed, decomposing $u \in \mathcal{D}_\mu^{2,\alpha}(\Sigma) \oplus \mathcal{W}_\Sigma$ as

$$u = u' + u'',$$

with $u' \in \mathcal{D}_\mu^{2,\alpha}(\Sigma)$ and $u'' \in \mathcal{W}_\Sigma$ we then let Σ_u denote the surface which is the normal graph by the function u' over the surface obtained by slightly deforming the ends of Σ in the manner prescribed by the components of $u'' \in \mathcal{W}_\Sigma$. (More precisely, one defines a $6k$ -dimensional real analytic parameter space \mathcal{P} of geometric deformations of Σ such that the differentials of curves in \mathcal{P} through Σ correspond to the geometric Jacobi fields $\Phi_\ell^{j,\pm}$.) The mean curvature of Σ_u is identified with some function $H(u)$ defined on Σ , and by Proposition 7, when Σ is nondegenerate, the differential of this map with respect to u at $u = 0$ is surjective.

3.2 CMC surfaces close to a k -unduloid

3.2.1 Geometric preparations

Let $\Sigma \in \mathcal{M}_{g,k}^{\tau^*}$ be nondegenerate and fix a point $p_0 \in \Sigma$. Assume by a rigid motion that $p_0 = 0$ and the oriented normal vector to Σ at that point is $(0, 0, -1)$, so that the tangent plane $T_0\Sigma$ is the $x y$ -plane. Then in some neighborhood of 0, Σ can be represented as a vertical graph $z = u_0(x, y)$ for some function u_0 defined on a ball B_ρ . By the assumptions above, $u_0(0) = \nabla u_0(0) = 0$.

Now for any sufficiently small vector $a = (a_{-1}, a_0, a_1) \in \mathbb{R}^3$, we rotate and translate Σ slightly so that this new surface Σ_a is the graph of a function u_a , also defined on B_ρ , such that

$$u_a(0) = a_0, \quad \partial_x u_a(0) = a_1, \quad \text{and} \quad \partial_y u_a(0) = a_{-1}.$$

Also, for $0 < r \leq \rho$, let $\Sigma_{a,r}$ denote the complement in Σ_a of the graph of u_a on B_r . Finally, set

$$p_a := (0, 0, u_a(0)).$$

Next, let us modify the unit normal N_a on Σ_a to a new unit vector field \bar{N}_a on Σ_a such that in $\Sigma_a - \Sigma_{a,\rho/2}$, $\bar{N}_a \equiv (0, 0, -1)$, while on $\Sigma_{a,\rho}$, $\bar{N}_a = N_\Sigma$. The linearization of the mean curvature operator with respect to this new vector field, $\bar{\mathcal{L}}_a$, is a slight perturbation of \mathcal{L}_{Σ_a} . Any mapping property for \mathcal{L}_Σ immediately transforms to one for $\bar{\mathcal{L}}_a$, hence in particular Proposition 7 holds when \mathcal{L}_Σ is replaced by $\bar{\mathcal{L}}_a$.

3.2.2 The mean curvature operator for graphs

The mean curvature operator for the vertical graph of a function $u : \mathbb{R}^2 \supset B_\rho \rightarrow \mathbb{R}$ (with downward pointing normal) is given by

$$H(u) := -\frac{1}{2} \operatorname{div} \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}} \right).$$

Since $\bar{\mathcal{L}}_a w = D_u H|_{u=u_a}(w)$ in $\Sigma_a - \Sigma_{a,\rho/2}$, we obtain that

$$\bar{\mathcal{L}}_a w = \frac{1}{2} \operatorname{div} \left(\frac{\nabla w}{(1 + |\nabla u_a|^2)^{1/2}} \right) - \frac{1}{2} \operatorname{div} \left(\frac{\nabla w \cdot \nabla u_a}{(1 + |\nabla u_a|^2)^{3/2}} \nabla u_a \right)$$

there. The main point is that $\bar{\mathcal{L}}_a$ is close to the standard Laplace operator. More precisely, we have the

Lemma 3 *Fix any $\nu \in \mathbb{R}$. There exist $a_*, c > 0$ such that when $|a_{\pm 1}| < a_*$, then for $0 < \rho_1 < \rho_2$ and any $w \in \mathcal{C}_\nu^{2,\alpha}(\overline{B_{\rho_2}} - B_{\rho_1})$, we have*

$$\|\bar{\mathcal{L}}_a w - \Delta w\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{B_{\rho_2}} - B_{\rho_1})} \leq c(\rho_2(|a_1| + |a_{-1}|) + \rho_2^2) \|w\|_{\mathcal{C}_\nu^{2,\alpha}(\overline{B_{\rho_2}} - B_{\rho_1})},$$

A brief calculation also shows that the equation $H(u) = 1$ can be written as

$$\Delta u + \left(\Delta u |\nabla u|^2 - \frac{1}{2} \nabla^2 u (\nabla u, \nabla u) \right) + 2 \left((1 + |\nabla u|^2)^{3/2} - 1 - \frac{3}{2} |\nabla u|^2 \right) = 1. \quad (20)$$

3.2.3 Mapping properties of $\bar{\mathcal{L}}_a$

The proof of the following result can be found in [2] :

Lemma 4 Fix $-1 < \nu < 0$ and $0 < \rho_1 < \rho_2/2$. Then there exists an operator

$$G_{\rho_1, \rho_2} : \mathcal{C}_{\nu-2}^{0, \alpha}(\overline{B_{\rho_2}} - B_{\rho_1}) \longrightarrow \mathcal{C}_{\nu}^{2, \alpha}(\overline{B_{\rho_2}} - B_{\rho_1})$$

with the following properties. For any $f \in \mathcal{C}_{\nu-2}^{0, \alpha}(\overline{B_{\rho_2}} - B_{\rho_1})$, the function $w = G_{\rho_1, \rho_2}(f)$ is a solution of the problem

$$\Delta w = f \quad \text{in } B_{\rho_2} - \overline{B_{\rho_1}}$$

with boundary data $w = 0$ on ∂B_{ρ_2} and w equal to a constant on ∂B_{ρ_1} . In addition,

$$\|G_{\rho_1, \rho_2}(f)\|_{\mathcal{C}_{\nu}^{2, \alpha}} \leq c \|f\|_{\mathcal{C}_{\nu-2}^{0, \alpha}}$$

for some constant $c > 0$ independent of ρ_1 and ρ_2 .

We now define the function spaces suitable for the analysis of these operators on the punctured surfaces $\Sigma_a - \text{setminus}\{p_a\}$. We identify all functions defined near p_a on Σ_a with functions on $B_\rho \subset \mathbb{R}^2$ using the fixed coordinates induced by the functions u_a .

Definition 9 Let $r \in \mathbb{N}$, $0 < \alpha < 1$ and $\nu, \mu \in \mathbb{R}$. Then $\mathcal{D}_{\mu, \nu}^{r, \alpha}(\Sigma_a - \{p_a\})$ is the space of functions $w \in \mathcal{C}_{\text{loc}}^{r, \alpha}(\Sigma_a - \{p_a\})$ for which

$$\|w\|_{\mathcal{D}_{\mu, \nu}^{r, \alpha}} := \|w\|_{\mathcal{D}_{\mu}^{r, \alpha}(\Sigma_{a, \rho})} + \|w \circ u_a^{-1}\|_{\mathcal{C}_{\nu}^{r, \alpha}(B_\rho - \{0\})} < \infty.$$

Thus μ is the weight on the other ends of Σ_a and ν regulates growth or decay near p_a . As usual, if Ω is any closed subset of $\Sigma_a - \{p_a\}$, then $\mathcal{D}_{\mu, \nu}^{r, \alpha}(\Omega)$ is the space of restriction of functions in $\mathcal{D}_{\mu, \nu}^{r, \alpha}(\Sigma_a - \{p_a\})$ to Ω , endowed with the induced norm.

Proposition 8 Fix $\mu \in (-\inf_\ell \gamma_{\tau_\ell, 2}, 0)$, $-1 < \nu < 0$ and $0 < \alpha < 1$. Recall the number r_τ defined in §2.2. Then for τ in some small interval $(0, \tau_0)$, there exists an operator

$$\bar{G}_\tau : \mathcal{D}_{\mu, \nu-2}^{0, \alpha}(\Sigma_{a, r_\tau}) \longrightarrow \mathcal{D}_{\mu, \nu}^{2, \alpha}(\Sigma_{a, r_\tau}) \oplus \mathcal{K}_\Sigma,$$

such that for all $f \in \mathcal{D}_{\mu, \nu-2}^{0, \alpha}(\Sigma_{a, r_\tau})$, the function $w = \bar{G}_\tau(f)$ is a solution of

$$\bar{\mathcal{L}}_a w = f$$

in Σ_{a, r_τ} , with w constant on $\partial \Sigma_{a, r_\tau}$. Furthermore, there exists $c > 0$ which is independent of τ and f , such that

$$\|w\|_{\mathcal{D}_{\mu, \nu}^{2, \alpha} \oplus \mathcal{K}} \leq c \|f\|_{\mathcal{D}_{\mu, \nu-2}^{0, \alpha}}.$$

Proof: First choose $\rho_* \in (0, \rho]$ and define

$$w := \chi G_{\rho_*, r_\tau}(f),$$

where G_{ρ_*, r_τ} is the operator in Lemma 4 and where χ is a radial cutoff function identically equal to 1 in $B_{\rho_*/2}$ and equal to 0 outside B_{ρ_*} . By construction the support of the function g defined by $g := \Delta w - f$ is disjoint from B_{ρ_*} . Now, using Proposition 7, we define

$$w' := \chi' \bar{\mathcal{L}}_a^{-1}(g),$$

where χ' is another radial cutoff function equal to 1 outside $B_{2r\tau}$ and vanishing in $B_{r\tau}$.

It is easy to check that

$$\|f - \bar{\mathcal{L}}_a w'\|_{\mathcal{D}_{\mu,\nu-2}^{0,\alpha}} \leq c(\rho_*^2 + |a_{\pm 1}| \rho_* + r_\tau^{-\nu}) \|f\|_{\mathcal{D}_{\mu,\nu-2}^{0,\alpha}},$$

and also that

$$\|w'\|_{\mathcal{D}_{\mu,\nu}^{2,\alpha}} \leq c \|f\|_{\mathcal{C}_{\mu,\nu-2}^{0,\alpha}},$$

for some constant $c > 0$ which depends neither on τ nor on ρ_* . The result follows immediately by a simple perturbation argument, provided ρ_* and τ are chosen small enough. \square

We conclude this subsection with a simple result regarding the Poisson operator for Δ which is close to Lemma 1 in spirit.

Lemma 5 *For any $g \in \mathcal{C}^{2,\alpha}(S^1)$ such that $\int_{S^1} g = 0$, there exists a unique function $w \in \mathcal{C}_{-1}^{2,\alpha}(\mathbb{R}^2 - B_1)$ which is a solution of*

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^2 - B_1 \\ w = g & \text{on } \partial B_1, \end{cases} \quad (21)$$

and which satisfies $\|w\|_{\mathcal{C}_{-1}^{2,\alpha}} \leq c \|g\|_{\mathcal{C}^{2,\alpha}}$ for some constant $c > 0$ which does not depend on g .

We write this solution of (21) as $P(g)$.

3.2.4 The nonlinear Poisson problem

Using Proposition 7 let γ_0 be the solution of

$$\bar{\mathcal{L}}_a \gamma = -2\pi \delta_0, \quad \text{in } \Sigma_a - \{p_a\}, \quad (22)$$

with $\gamma + \chi \log r \in \mathcal{D}_\mu^{2,\alpha}(\Sigma_a) \oplus \mathcal{K}_{\Sigma_a}$, where χ is, as usual, a cutoff function equal to 1 in $\Sigma_a - \Sigma_{a,\rho/2}$ and vanishing in $\Sigma_{a,\rho}$ and $r = |(x, y)|$.

We now observe that there are three global Jacobi fields Φ_j , $j = 0, \pm 1$, on Σ_a such that

$$\Phi_0 = 1 + \mathcal{O}(r), \quad \Phi_1 = r \cos \theta + \mathcal{O}(r^2), \quad \text{and} \quad \Phi_{-1} = r \sin \theta + \mathcal{O}(r^2)$$

in B_ρ . Indeed, Φ_0 is obtained by projecting the constant Killing field $(0, 0, 1)$ (corresponding to a vertical translation) onto the normal vector field, while $\Phi_{\pm 1}$ are obtained by projecting the Killing fields $(z, 0, -x)$ and $(0, z, -y)$ (corresponding to the two rotations about the horizontal axes). By adding an appropriate linear combination of these three Jacobi fields to γ , we may also assume that $\gamma + \chi \log r$ and its gradient vanishes at 0.

Lemma 6 *If γ is defined as above, then for all $k \geq 0$, there exists a constant $c_k > 0$ such that*

$$|\nabla^k (\gamma + \log r)| \leq c_k r^{2-k} |\log r| \quad (23)$$

in B_ρ , where $r := |x|$.

The proof is straightforward.

Now fix $\mu \in (-\inf_\ell \gamma_{\tau_\ell, 2}, 0)$, $-2/3 < \nu < 0$ and $\delta > 0$. Suppose that $a_j \in \mathbb{R}$, $j = 0, \pm 1$, satisfy

$$|a_0| + \tau^{3/2} |a_{\pm 1}| \leq \delta \tau^3.$$

Finally, for any $g \in \mathcal{C}^{2,\alpha}(S_1)$ such that $\int_{S^1} g = 0$, and $\|g\|_{\mathcal{C}^{2,\alpha}} \leq \delta \tau^3$, we define

$$w_g := P(g)(r_\tau^{-1} u_a^{-1}(\cdot)).$$

This is the bounded harmonic extension of g to $\mathbb{R}^2 - B_{r_\tau}$. By Lemma 5 there exists a constant $c > 0$ such that

$$\|w_g\|_{\mathcal{C}^{2,\alpha}_{\mu,-1}} \leq c r_\tau \|g\|_{\mathcal{C}^{2,\alpha}}. \quad (24)$$

Now define

$$\tilde{w}_{\tau,g} := \chi w_g - \frac{\tau^2}{4} \gamma,$$

where χ is the same cutoff function used in the definition of γ .

We wish to find $v \in \mathcal{D}_{\mu,\nu}^{2,\alpha}(\Sigma_{a,r_\tau}) \oplus \mathcal{K}_{\Sigma_a}$ so that the graph over Σ_a of the function $w := \tilde{w}_{\tau,g} + v$, using the vector field \bar{N}_a is CMC. (Note that, just as in §3.1, we do not actually consider the graph of w over Σ_{a,r_τ} , but rather decompose $w = w' + w''$ and consider the graph of w' over some deformation $\Sigma_{a,r_\tau,w''}$ of Σ_{a,r_τ} .) This is equivalent to finding the solution of some nonlinear elliptic operator which we write formally simply as

$$\bar{\mathcal{L}}_a v = Q(\tilde{w}_{g,\tau} + v) - \bar{\mathcal{L}}_a \tilde{w}_{g,\tau}.$$

As usual, we use Proposition 8 to rephrase this as a fixed point problem for the operator

$$\mathcal{M}_\tau(v) := \bar{\mathcal{G}}_\tau(Q(\tilde{w}_{g,\tau} + v) - \bar{\mathcal{L}}_a \tilde{w}_{g,\tau}).$$

From the estimates (23) and (24), along with (20), it is easy to show that

$$\|\bar{\mathcal{L}}_a \tilde{w}_{g,\tau}\|_{\mathcal{D}_{\mu,\nu-2}^{2,\alpha}} \leq c \tau^4, \quad \text{and} \quad \|Q(\tilde{w}_{g,\tau})\|_{\mathcal{D}_{\mu,\nu-2}^{2,\alpha}} \leq c_\delta \tau^{9/2},$$

where $c > 0$ does not depend on δ , provided τ is small enough.

It is then routine to show that for any fixed $\delta > 0$, there exist $c_* > 0$ and $\tau_0 > 0$ such that when $\tau \in (0, \tau_0)$, the nonlinear mapping \mathcal{M}_τ is a contraction mapping in the ball

$$\hat{\mathcal{B}} := \{v \mid \|v\|_{\mathcal{D}_{\mu,\nu}^{2,\alpha} \oplus \mathcal{K}} \leq c_* \tau^4\},$$

and thus has a unique fixed point in this ball. Again, τ_0 depends on δ while c_* does not.

In summary, we have proved the

Theorem 8 *Fix constants $\mu \in (-\inf_t \gamma_{\tau_t,2}, 0)$, $\nu \in (-2/3, 0)$, $\alpha \in (0, 1)$ and $\delta > 0$. Suppose that τ sufficiently small and $a \in \mathbb{R}^3$ with $|a_0| + \tau^{3/2} |a_{\pm 1}| \leq \delta \tau^{3/2}$. Then for any $g \in \mathcal{C}^{2,\alpha}(S^1)$ with $\int_{S^1} g = 0$ and $\|g\|_{\mathcal{C}^{2,\alpha}} \leq \delta \tau^3$, there exists a CMC surface with boundary which is close to Σ_{a,r_τ} and such that a collar neighbourhood of its boundary can be parameterized as a vertical graph*

$$\overline{B_{2r_\tau}} - B_{r_\tau} \ni x \longrightarrow -\frac{\tau^2}{4} \log r + a_0 + a_1 r \cos \theta + a_{-1} \sin \theta - \widehat{W}_g(x) + \widehat{V}_{\tau,a,g}(x). \quad (25)$$

Here \widehat{W}_g is the unique bounded harmonic extension of g to $\mathbb{R}^2 - B_{r_\tau}$ and $\widehat{V}_{\tau,a,g}$ is a function which is bounded in $\mathcal{C}_0^{2,\alpha}(\overline{B_{2r_\tau}} - B_{r_\tau})$ by $c\tau^3$, where $c > 0$ does not depend on δ or τ .

It is only in the final estimate that we need to restrict ν to lie in $(-2/3, 0)$.

We can also see from this that $\widehat{V}_{\tau,a,g}$ depends smoothly on the parameters (τ, a, g) . In fact, with a little work, we find that

$$\|\widehat{V}_{\tilde{\tau},\tilde{a},\tilde{g}} - V_{\tau,a,g}\|_{\mathcal{C}_0^{2,\alpha}} \leq c \tau^{3\nu/2} (\tau^{3/2} \|\tilde{g} - g\|_{\mathcal{C}^{2,\alpha}} + \tau^4 |(\tilde{a}_{-1}, \tilde{a}_1) - (a_{-1}, a_1)| + \tau^3 |\tilde{\tau} - \tau|) \quad (26)$$

for some constant $c > 0$ which not depending on δ or τ .

Remark 1 *A similar construction holds mutatis mutandis if one replaces γ by $-\gamma$ in the formula defining $\tilde{w}_{\tau,g}$. We denote by $\Sigma^\pm(\tau, a, g)$ the surface constructed here, where for Σ^+ we have used $+\gamma$ and for Σ^- we have used $-\gamma$ in the construction.*

4 Adding a Delaunay end

We now assemble the results established in the previous sections and prove the main gluing theorem

Theorem 9 *Let $\Sigma \in \mathcal{M}_{g,k}^{\tau^*}$ be nondegenerate. Then there exists $\tau_0 > 0$, \mathcal{U} a neighborhood of p in Σ and a smooth gluing map*

$$\mathfrak{G}_\Sigma : \mathcal{U} \times \{\tau \mid 0 < |\tau| < \tau_0\} \longrightarrow \mathcal{M}_{g,k+1}^{\tau^*}$$

where (p, τ) is mapped to the surface obtained by gluing D_τ^+ to Σ at p . The surfaces in the range of this map \mathfrak{G} are all nondegenerate.

We prove this theorem in the next two subsections.

4.1 Step 1 : existence

Let $\Sigma \in \mathcal{M}_{g,k}^{\tau^*}$ be nondegenerate and fix a point $p \in \Sigma$ and $\delta > 0$ sufficiently large enough, and apply the results of the previous sections. These give the existence of a $\tau_0 > 0$ such that when $0 < |\tau| < \tau_0$, there are two families of CMC surfaces with boundary : $D_\tau^+(h)$ and $\Sigma^\pm(\tilde{\tau}, a, g)$. The first of are the perturbed half-Delaunay surfaces, where the boundary values $h \in \mathcal{C}^{2,\alpha}(S^1)$ satisfies

$$\int_{S^1} h = \int_{S^1} h e^{\pm i\theta} = 0, \quad \text{and} \quad \|h\|_{2,\alpha} \leq \kappa \tau^3.$$

The second are the CMC perturbations of $\Sigma_{a,\rho/2}$, where the parameters are defined as follows. First,

$$\tilde{\tau} := \sqrt{\tau^2 + 4t},$$

where $t \in \mathbb{R}$ is any number such that

$$|\log \tau| |t| \leq \delta \tau^3;$$

next $a \in \mathbb{R}^3$ satisfies

$$|a_0| + \tau^{3/2} |a_{\pm 1}| \leq \delta \tau^3;$$

finally, $g \in \mathcal{C}^{2,\alpha}(S^1)$ is such that

$$\int_{S^1} g = 0 \quad \text{and} \quad \|g\|_{2,\alpha} \leq \kappa \tau^3.$$

The superscript on $\Sigma^\pm(\tilde{\tau}, a, g)$ should be taken to be + when $\tau > 0$ and - when $\tau < 0$. On the other hand the + superscript on $D_\tau^+(h)$ only serves to remind us that we are dealing with a half-Delaunay surface, since τ (and hence its sign) appears explicitly.

If we can now find choices of g, h, t and a so that

$$\Sigma^\pm(\tilde{\tau}, a, g) \sqcup D_\tau^+(h)$$

is \mathcal{C}^1 across the joining interface, then the existence will be established. This is because the equation is noncharacteristic at the boundaries where these surfaces intersect and standard regularity theory for the mean curvature equation shows that this union is then a \mathcal{C}^∞ surface. We denote the CMC surfaces obtained in this way by $\mathfrak{G}_\Sigma(p, \tau)$. These look like Σ with an additional Delaunay end, unduloidal when $\tau > 0$ and nodoidal when $\tau < 0$, attached at the point $p \in \Sigma$.

We now show that we can match the Cauchy data by choosing these parameters correctly. By construction, near each of their respective boundaries, the surfaces $\Sigma^\pm(\tilde{\tau}, a, g)$ and $D_\tau^+(h)$ are vertical graphs over the xy -plane. Therefore, in view of (15) and (25), it suffices to solve the equations

$$\begin{cases} \mp t \log r + a_0 + a_1 r \cos \theta + a_{-1} r \sin \theta - \widehat{W}_g + \widehat{V}_{\tilde{\tau}, a, g} &= -W_h + V_{\tau, h} \\ \mp t + a_1 r \cos \theta + a_{-1} r \sin \theta - r \partial_r \widehat{W}_g + r \partial_r \widehat{V}_{\tilde{\tau}, a, g} &= -r \partial_r W_h + r \partial_r V_{\tau, h}; \end{cases} \quad (27)$$

all functions here are evaluated on ∂B_{r_τ} . These identities correspond to the coincidence of the Dirichlet and Neumann data, respectively, of the two graphs.

To solve these, recall that the mapping

$$\mathcal{P} : \mathcal{C}^{2, \alpha}(S^1) \ni h \longrightarrow r_\tau \partial_r (W_h - \widehat{W}_h)(r_\tau \cdot) \in \mathcal{C}^{1, \alpha}(S^1)$$

is an isomorphism such that both it and its inverse have norm uniformly bounded in τ . To see this first observe that this mapping does not depend on r_τ , so we may as well assume that $r_\tau \equiv 1$. Next, note that \mathcal{P} is a linear first order elliptic self-adjoint pseudodifferential operator with principal symbol $-2|\xi|$. Thus, it is enough to check that it is injective. Now, if $\mathcal{P}(h) = 0$ then the function w which equals \widehat{W}_h in $\mathbb{R}^2 - B_1$ and W_h in B_1 is a global solution of $\Delta w = 0$ on all of \mathbb{R}^2 ; furthermore, w belongs to $\mathcal{C}^{2, \alpha}(B_1) \cap [\mathcal{C}_{-1}^{2, \alpha}(\mathbb{R}^2 - B_1) \oplus \text{Span}\{\log r\}]$. No such function exists, and so first w and then h must be trivial.

Now define

$$h^- := g \pm t \log r_\tau \quad \text{and} \quad h^+ := h + a_0 + a_1 r_\tau \cos \theta + a_{-1} r_\tau \sin \theta.$$

It is easy to see that (27) reduces to a fixed point problem

$$(h^+, h^-) = \mathbf{C}_\tau(h^+, h^-)$$

in the space $\mathcal{E} := (\mathcal{C}^{2, \alpha}(S^1))^2$. But (15) and (26) imply that $\mathbf{C}_\tau : \mathcal{E} \longrightarrow \mathcal{E}$ is a contraction mapping defined in the ball of radius $\delta \tau^3$ in \mathcal{E} into itself, provided δ is sufficiently large τ is sufficiently small. This gives the fixed point and completes the proof of the existence of the gluing map.

4.2 Step 2 : nondegeneracy

We now prove that the surfaces $\mathfrak{G}_\Sigma(p, \tau)$ constructed above are nondegenerate when τ is small enough. The proof is by contradiction. Assume that this is not the case, so that there exists a sequence $\tau_n \rightarrow 0$, a sequence $p_n \in \Sigma$ tending to p and a sequence of surfaces

$$\Sigma_n := \mathfrak{G}_\Sigma(p_n, \tau_n),$$

for which the operators \mathcal{L}_{Σ_n} are not injective on $\mathcal{D}_{\mu_n}^{2, \alpha}(\Sigma_n)$, for some $\mu_n < 0$. In other words, for each n there exists a nontrivial function $w_n \in \mathcal{D}_{\mu_n}^{2, \alpha}(\Sigma_n)$ such that $\mathcal{L}_{\Sigma_n} w_n = 0$. Without loss of generality, we can assume that $p = 0$ and that the tangent plane of Σ at 0 is the xy -plane.

The set indicial roots of \mathcal{L}_{Σ_n} decomposes into two groups :

- (i) Those indicial roots associated to the ends of Σ_n which converge to the ends of Σ ;
- (ii) Those indicial roots associated to the end of Σ_n which is a perturbation of D_{τ_n} .

As $n \rightarrow \infty$, elements of the first subset converge to the corresponding indicial roots of Σ , while elements of the second subset converge to elements in $\mathbb{Z} - \{\pm 1\}$. Hence there exist $\mu \in (-\inf_\ell \gamma_{\tau_\ell, 2}, 0)$

and $\tilde{\mu} \in (-2, -1)$ such that when n is large, w_n is bounded by a multiple of $e^{\mu s}$ on each end of the ‘original’ ends of Σ_n and by a multiple of $e^{\tilde{\mu} s}$ on the new end of Σ_n .

Choose $\mu' \in (\mu, 0)$ and $\tilde{\mu}' \in (\tilde{\mu}, -1)$. By construction, Σ_n decomposes into the union of surfaces, one of which we denote by $\tilde{\Sigma}_n$ and has boundary $\partial\tilde{\Sigma}_n$ which is a normal graph a compact portion of Σ . The other, which we denote by \tilde{D}_n , is a normal graph over the Delaunay surface $D_{\tau_n}^+$. Now define a weight function $\zeta_n > 0$ on Σ_n , such that

- $\zeta_n \sim e^{\mu s}$ on each end of $\tilde{\Sigma}_n$,
- $\zeta_n \sim r^{-\tilde{\mu}}$ near $\partial\tilde{\Sigma}_n$,
- $\zeta_n \sim \tau^{-2\tilde{\mu}} e^{\tilde{\mu} s}$ on \tilde{D}_n .

Here $f \sim g$ means that $1/2 \leq f/g \leq 2$. We use the usual cylindrical coordinates to parametrize the various ends of $\tilde{\Sigma}_n$, while in a small annulus near $\partial\tilde{\Sigma}_n$ we use polar coordinates.

Normalize the sequence w_n so that

$$\sup_{\Sigma_n} \zeta_n^{-1} w_n = 1.$$

By the choices of μ and $\tilde{\mu}$, these suprema are achieved at some point $q_n \in \Sigma_n$. We distinguish a few cases according to the behavior of the sequence of points q_n . Note that as $n \rightarrow \infty$, the surfaces Σ_n converge to the union of the original surface Σ and an infinite union of spheres of radius 1 centered at the points $(0, 0, 2j + 1)$, for $j \in \mathbb{N}$.

Case 1 : Suppose that (some subsequence of) the points q_n remain in a fixed end of $\tilde{\Sigma}_n$ and tend to infinity. Then $q_n = (s_n, \theta_n) \in [0, \infty) \times S^1$, where $s_n \rightarrow \infty$. Possibly extracting another subsequence, we may assume that

$$\tilde{w}_n := e^{-\mu s_n} w_n(s_n + \cdot, \cdot)$$

converges uniformly on any compact subset of $\mathbb{R} \times S^1$ to a limiting function w_∞ which is a solution of

$$\mathcal{L}_{\tau_\ell} w_\infty = 0$$

on the full Delaunay surface D_{τ_ℓ} . Here τ_ℓ is the Delaunay parameter for that end in the surface Σ . In addition, $|w_\infty| \leq c e^{\mu s}$. To see this is impossible, decompose $w_\infty = \sum_j w_j e^{i j \theta}$. By the choice of μ we have $w_0 = w_{\pm 1} = 0$ since the nontrivial solutions of $\mathcal{L}_{\tau_\ell} w = 0$ in these eigenspaces are either bounded or blow up linearly at both ends of D_{τ_ℓ} . On the other hand, the restriction of \mathcal{L}_{τ_ℓ} to the eigenspaces with $|j| \geq 2$ satisfies the maximum principle. This implies that $w_j = 0$ for all other values of j . This is a contradiction.

Case 2 : Next, suppose that the sequence q_n converges to a point $q_\infty \in \Sigma - \{0\}$. We may clearly assume that w_n converges uniformly on any compact of $\Sigma - \{0\}$ to a solution of

$$\mathcal{L}_\Sigma w_\infty = 0$$

defined on $\Sigma - \{0\}$. But $|w_\infty| \leq c r^{-\tilde{\mu}}$ as $r \rightarrow 0$ and $|w_\infty| \leq c e^{\mu s}$ at all other ends of Σ . Hence it is a Jacobi field which decays exponentially at all ends, and so must vanish by nondegeneracy.

Case 3 : Finally, suppose that p_n tends to a point on the union of spheres centered at the points $(0, 0, 2j + 1)$, for $j \in \mathbb{N}$. But p_n corresponds to (s_n, θ_n) in the parameterization of Delaunay end of parameter τ_n . Both \tilde{D}_n and $\tilde{\Sigma}_n - \tilde{\Sigma}_{n,\rho}$ are normal graphs over D_{τ_n} when ρ is small enough. Thus we may define the rescaled sequence

$$\tilde{w}_n(s, \theta) := e^{\mu s_n} w_n(s + s_n, \theta).$$

It is proved in [8] that, as $\tau_n \rightarrow 0$, the term of order 0 in \mathcal{L}_{τ_n} converges either to 0 or to $2 \cosh^{-2} s$ on compact subsets. It is then easy to see that, some subsequence of $(\tilde{w}_n)_n$ converges to w_∞ , a nontrivial solution of one of the following equations :

$$\Delta_0 w_\infty = 0,$$

or

$$\Delta_0 w_\infty + \frac{2}{\cosh^2(s + \bar{s})} w_\infty = 0, \quad \text{for some } \bar{s} \in \mathbb{R},$$

on $\mathbb{R} \times S^1$. In addition, $|w| \leq c e^{\tilde{\mu}s}$. This is once again impossible. To see this, decompose $w_\infty = \sum_j w_j(s) e^{ij\theta}$. By the choice of $\tilde{\mu}$ we get $w_0 = w_{\pm 1} = 0$ since nontrivial solutions of the homogeneous problems on the eigenspaces $j = 0, \pm 1$ decay at most like $\cosh^{-1} s$ at ∞ , whereas $\tilde{\mu} \in (-2, -1)$. On the other hand, the restrictions of these two operators to the eigenspaces with $|j| \geq 2$ satisfy the maximum principle, and so all the remaining components $w_j = 0$. This is again a contradiction.

We have now ruled out all possibilities, and so the surfaces $\mathfrak{G}_\Sigma(p, \tau)$ are nondegenerate when τ is sufficiently small.

5 Analysis of the forgetful map

In the remainder of this paper we apply this gluing construction to study some aspects of the topology of $\mathcal{M}_{g,k}$; this will be accomplished using a natural mapping from this space into the more familiar and better understood Teichmüller space of closed Riemann surfaces of genus g with an ordered k -tuple of points deleted. We let $\mathcal{T}_{g,k}$ denote this latter space; any element has the form $\bar{\Sigma} - \{p_1, \dots, p_k\}$, where $\bar{\Sigma}$ is a compact Riemann surface and the p_j are distinct points on it.

Any CMC surface $\Sigma \in \mathcal{M}_{g,k}$ is conformally equivalent to some element of this Teichmüller space. This allows us to define a forgetful map

$$\mathcal{F}_{g,k} : \mathcal{M}_{g,k} \longrightarrow \mathcal{T}_{g,k}, \quad (28)$$

which is given by forgetting the geometric (i.e. CMC) structure of a surface and remembering only its conformal structure. For convenience, we shall usually drop the subscripts (g, k) from the notation for this map.

Our basic goal is to understand the image $\mathcal{I}_{g,k} = \mathcal{I}$ of this mapping. We first show that \mathcal{F} is real analytic. A recent result of Kusner [5] states that (a slight modification of) \mathcal{F} is proper. Together, these results show that \mathcal{I} is a closed, subanalytic set. As such, it is stratified by real analytic submanifolds, and so we may define the codimension of \mathcal{I} as the codimension of its stratum of maximal dimension. Using the preceding gluing construction, we prove that \mathcal{F} is surjective when $g = 0$ while for each $g > 0$ its image has codimension which is uniformly bounded (depending on g) as $k \rightarrow \infty$. Examined more carefully, this argument also shows that $\mathcal{M}_{g,k}$ detects much of the topology of this Teichmüller space. We conclude by summarizing the ramifications of these results for the differential topological structure of the CMC moduli space.

We stress that the basic ideas here very intuitive, granting the main gluing theorem. Roughly speaking, the fact that $\mathcal{F}_{0,k}$ is surjective when $g = 0$ follows inductively from the fact that we can glue a half-Delaunay surface at any point p of any fixed nondegenerate $\Sigma \in \mathcal{M}_{0,k'}$, $3 \leq k' \leq k$, without changing the conformal structure much away from p . The result is complicated when $g > 0$ by the fact that there may be constraints on the image of \mathcal{F} which we do not see directly; this is where the real analyticity enters, for it implies that the image does lie in a well-behaved set. The uniform boundedness of the codimension of the image means essentially that the only serious constraints on the image occur when k is small, and that when k is large enough, the (conformal) location of the ends may be chosen freely.

5.1 Analyticity of \mathcal{F}

We show now that \mathcal{F} is analytic, not only when near smooth points of $\mathcal{M}_{g,k}$, but even as a map of (possibly singular) real analytic spaces.

To state this result precisely, fix $\Sigma \in \mathcal{M}_{g,k}$ and let g denote its induced metric. We parametrize a neighborhood of Σ in the space of all nearby surfaces in the usual way, using a finite dimensional family of deformations of Σ which preserve the CMC structure of the ends and then taking normal perturbations of these. Part of the main theorem in [7] is that for any $\Sigma \in \mathcal{M}_{g,k}$, there is a neighborhood in the CMC moduli space which lies in some open finite dimensional analytic submanifold \mathcal{Q} of this infinite dimensional space. When Σ is nondegenerate, then we may assume that \mathcal{Q} is a real analytic coordinate chart in $\mathcal{M}_{g,k}$, but in general, the CMC moduli space is the zero set of a real analytic function on \mathcal{Q} . To be definite, we regard each element of q as an embedding of a fixed surface Σ_0 into \mathbb{R}^3 ; we let q_0 denote the base embedding, corresponding to the original CMC surface Σ . From that construction, we may even assume that every $q \in \mathcal{Q}$ is an analytic embedding, but this is not so important here.

Proposition 9 *Suppose $\Sigma \in \mathcal{M}_{g,k}$, and let the finite dimensional analytic manifold \mathcal{Q} be chosen as above. Then the natural extension of the forgetful map \mathcal{F} which assigns to any $q \in \mathcal{Q}$ the element of $\mathcal{T}_{g,k}$ determined by $q(\Sigma_0)$ is a real analytic mapping.*

Since the CMC moduli space (locally) lies in \mathcal{Q} , this result gives what is perhaps the most natural meaning to the statement that \mathcal{F} is real analytic on $\mathcal{M}_{g,k}$ near singular points of this moduli space.

Proof: Let g_0 denote the base metric on Σ_0 , i.e. $g_0 = q_0^*(\delta)$, where δ is the Euclidean metric. Similarly, for $q \in \mathcal{Q}$ we let $g_q = q^*(\delta)$. For each one of these metrics, there is a uniquely determined conformal factor $e^{2\phi_q}$ such that $h_q = e^{2\phi_q}g_q$ is a hyperbolic metric of finite area. These hyperbolic metrics parametrize the Teichmüller space $\mathcal{T}_{g,k}$, and so the theorem will be proved if we show that the map $q \mapsto \phi_q$ is real analytic.

For each end E_j of Σ_0 , fix isothermal coordinates (s_j, θ_j) . The model Delaunay surface D_{τ_j} for this end has the metric

$$g_{\tau_j} = (\tau_j^2 e^{2\sigma_{\tau_j}})(ds_j^2 + d\theta_j^2),$$

and hence

$$g_q = (\tau_j^2 e^{2\sigma_{\tau_j}})(ds_j^2 + d\theta_j^2 + \mathcal{O}(e^{-\alpha s_j}))$$

there, for some $\alpha > 0$. The Delaunay parameters τ_j and the functions σ_{τ_j} depend real analytically on q . We can even modify this conformal factor further and thereby find a function μ_q which depends real analytically on q such that

$$g_{c,q} := e^{2\mu_q}g_q = ds_j^2 + d\theta_j^2 \quad \text{on } E_j.$$

This is a metric with cylindrical ends. Accordingly, decompose the sought-after conformal factor ϕ_q as $\mu_q + \psi_q$. We must prove that ψ_q depends analytically on q .

Letting Δ_q and K_q denote the Laplace operator and Gauss curvature function for $g_{c,q}$, then ψ_q is the unique solution of the PDE

$$\Delta_q \psi_q - e^{2\psi_q} = K_q.$$

Write ψ_0 for the unique solution when $q = q_0$. Note that $K_q = 0$ on all ends, and so $-\log s_j$ is a solution along each E_j .

We shall define natural function spaces X and Y below (these will be certain weighted Sobolev spaces), on which the mapping

$$\mathcal{N} : \mathcal{Q} \times X \longrightarrow Y,$$

defined by

$$(q, \psi) \longmapsto \Delta_q \psi - e^{2\psi} - K_q$$

is locally surjective near (q_0, ψ_0) . To this end, observe that the differential in the second factor is

$$Lw := D_2 \mathcal{N}_{q_0, \psi_0}(w) = \Delta_0 w - 2e^{2\psi_0} w.$$

In terms of the coordinates (s_j, θ_j) on E_j ,

$$L|_{E_j} = \partial_{s_j}^2 + \partial_{\theta_j}^2 - 2/s_j^2.$$

It remains to choose the function spaces X and Y which have the correct properties, in particular that $L : X \mapsto Y$ is surjective. While this is not too difficult, and equivalent theorems surely exist elsewhere in the literature, we sketch the proof for completeness.

As a first attempt, we can try to use a procedure similar to the one outlined in §3.1; namely, if we define the weighted Sobolev spaces $e^{ms} H^t(\Sigma)$ (so u is in this space if $u \in H_{loc}^t$ and on each end E_j , $u = e^{ms_j} v$ where v is in the ordinary Sobolev space $H^t(E_j)$), then it is straightforward to show that

$$L : e^{ms} H^{t+2}(\Sigma) \longrightarrow e^{ms} H^t(\Sigma) \quad (29)$$

is Fredholm whenever $m \notin \mathbb{Z}$. Furthermore, using the maximum principle, which applies since the term of order zero in L is strictly negative, the mapping (29) is injective when $m < 0$; by duality and elliptic regularity, it is then surjective when $m > 0$, $m \neq 1, 2, \dots$. Again as in §3.1, there is a deficiency space consisting of the linear span of cutoffs of a special set of temperate solutions to $Lu = 0$ along each end. We can determine these temperate solutions using separation of variables and see that they are linear combinations of the functions s^2 and s^{-1} . (For convenience, in many places in the remainder of this proof we drop references to the various ends E_j and also omit the subscript j .) Therefore, if $-1 < m < 0$, then for any $f_1 \in e^{ms} H^t(\Sigma)$ it is possible to find a function

$$u_1 = as^2 + bs^{-1} + \tilde{u}, \quad \tilde{u} \in e^{ms} H^{t+2}(\Sigma),$$

where a and b are constants, and such that $Lu_1 = f_1$ on Σ . The deficiency space W consists of all linear combinations of these solutions s^2 and s^{-1} on all ends, and thus is $2k$ -dimensional. We have shown that

$$L : e^{ms} H^{t+2}(\Sigma) \oplus W \longrightarrow e^{ms} H^t(\Sigma) \quad (30)$$

is surjective. A relative index calculation (cf. [7]) shows that the nullspace \mathcal{B} of (30) is k -dimensional.

Unfortunately, this is not the end of the story because the nonlinear operator \mathcal{N} does not carry this domain space into $e^{ms} H^t(\Sigma)$; in fact, there is no evident way to use the geometric context to regularize this mapping. Therefore we proceed further.

We first require Sobolev spaces with polynomial rather than exponential weights. Thus for $t \in \mathbb{N}$ and $\nu \in \mathbb{R}$, let $H_\nu^t(\mathbb{R}^+ \times S^1)$ be the space of functions in H_{loc}^t such that $s^{-1/2+k-\nu} \nabla^k u \in L^2(\mathbb{R}^+ \times S^1)$, $k = 0, \dots, t$. We also define $H_\nu^t(\Sigma)$ as the space of $H_{loc}^t(\Sigma)$ functions which lie in $H_\nu^t(\mathbb{R}^+ \times S^1)$ on each end.

By separation of variables, it is easy to produce a map

$$G_0 : H_{\nu-2}^t(\mathbb{R}^+ \times S^1) \longrightarrow H_\nu^{t+2}(\mathbb{R}^+ \times S^1)$$

for any $\nu \in \mathbb{R}$, $\nu \neq -1, 2$, such that $(\partial_s^2 + \partial_\theta^2 - 2/s^2)G_0 f_0 = f_0$. In other words, G_0 is a right inverse for L . Observe that we do not impose any boundary data, in particular G_0 is not unique. Using G_0 and cutoff functions χ_j on each ends, we can produce an operator

$$\tilde{G}_0 : H_{\nu-2}^t(\Sigma) \longrightarrow H_{\nu}^{t+2}(\Sigma)$$

such that $u_0 := \tilde{G}_0 f_0$ has the property that Lu_0 vanishes outside a compact set of Σ .

Now, if $f \in H_{-2}^t(\Sigma)$ and $t \geq 2$, f is continuous. Since the constant function 1 is a subsolution for the operator L – in fact $L(1) = -2/s^2$ on each end – we can solve a sequence of equations $Lu_j = f$ in Σ_j with $u_j = 0$ on the boundaries $\partial\Sigma_j$, where Σ_j is a smooth exhaustion of Σ by compact sets; the limit of this sequence is a *bounded* solution u of $Lu = f$. By earlier remarks, this is the only bounded solution of this equation. If $f = 0$ on an end, then we know that u must be a linear combination of s^2 and $1/s$ and a term which exponentially decreases on that end, and so by boundedness, $u = u_1 + v$ where $u_1 = a/s$ and $v \in e^{-s} H^{t+2}$.

Finally, let $\nu \in (-1, 0]$, $t \geq 2$ and define

$$X = H_{\nu}^{t+2}(\Sigma), \quad \text{and} \quad Y = H_{\nu-2}^t(\Sigma).$$

We claim that $L : X \rightarrow Y$ is an isomorphism and $\mathcal{N} : \mathcal{Q} \times X \rightarrow Y$ is smooth. To prove these, first suppose that $f \in Y$. Let $u_0 := \tilde{G}_0 f$, then $L(u - u_0) = \tilde{f}$ has compact support. Next, there is also a unique bounded solution \bar{u} of $L\bar{u} = \tilde{f}$. As explained above, separating variables on the end, we see that $\tilde{v} := u_0 + \bar{u}$ is the sum of some multiple of $1/s$ and a function in $H_{\nu}^{t+2}(\Sigma)$. Tracing through this procedure, we have found a bounded map $G : Y \rightarrow X$ such that $LG = I$. Since L does not have any nullspace in X , this map is an isomorphism. So far, we have only used the fact that $t \geq 2$ and $\nu \in (-1, 2)$.

We also note that \mathcal{N} is a real analytic mapping from a neighbourhood of 0 in X to Y . This follows from the considerations above concerning the linear part L of \mathcal{N} , as well as the fact that the nonlinear error term, which has the form $s^{-2}(e^{2w} - 1 - 2w)$ on each end. This is where the restriction that $\nu \leq 0$ is required.

In any case, we may now apply the real analytic version of the implicit function theorem to get the existence of a real analytic mapping $\Psi : \mathcal{Q} \rightarrow X$ such that $\mathcal{N}(q, \Psi(q)) \equiv 0$. Since $\psi_q = \Psi(q)$, we have proved the theorem. \square

5.2 The image of \mathcal{F}

Recall that a connected component of $\mathcal{M}_{g,k}$ is said to be nondegenerate if it contains an element Σ which is (analytically) nondegenerate in the sense of Definition 1. It is proved in [11] that $\mathcal{M}_{g,k}$ contains a nondegenerate component for every (g, k) with $g \geq 0$ and $k \geq 3$. The principal stratum in a nondegenerate component of $\mathcal{M}_{g,k}$ has dimension $3k$. On the other hand, $\mathcal{T}_{g,k}$ is a real analytic manifold of dimension $6g - 6 + 2k$. Therefore one might hope that \mathcal{F} is surjective, at least when k is sufficiently large.

We now give some results concerning the nature and size of the image \mathcal{I} . These require some preliminary definitions.

There is a tautological bundle $\mathbb{V}_{g,k}$ over $\mathcal{T}_{g,k}$ defined by

$$\mathbb{V}_{g,k} = \{([\bar{\Sigma}, p_1, \dots, p_k], p) \mid [(\bar{\Sigma}, p_1, \dots, p_k)] \in \mathcal{T}_{g,k}, p \in \bar{\Sigma} - \{p_1, \dots, p_k\}\}.$$

This is the domain of a natural augmentation map

$$\begin{aligned} \mathcal{A} : \mathbb{V}_{g,k} &\longrightarrow \mathcal{T}_{g,k+1} \\ ([(\bar{\Sigma}, p_1, \dots, p_k)], p) &\longmapsto [(\bar{\Sigma}, p_1, \dots, p_k, p)]. \end{aligned}$$

It is clear that \mathcal{A} is an isomorphism.

Next, there is also a tautological bundle over $\mathcal{M}_{g,k}$,

$$\mathbb{U}_{g,k} = \{(\Sigma, p) \mid \Sigma \in \mathcal{M}_{g,k}, p \in \Sigma\}.$$

For any $(\Sigma, p) \in \mathbb{U}_{g,k}$, let $(0, \tau^*(\Sigma, p))$ denote the largest open interval such that if $0 < \tau < \tau^*(\Sigma, p)$, then the gluing map which attaches a half-Delaunay end with Delaunay parameter τ to the point p exists. It follows from the gluing construction that $\tau^*(\Sigma, p)$ is bounded away from zero on compact sets of Σ , provided Σ lies in a compact set in a nondegenerate stratum of $\mathcal{M}_{g,k}$. Let

$$\mathcal{W}_{g,k} = \{(\Sigma, p, \tau) \mid (\Sigma, p) \in \mathbb{U}_{g,k}, 0 < \tau < \tau^*(\Sigma, p)\}.$$

This is the natural domain of the gluing map

$$\mathfrak{G} : \mathcal{W}_{g,k} \longrightarrow \mathcal{M}_{g,k+1}.$$

This map is continuous, and in fact, real analytic. We shall often omit the subscripts (g, k) from these bundles when the meaning is clear. Also, if C is any subset either of $\mathcal{M}_{g,k}$ or of $\mathbb{U}_{g,k}$, then we let $\mathcal{W}(C)$ denote the portion of \mathcal{W} lying over C ; in particular $\mathcal{W}(\Sigma)$ denotes the natural domain of the gluing map over a fixed surface Σ . We write the CMC surface $\mathfrak{G}_\Sigma(p, \tau)$ as $\Sigma_{p,\tau}$. Finally, note that

$$\lim_{\tau \rightarrow 0} \mathcal{F}(\Sigma_{p,\tau}) = \mathcal{A}(\mathcal{F}(\Sigma), p). \quad (31)$$

Theorem 10 *Suppose that $g = 0$ and $k \geq 3$. Then there is a nondegenerate component of $\mathcal{M}_{0,k}$ on which \mathcal{F} is surjective.*

Proof: We prove this by induction on k . When $k = 3$, then according to [3], $\mathcal{M}_{0,3}$ is homeomorphic to a 3-ball, hence in particular is connected; by [8] it contains a nondegenerate element. In other words, $\mathcal{M}_{0,3}$ contains a single component, and this component is nondegenerate. On the other hand, $\mathcal{T}_{0,3}$ consists of a single point. Hence $\mathcal{F}_{0,3}$ is obviously surjective.

Now suppose that $k \geq 3$, and C_k^0 is a nondegenerate stratum in some component $C_k \subset \mathcal{M}_{0,k}$ such that $\mathcal{F} : C_k^0 \rightarrow \mathcal{T}_{0,k}$ is surjective. Choose any point $[(S^2, p_1, \dots, p_k, p_{k+1})] \in \mathcal{T}_{0,k+1}$ (note that $\bar{\Sigma}$ must be S^2 when $g = 0$), and let $[(S^2, p_1, \dots, p_k), p_{k+1}]$ be the lift of this point to $\mathbb{V}_{0,k}$. Write p for p_{k+1} for simplicity.

By assumption, there is an element $\Sigma \in C_k^0$ such that $\mathcal{F}(\Sigma) = [(S^2, p_1, \dots, p_k)]$. Let \mathcal{B} be some neighbourhood of (Σ, p) in $\mathbb{U}_{0,k}$. Then for any $(\Sigma', q, \tau) \in \mathcal{B} \times (0, \eta)$, we obtain elements $\Sigma'_{q,\tau} \in \mathcal{M}_{0,k+1}$ and $(\mathcal{F}(\Sigma'), q) \in \mathbb{V}_{0,k}$. The theorem will be proved if we show that

$$\mathcal{A}(\mathcal{F}(\Sigma), p) \in (\mathcal{F} \circ \mathfrak{G})(\mathcal{B}, \eta),$$

when η is small enough. But this is clear from (31) using a straightforward degree theory argument.

This proof also shows that the nondegenerate stratum C_{k+1}^0 is obtained inductively by gluing half-Delaunay surfaces with very small necks located at arbitrary points $p \in \Sigma$ for all surfaces $\Sigma \in C_k^0$. \square

As remarked above, when $g > 0$ we no longer expect $\mathcal{F}_{g,k}$ to be surjective. We shall show instead that its codimension does not become unbounded as $k \rightarrow \infty$.

Theorem 11 *Suppose that $g \geq 1$ and that C_{k_0} is a nondegenerate component of \mathcal{M}_{g,k_0} , with nondegenerate stratum $C_{k_0}^0$. Then for each $k = k_0, k_0 + 1, \dots$ there is a nondegenerate component $C_k \subset \mathcal{M}_{g,k}$ such that codimension of the image $\mathcal{I}_k = \mathcal{F}(C_k)$ is bounded as $k \rightarrow \infty$.*

Proof: This is also proved by an inductive procedure. Let us suppose that for some $k \geq k_0$, there is a nondegenerate element $\Sigma \in \mathcal{M}_{g,k}$; suppose also that $\text{rank } D\mathcal{F}|_{\Sigma} \equiv r$ is maximal amongst all such elements. Let C_k be the component of $\mathcal{M}_{g,k}$ containing Σ and \mathcal{I}_k its image by \mathcal{F} in $\mathcal{T}_{g,k}$; The dimension of the stratum of \mathcal{I}_k through $\mathcal{F}(\Sigma)$ is r , and so $d_k = \text{codim}(\mathcal{I}_k) \leq 6g - 6 + 2k - r$. Let us furthermore choose an r -dimensional analytic submanifold \mathcal{S}_k through Σ such that the restriction of \mathcal{F} to it is an analytic diffeomorphism onto its image. Let $\mathbb{U}(\mathcal{S}_k)$ be the portion of the bundle \mathcal{U} lying over \mathcal{S}_k .

We now consider, for some small $\eta > 0$, the restriction of the gluing map

$$\mathfrak{G}_\eta : \mathbb{U}(\mathcal{S}_k) \longrightarrow \mathcal{M}_{g,k+1}.$$

We first claim that the dimension of the image of \mathfrak{G}_η is $r + 2$. This is straightforward, since the differential of \mathfrak{G}_η in the directions of the fibres of $\mathbb{U}(\mathcal{S}_k)$, i.e. letting p vary and $\Sigma' \in \mathcal{S}$ remain fixed, is injective. This produces an $(r + 2)$ -dimensional submanifold \mathcal{S}_{k+1} in $\mathcal{M}_{g,k+1}$, consisting entirely of nondegenerate points. Using (31) again, when η is small enough, $\mathcal{F}(\mathcal{S}_{k+1})$ is $(r + 2)$ -dimensional. However, since $\dim(\mathcal{T}_{g,k+1}) - \dim(\mathcal{T}_{g,k}) = 2$, we see that the codimension of $\mathcal{F}(\mathcal{S}_{k+1})$ is again $6g - 6 + 2k - r$, and so the codimension of the image of the component of $\mathcal{M}_{g,k+1}$ containing \mathcal{S}_{k+1} is bounded by this same number. This proves the theorem. \square

Regarding the global structure of the image $\mathcal{I}_{g,k}$ of $\mathcal{M}_{g,k}$ by \mathcal{F} , we quote a recent nice result of Kusner. To state it, recall that the necksize parameter τ of an end E of $\Sigma \in \mathcal{M}_{g,k}$ is the Delaunay parameter of the Delaunay surface to which this end is asymptotic. This value may be determined using the force integral.

Proposition 10 (Kusner [5]) *Let Σ_ℓ be a sequence of elements in $\mathcal{M}_{g,k}$ and suppose that the necksize parameters $\tau_j^{(\ell)}$ of the ends E_j of Σ_ℓ are all bounded below by some $\eta > 0$. Then either the conformal structures $\mathcal{F}(\Sigma_\ell)$ diverge in $\mathcal{T}_{g,k}$ or else the surfaces Σ_ℓ converge, up to rigid motion, to some limiting surface $\Sigma_\infty \in \mathcal{M}_{g,k}$.*

Kusner's result is somewhat more general, and he phrases it in terms of properness of \mathcal{F} .

Combining Propositions 9 and 10, we obtain

Proposition 11 *The image $\mathcal{I}_{g,k}$ of the forgetful map \mathcal{F} , restricted to any (not necessarily nondegenerate) component in $\mathcal{T}_{g,k}$ is a closed subanalytic set.*

We conclude this subsection with a discussion of the fundamental group of $\mathcal{M}_{g,k}$. While we do not determine this group precisely, we examine the homomorphism

$$\mathcal{F}_* : \pi_1(\mathcal{M}_{g,k}) \longrightarrow \pi_1(\mathcal{T}_{g,k}).$$

The group on the right here is well understood and rather complicated. We show that the image of \mathcal{F}_* is a fairly large subgroup.

We first review some facts about the group $\pi_1(\mathcal{T}_{g,k})$, referring to [1] for more details. There is a subsidiary forgetful map

$$F' : \mathcal{T}_{g,k} \longrightarrow \mathcal{C}(g,k).$$

The space on the right here is the Teichmüller space of conformal structures on a compact surface of genus g (i.e. the classical Teichmüller space) and the configuration space of k points on a compact surface of genus g . To define it, recall that we may identify an element of $\mathcal{T}_{g,k}$ with a hyperbolic metric on the compact surface $\bar{\Sigma}$ along with an ordered k -tuple of distinct points (p_1, \dots, p_k) on $\bar{\Sigma}$ (rather than finite area complete hyperbolic metrics on $\bar{\Sigma} - \{p_1, \dots, p_k\}$). Then F' is defined by forgetting the conformal structure. It is well-known that

$$\mathcal{F}'_* : \pi_1(\mathcal{T}_{g,k}) \longrightarrow \pi_1(\mathcal{C}(g,k))$$

is an isomorphism. We write $\mathcal{F}' = F' \circ \mathcal{F}$.

The space $\mathcal{C}(g, k)$ has a rather interesting topology. Its fundamental group is known as the pure braid group of the surface of genus g on k braids, and we denote it by $B(g, k)$. It is a finitely generated group; the loops Γ_{ij} , $1 \leq i, j \leq k$, $i \neq j$, corresponding to the point p_j traversing a small loop winding around the point p_i once, with all other p_ℓ fixed, comprise a generating set.

Theorem 12 *When $k \geq 3$, the map*

$$\mathcal{F}'_* : \pi_1(\mathcal{M}_{0,k}) \longrightarrow \pi_1(\mathcal{C}(0, k))$$

is an epimorphism.

Theorem 13 *Suppose $g \geq 1$ and \mathcal{M}_{g,k_0} contains a nondegenerate component. Then for any $k \geq k_0$, $\mathcal{M}_{g,k}$ contains a nondegenerate component C_k such that the image of the homomorphism*

$$\mathcal{F}'_* : \pi_1(C_k) \longrightarrow \pi_1(\mathcal{C}(g, k))$$

contains the subgroup of $B(g, k)$ generated by the collection of loops Γ_{ij} , $j > k_0$.

The proofs of these two theorems are nearly identical. They rely only on the simple observations that any of the loops Γ_{ij} are in the image of \mathcal{F}'_* when $g = 0$, while when $g > 0$, at least those loops with $j > k_0$ are in the image.

5.3 The structure of the CMC moduli space

We conclude this paper by describing informally what we know at this point about the CMC moduli spaces $\mathcal{M}_{g,k}$.

As before, we let $\mathcal{I}_{g,k}$ denote the image of $\mathcal{M}_{g,k}$ under the forgetful map \mathcal{F} . Then it is tempting to think of

$$\mathcal{F} : \mathcal{M}_{g,k} \longrightarrow \mathcal{I}_{g,k} \subset \mathcal{T}_{g,k}$$

as a sort of singular fibration. We have shown that all spaces here are real analytic or subanalytic, hence stratified, and \mathcal{F} is a real analytic mapping. The image $\mathcal{I}_{g,k}$ detects at least some fairly large portion of the fundamental group of $\mathcal{T}_{g,k}$ when $g \geq 1$ and k is large; it detects all of it when $g = 0$ and $k \geq 3$. If $\mathcal{M}_{g,k}(\eta)$ denotes the subset of surfaces $\Sigma \in \mathcal{M}_{g,k}$ with necksizes of all ends of Σ no smaller than η , then by Kusner's theorem, the restriction of \mathcal{F} to this subset is proper. As already noted, one way to interpret this is that if Σ_j is a divergent sequence of surfaces in $\mathcal{M}_{g,k}$ with no end necksizes tending to zero, then necessarily the conformal structures $\mathcal{F}(\Sigma_j)$ must be degenerating. This behaviour does indeed occur; for example, the connected sum construction of [10], cf. also [11], shows that it is possible to construct sequences of surfaces with no end necksizes tending to zero, but with some interior necks pinching off. This corresponds to degeneration in $\mathcal{T}_{g,k}$, and these examples exist even when $g = 0$.

We conclude with a number of open questions :

- Is $\mathcal{M}_{g,k}$ connected? The only case where this is understood (and known to be true) is when $(g, k) = (0, 3)$ by [3].
- Do the fibres of \mathcal{F} ever have nontrivial topology; for example, do they ever contain homotopically nontrivial loops? This does not occur in $\mathcal{M}_{0,3}$, and if the fibres are always contractible, then the image $\mathcal{I}_{g,k}$ of any component of $\mathcal{M}_{g,k}$ would be a retract of that component.

- Is $\mathcal{F}_{g,k}$ ever surjective when $g > 0$? A heuristic argument against this might be made by considering configurations $(\overline{\Sigma}, p_1, \dots, p_k)$ where all of the points p_j are contained in a small neighbourhood (for example, a small ball relative to the conformally equivalent flat or hyperbolic metric on $\overline{\Sigma}$); it seems likely that the CMC balancing formulæ would rule out CMC realizations of such configurations.
- Construct, or prove the existence of, a degenerate CMC surface Σ in some $\mathcal{M}_{g,k}$. Although the possibility of their existence adds significant complications throughout the theory, to date none are known to exist.
- In a related direction, it seems quite likely that every element in $\mathcal{M}_{0,3}$ is nondegenerate, and it would be very useful to know whether this is true. One motivation is that the (otherwise) very explicit geometric knowledge about these surfaces makes them ideally suited as building blocks in gluing constructions, but to use them in this way requires knowing that they are nondegenerate.
- We have not discussed the geometric structure on these CMC moduli spaces. Along these lines, it is proved in [7] that on the infinitesimal level, each $\mathcal{M}_{g,k}$ has the structure of a Lagrangian submanifold of a larger symplectic submanifold. More precisely, the tangent space of $\mathcal{M}_{g,k}$ at any nondegenerate point is a Lagrangian subspace of a natural symplectic vector space. It is not too difficult to make this global picture more precise, but the more compelling question is : what can be done with it? Some simple examples and other evidence point to the possibility that there is a tautological one-form on some large subset of this ambient symplectic manifold which becomes singular on the subvariety of surfaces with at least one (asymptotically) cylindrical end. The periods of this one-form appear to have direct geometric meaning. Is there anything else which can be done with this Lagrangian structure?

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